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PERIODIC SOLUTIONS IN SUPERLINEAR PARABOLIC PROBLEMS

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Abstract. Consider the Dirichlet problem for the parabolic equation $u_t = \Delta u + m(t)g(x,u)$ in $\Omega \times (0,\infty)$ where Ω is a smoothly bounded, convex domain in $\mathbb{R}^n$ and g has superlinear subcritical growth in u. If m is periodic, positive and m, g satisfy some technical conditions then we prove the existence of a positive periodic solution.

1. Introduction

Consider the following parabolic semilinear problem

$$\text{(P)} \qquad \begin{cases} u_t = \Delta u + m(t)g(x,u), & x \in \Omega,\ t \in (0,T), \\ u = 0, & x \in \partial\Omega,\ t \in (0,T), \\ u(\cdot,0) = u(\cdot,T), & x \in \bar{\Omega}, \\ u > 0, & x \in \Omega,\ t \in (0,T) \end{cases}$$

where Ω is a smoothly bounded, convex domain in $\mathbb{R}^n$, m is a positive, periodic function and $g(x,\cdot)$ is superlinear with subcritical growth. The goal of this paper is to prove the existence of at least one solution of problem (P).

Let us briefly mention what is known about the existence of solutions of (P) in the case when $g(x,u) = g(u)$ is independent of x. In [**2**], M.J. Esteban derived the existence of at least one solution of (P) under the following assumption on the growth of g (and certain technical conditions on m, g)

$$\limsup_{u\to+\infty} \frac{g(u)}{u^\sigma} = 0 \qquad \text{for some} \qquad \sigma < \frac{3n+8}{3n-4}\ .$$

In the recent paper [**5**], P. Quittner improved this result considerably by obtaining the existence with the optimal growth assumption on g, i.e. $|g(u)| \le c(1+|u|^p)$, $1 < p < p_S$ where

$$(1) \qquad p_S = \frac{n+2}{n-2} \quad \text{if} \quad n > 2, \qquad p_S = +\infty \quad \text{for} \quad n \le 2.$$

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Here, we prove the existence of at least one solution of (P) in the general case when $g = g(x,u)$. In order to do so, we impose some technical conditions on m, g. Our assumption on the growth of $g(x,\cdot)$ is optimal.

Three essential steps are used to obtain the results. We first prove that solutions of (P) are uniformly bounded in $L^4((0,T); H_0^1(\Omega))$. In this step we strongly rely on ideas used by M.J. Esteban in [**2**]. In the second step we obtain a priori estimates in $L^\infty((0,T); H_0^1(\Omega))$. Using the results of P. Quittner in [**4**], these estimates can be achieved under optimal growth condition on g in u variable. In the end we apply essentially the same topological degree argument as in [**2**] to infer the existence of at least one solution of (P). The proofs of the results stated in this paper are just sketched; detailed proofs can be found in [**6**].

In this paper we denote by $|\cdot|_q$, $|\cdot|_{k,q}$ the usual norms in $L^q(\Omega)$, $W^{k,q}(\Omega)$, respectively. We shall use the fact that $|u|_{1,2} = (\int_\Omega |\nabla u|^2)^{\frac{1}{2}}$ defines an equivalent norm in $H_0^1(\Omega)$. Let $g \in C^1(\bar{\Omega} \times \mathbf{R}, \mathbf{R})$ satisfy the following inequalities

$$|g(x,u)| \le c_2|u|^{p_2} + a_2(x), \qquad p_2 < p_S,\ a_2 \in L^{(p_2+1)/p_2}(\Omega), \tag{2}$$

$$g(x,u)\mathrm{sign(u)} \ge c_1|u|^{p_1} - a_1(x), \quad 1 < p_1 \le p_2,\ a_1 \in L^{(p_1+1)/p_1}(\Omega), \tag{3}$$

$$g(x,u)u \ge \mu G(x,u) - a_3(x), \qquad \mu > 2,\ a_3 \in L^1(\Omega), \tag{4}$$

$$ug(x,u) \le \theta G(x,u) + a_4(x), \qquad \theta \in (\mu, p_S + 1), a_4 \in L^1(\Omega), \tag{5}$$

$$|g(x,u) - g(x,v)| \le c_5(a_5(x) + |u|^{r-1} + |v|^{r-1})|u - v|, \tag{6}$$
$$r < p_S,\ a_5 \in L^\xi(\Omega),\ \xi > \frac{n}{2},\ \xi \ge 1$$

where μ, c_1, c_2, c_5 are some positive constants, p_S is from (1), $a_i, i = 1,\dots,5$ are nonnegative functions and $G(x,u) = \int_0^u g(x,v)\,dv$. Function g satisfying (2)-(6) can, roughly speaking, oscillate between the power nonlinearities $|u|^{p_1-1}u$ and $|u|^{p_2-1}u$ as $|u| \to \infty$. Let

$$m \in W^{1,\infty}(0,T), \quad \inf_{t\in[0,T]} m(t) = m_0 > 0,\ m(0) = m(T). \tag{7}$$

To formulate our assumptions on the behaviour of $g(\cdot,u)$ near the boundary $\partial\Omega$, we need to introduce some notation. Denote by $n(x_0)$ the unit outward normal vector to $\partial\Omega$ at the point $x_0 \in \partial\Omega$. The hyperplane $T_{\lambda,n(x_0)} := \{x \in \mathbb{R}^n; x\cdot n(x_0) = \lambda\}$ where $\cdot$ denotes the inner product in $\mathbb{R}^n$, does not intersect $\bar{\Omega}$ for $\lambda \gg 1$. Let $\lambda_0(x_0)$ be such that $T_{\lambda_0(x_0),n(x_0)} \cap \bar{\Omega} \ne \emptyset$ and $T_{\lambda,n(x_0)} \cap \bar{\Omega} = \emptyset$ for all $\lambda > \lambda_0(x_0)$. Define $\Sigma_{\lambda,n(x_0)} := \bigcup_{\gamma>\lambda} T_{\gamma,n(x_0)} \cap \Omega$ and denote by $\Sigma'_{\lambda,n(x_0)}$ the reflection of $\Sigma_{\lambda,n(x_0)}$ in $T_{\lambda,n(x_0)}$. Similarly, let x^λ be the reflection of the point $x \in \Omega$ in $T_{\lambda,n(x_0)}$. Let g, in addition to (2)-(6), satisfy the following conditions: there exists $\varepsilon_0 > 0$ such that for any $x_0 \in \partial\Omega$,

$$g(x^\lambda, u) \ge g(x,u), \qquad x \in \Sigma_{\lambda,n(x_0)},\ \lambda \in [\lambda_0(x_0) - \varepsilon_0, \lambda_0(x_0)),\ u \ge 0, \tag{8}$$

$$g(x_0, 0) = 0, \tag{9}$$

and there exists a nonnegative function $a_6 \in L^1(\Omega)$ such that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ with

$$\sum_{i=1}^{n} x_i \left(\int_0^u g_{x_i}(x,v)\,dv \right) \geq -C_\varepsilon a_6(x) - \varepsilon G(x,u), \qquad x \in \Omega,\ u \geq 0. \tag{10}$$

Moreover, let m satisfy

$$\sup_{0 \leq t \leq T} \frac{(m'(t))^-}{m(t)} < \frac{2n-(n-2)\theta}{r^2(\Omega)} \tag{11}$$

where $n \geq 2$, θ is from (5), $(m'(t))^- = \max\{0, -m'(t)\}$ and $r(\Omega)$ is the radius of the smallest ball containing Ω. For $n = 1$ the condition (11) takes the form

$$\sup_{0 \leq t \leq T} \frac{(m'(t))^-}{m(t)} < \frac{2+\mu}{r^2(\Omega)} \tag{12}$$

where μ is from (4).

Remark 1.1. *These technical assumptions on m can be skipped in the case when $p_1 = p_2 = p$ and $p(n-2) < n$. This follows from* [**3**].

2. Boundedness in $L^4((0,T); H_0^1(\Omega))$

We first prove a uniform bound of $\|u\|_{H^1(Q_T)}$, $(Q_T = \Omega \times (0,T))$, and with the aid of this estimate we derive the bound for u in $L^4((0,T); H_0^1(\Omega))$. From now on, we shall always assume that m and g satisfy all assumptions mentioned above.

Lemma 2.1. *There exists a fixed neighbourhood of $\partial\Omega \times (0,T)$ and a positive constant C such that if u is a solution of* (P) *then $|u|, |\nabla u| \leq C$ on this neighbourhood.*

Proof. For the proof, see [**2**, Lemma 10] or the detailed proof in [**6**]. □

Put $Q_T = \Omega \times (0,T)$ and $S_T = \partial\Omega \times (0,T)$.

Lemma 2.2. *For any solution u of* (P) *the following identity holds:*

$$\begin{aligned}
&\iint_{Q_T} \left[nG(x,u) - \frac{(n-2)}{2} g(x,u)u \right] m(t)\,dx\,dt \\
&\quad = \iint_{Q_T} \left(m'(t)G(x,u) + u_t^2 \right) \frac{|x|^2}{2}\,dx\,dt \\
&\quad - \iint_{Q_T} m(t) \sum_{i=1}^{n} x_i \left(\int_0^u g_{x_i}(x,v)\,dv \right) dx\,dt \\
&\quad + \iint_{S_T} \frac{1}{2} |\nabla u|^2 (x \cdot \vec{n})\,dx\,dt\,.
\end{aligned} \tag{13}$$

Proof. The proof follows by multiplying the equation in (P) by the term $\left(\sum_{i=1}^{n} x_i u_{x_i} - u_t \frac{|x|^2}{2}\right)$ and integrating by parts. □

Corollary 2.1. *Let u be a solution of* (P). *Then u satisfies the following inequality:*

$$\begin{aligned}
&\frac{1}{2}\iint\limits_{Q_T} (r^2(\Omega) - |x|^2) m'(t) G(x,u)\,dx\,dt \\
&+ \iint\limits_{Q_T} m(t)\Big(nG(x,u) - \frac{(n-2)}{2} u g(x,u)\Big)\,dx\,dt \\
&+ \iint\limits_{Q_T} m(t) \sum_{i=1}^{n} x_i \left(\int_0^u g_{x_i}(x,v)\,dv\right)\,dx\,dt \le C
\end{aligned} \tag{14}$$

where C does not depend on u.

Proof. It is sufficient to use identity (13), Lemma 2.1 and proceed in the same way as in the proof of Corollary 16 in [**2**]. □

Proposition 2.1. *There exists $C > 0$ such that*

$$\|u\|_{H^1(Q_T)} \le C \tag{15}$$

for any solution u of (P).

Proof. We multiply the first equation in (P) by u and u_t, integrate by parts and add the resulting equalities to get

$$\|u\|^2_{H^1(Q_T)} = \iint\limits_{Q_T} m(t) g(x,u) u\,dx\,dt - \iint\limits_{Q_T} m'(t) G(x,u)\,dx\,dt. \tag{16}$$

Now (16), (10), Corollary 2.1 and the integrability of the functions a_i, $i = 1,3,4,6$ from (3), (4), (5), (10) imply that in order to be able to prove the uniform bound in $H^1(Q_T)$ it is sufficient to prove this inequality

$$\begin{aligned}
m(t)g(x,u)u - m'(t)G(x,u) \le C \Bigg[&\frac{(r^2(\Omega) - |x|^2)}{2} m'(t) G(x,u) \\
+ m(t)\left((n-\varepsilon)G(x,u) - \frac{(n-2)}{2} u g(x,u)\right)\Bigg] &+ \tilde{C}(a_1(x)^{1+\frac{1}{p_1}} + a_3(x) + a_4(x))
\end{aligned} \tag{17}$$

where ε is some (small) positive number, $C, \tilde{C}$ are appropriate constants and $t \in [0,T], u \ge 0, x \in \Omega$. Let $\bar{x} \in \mathbb{R}^n$ be the center of the ball with the radius $r(\Omega)$ containing Ω. Choose this point as origin. Now, the inequality (17) is easily verified since m satisfies (11) (or (12) if $n = 1$). Thus, the proof of Proposition 2.1 is complete. □

Proposition 2.2. *There exists $C > 0$ such that $\int_0^T |u(s)|^4_{1,2}\,ds \le C$ for any solution u of* (P).

Proof. For $t \in [0,T]$ set

$$S(u(t)) = \frac{1}{2}\int_\Omega |\nabla u(t)|^2\,dx - \int_\Omega m(t)G(x,u(t))\,dx. \tag{18}$$

We first prove that $S(u)$ is uniformly bounded in $L^\infty(0,T)$. From the equation in (P) we find

$$\iint\limits_{Q_T} |\nabla u|^2\,dx\,dt = \iint\limits_{Q_T} m(t)g(x,u)u\,dx\,dt. \tag{19}$$

Using (4), the Young inequality in (3) and (5), we arrive at

$$|G(x,u)| \le \frac{1}{\mu}ug(x,u) + C(a_1(x)^{1+\frac{1}{p_1}} + a_1(x) + a_3(x) + a_4(x)) \tag{20}$$

for all $u \ge 0, x \in \Omega$ and some positive constant C. With the aid of (19), (20) and (15) we conclude

$$\begin{aligned}\iint\limits_{Q_T} m(t)|G(x,u)|\,dx\,dt &\le C(1 + \iint\limits_{Q_T} m(t)g(x,u)u\,dx\,dt)\\ &\le C(1+\|u\|^2_{H^1(Q_T)}) \le C.\end{aligned}$$

Therefore $S(u(\cdot)) \in L^1(0,T)$ and $\|S(u(\cdot))\|_{L^1(0,T)} \le C$. For all $t,s \in [0,T]$ we have

$$\begin{aligned}S(u(t)) - S(u(s)) &= \frac{1}{2}\int_s^t \frac{\partial}{\partial\tau}\left(\int_\Omega |\nabla u(\tau)|^2\,dx\right)d\tau\\ &- \int_\Omega\int_s^t m'(\tau)G(x,u(\tau))\,d\tau\,dx - \int_\Omega\int_s^t m(\tau)g(x,u(\tau))u_\tau(\tau)\,d\tau\,dx\end{aligned}$$

and using (P) we find

$$S(u(t)) - S(u(s)) = -\int_s^t\int_\Omega |u_\tau(\tau)|^2\,dx\,d\tau - \int_s^t\int_\Omega m'(\tau)G(x,u(\tau))\,dx\,d\tau.$$

Obviously

$$\int_s^t\int_\Omega |u_\tau(\tau)|^2\,dx\,d\tau \le \|u\|^2_{H^1(Q_T)} \le C.$$

Multiplying (3) by u ($u \ge 0$) and using the Young inequality we get

$$ug(x,u) + Ca_1(x)^{1+\frac{1}{p_1}} \ge 0 \tag{21}$$

for some positive constant C. Boundedness of $\|u\|_{H^1(Q_T)}$ together with (20) and (21) provide for this estimate

$$\int_s^t\int_\Omega |m'(\tau)G(x,u(\tau))|\,dx\,d\tau \le |m|_{1,\infty}\int_s^t\int_\Omega |G(x,u(\tau))|\,dx\,d\tau$$

$$\leq C\left(1+\int_s^t\int_\Omega m(\tau)u(\tau)g(x,u(\tau))\,dx\,d\tau\right)\leq C\left(1+\iint\limits_{Q_T}|\nabla u|^2\,dx\,dt\right)\leq C.$$

Thus, $|S(u(t))-S(u(s))|\leq C$ for all $s,t\in[0,T]$. Combining this fact with the uniform $L^1(0,T)$ integrability of $S(u(\cdot))$, we get $\sup\limits_{t\in[0,T]}|S(u(t))|\leq C$.

Bound (15) implies the existence of such a $t_0\in[0,T]$ that $|u(t_0)|_2\leq C$ and $\|u_t\|_{L^2(Q_T)}\leq C$. This gives us $\|u\|_{L^\infty((0,T);L^2(\Omega))}\leq C$. Finally, we multiply the equation in (P) by u and integrate over Ω. We find

$$\int_\Omega u(t)u_t(t)\,dx+\int_\Omega|\nabla u(t)|^2\,dx=\int_\Omega m(t)g(x,u(t))u(t)\,dx. \tag{22}$$

Assumption (4) and boundedness of $S(u)$ yield

$$\int_\Omega\frac{m(t)g(x,u(t))u(t)}{\mu}\,dx\geq\int_\Omega m(t)G(x,u(t))\,dx-C\geq\frac{1}{2}\int_\Omega|\nabla u(t)|^2\,dx-C$$

and using (22) and boundedness of u in $L^\infty((0,T);L^2(\Omega))$ we obtain

$$|\nabla u(t)|_2^2\leq C(1+|u(t)u_t(t)|_2)\leq C(1+|u_t(t)|_2). \tag{23}$$

Estimates (23) and (15) guarantee $\int_0^T|u(s)|_{1,2}^4\,ds\leq C$. □

3. Boundedness in $L^\infty((0,T);H_0^1(\Omega))$

Theorem 3.1. *Let p_1, p_2 satisfy*

$$p_2-p_1<\kappa_2(p_2) \tag{24}$$

where κ_2 is defined in [**4**, (2.29)]. *Let u be a solution of* (P). *Then*

$$\sup_{t\in[0,T]}|u(t)|_{1,2}\leq C$$

where C does not depend on u.

Proof. For the proof, see [**6**]. □

Remark 3.1. *As a consequence of the latter estimate we have that solutions of* (P) *are uniformly bounded in $L^\infty(Q_T)$.*

4. Existence

Suppose, in addition to all previous assumptions on m and g, that

$$m(t)g(x,u)+a(x)u\geq 0\qquad (x,t)\in\Omega\times(0,T),\ u\geq 0, \tag{25}$$

$$\limsup_{u\to 0+}\left(\sup_{x\in\Omega}\frac{g(x,u)}{u}\right)<\lambda_1(m) \tag{26}$$

where $\lambda_1(m)$ is the unique positive eigenvalue of the problem

$$\begin{aligned} u_t - \Delta u &= \lambda m(t) u, && x \in \Omega,\ t \in \mathbb{R}, \\ u &= 0, && x \in \partial\Omega,\ t \in \mathbb{R}, \\ u(\cdot, 0) &= u(\cdot, T), && x \in \bar{\Omega}, \end{aligned}$$

having a positive eigenfunction (see [**1**]) and a is positive, α_1-Hölder continuous function for some $\alpha_1 \in (0,1)$, i.e. $a \in C^{0,\alpha_1}(\bar{\Omega})$. The main result of this paper is the following theorem.

Theorem 4.1. *Let p_1, p_2 satisfy* (24). *Then there exists at least one solution of* (P).

Proof. Once we have obtained a priori estimates for solutions of (P), we proceed in the similar way as in the proof of Theorem 4 in [**2**]. First we prove that there exists $\rho > 0$ such that if u is a positive solution of

$$\begin{aligned} u_t - \Delta u &= \lambda m(t) g(x, u), && x \in \Omega,\ t \in (0,T), \\ u &= 0, && x \in \partial\Omega,\ t \in (0,T), \\ u(\cdot, 0) &= u(\cdot, T), && x \in \bar{\Omega},\ \lambda \in [0,1], \end{aligned}$$

then $\|u\|_{L^\infty(Q_T)} > \rho$ and ρ is independent of u and $\lambda \in [0,1]$. This is done in [**2**] in the case when $g(x,u) = u^\alpha$ and the proof is the same in the general case of $g = g(x,u)$ with g satisfying (26).

Fix $q > n+1$ for the rest of our proof. Let $W^{1,q}_T(Q_T)$ be the subspace of $T-$periodic functions from $W^{1,q}(Q_T)$. Define $K_\lambda : W^{1,q}_T(Q_T) \to W^{1,q}_T(Q_T)$ by $K_\lambda u = v$ where

$$\begin{aligned} v_t - \Delta v + \lambda a(x) v &= \lambda(m(t) g(x,u) + a(x) u), && x \in \Omega,\ t \in (0,T), \\ v &= 0, && x \in \partial\Omega,\ t \in (0,T), \\ v(\cdot, 0) &= v(\cdot, T), && x \in \bar{\Omega}, \end{aligned}$$

$\lambda \in [0,1]$ and a is from (25). Operator K_λ is well defined and completely continuous. Let $P : W^{1,q}_T(Q_T) \to \{u \in W^{1,q}_T(Q_T);\, u \geq 0\} =: K^+$ denote the projection of $W^{1,q}_T(Q_T)$ onto the positive cone K^+. This projection is well defined and locally uniformly continuous.

Let C_q denote the norm of the continuous embedding $W^{1,q}(Q_T) \hookrightarrow C^0(\bar{Q}_T)$. Put $\rho' = \rho / C_q$ where ρ is defined above. By $d(\cdot,\cdot,\cdot)$ we denote the Leray-Schauder topological degree. In the same way as in [**2**] we observe that $d(I - P \circ K_\lambda, B_{\rho'}, 0) \equiv$ *const.* for all $\lambda \in [0,1]$ where $B_{\rho'}$ is a ball with center in 0 and radius ρ' in $W^{1,q}_T(Q_T)$. But $K_0 = 0$. Hence, $d(I - P \circ K_\lambda, B_{\rho'}, 0) = 1$.

Let us now introduce a new family of operators from $W^{1,q}_T(Q_T)$ into itself. We say that $T_L u = v$ if v is a solution of

$$\begin{aligned} v_t - \Delta v + a(x)v &= m(t)(g(x,u) + Lu + L) + a(x)u, && x \in \Omega,\ t \in (0,T), \\ v &= 0, && x \in \partial\Omega,\ t \in (0,T), \\ v(\cdot,0) &= v(\cdot,T) && x \in \bar{\Omega} \end{aligned}$$

where $L \geq 0$ and function a is again from (25).

In exactly the same way as in [**2**] we observe that there exists $\tilde{\rho} > \rho'$ and $\tilde{L} \gg 1$ such that $d(I - P \circ T_l, B_{\tilde{\rho}}, 0) \equiv const.$ for all $l \in [0, \tilde{L}]$ and $d(I - P \circ T_{\tilde{L}}, B_{\tilde{\rho}}, 0) = 0$. Finally, we use the degree excision property and find that $d(I - P \circ K_1, B_{\tilde{\rho}} - \bar{B}_{\rho'}, 0) = -1$. This completes the proof of Theorem 4.1. □

5. Example

Let $1 < p_1 \leq p_2 < p_S$ and let p_1, p_2 satisfy (24). Define the function $\beta : \mathbb{R} \to [p_1, p_2)$ by

$$\beta(u) = p_1 + \frac{u^2}{u^2 + 1}(p_2 - p_1).$$

Set $\Omega := \{x \in \mathbb{R}^n;\ \sum_{i=1}^n x_i^2 < \frac{5}{2}\pi\}$. Let us by a^- denote the real number that is smaller than a but close to a. It is easy to verify that $g : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ defined by

$$g(x,u) = (\beta(u) + 1)|u|^{\beta(u)-1}u + |u|^{\beta(u)+1}\beta'(u)\ln|u| + \cos(\sum_{i=1}^n x_i^2)|u|^{p_2^- - 1}u$$

satisfies all assumptions that we needed to prove the existence of at least one solution of (P). The function m can be taken for example in this form: $m(t) = \alpha t(t - T) + 1$ for $\alpha > 0$ sufficiently small.

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