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ANALYSIS OF A SEMIDISCRETE SCHEME FOR SOLVING IMAGE SMOOTHING EQUATION OF MEAN CURVATURE FLOW TYPE

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ABSTRACT. Numerical approximation of a nonlinear diffusion equation of mean curvature flow type is discussed. Convergence and error analysis of a regularized problem is presented.

1. INTRODUCTION

In this paper we analyze a semidiscrete numerical method for solving nonlinear diffusion equation

$$u_t = g(|\nabla u|)|\nabla u|\nabla.(\frac{\nabla u}{|\nabla u|}) \tag{1.1}$$

in a domain $\Omega \subset \mathbb{R}^N$ accompanied with homogeneous Neumann boundary conditions and an initial condition. Equation (1.1) is useful in image processing for selective smoothing of images and shapes. Numerical experiments in processing of 2D and 3D images are presented in [**10**]. Here, we present analysis of a special semidiscrete scheme for solving (1.1).

Equation (1.1) is a degenerate parabolic equation and is related to the so-called *level set equation* ((1.1) with $g(s) \equiv 1$) which has been proposed by Osher & Sethian [**16**],[**21**] for computation of moving fronts in interfacial dynamics. The *level set equation* moves each level line (surface) of 2D (3D) image with the velocity proportional to its normal mean curvature field. This causes intrinsic smoothing of level sets. By means of the Perona-Malik function g (for which a typical choice is, e.g., $g(s) = 1/(1+s^2)$) we control the motion of level sets which are also edges. The smoothing of silhouettes on which the gradient of intensity is large can be slowed down by using g. In analysis and also in computations (see [**10**]) we use the following Evans-Spruck regularization,

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$$\frac{1}{\sqrt{\varepsilon+|\nabla u|^2}}u_t - g(|\nabla u|)\nabla.(\frac{\nabla u}{\sqrt{\varepsilon+|\nabla u|^2}}) = 0 \text{ in } I\times\Omega, \tag{1.2}$$

$$\partial_\nu u = 0 \text{ on } I\times\partial\Omega, \tag{1.3}$$

$$u(0,.) = u_0 \text{ in } \Omega, \tag{1.4}$$

where $1>\varepsilon>0$ is a (small) real number, fixed throughout the whole paper and constants in estimates can depend on it. $I=(0,T)$ is a time-scale interval and $\Omega\subset\mathbb{R}^N$. Using the ideas of Deckelnick and Dziuk [**5**] and Frehse's deformation technique ([**8**]) we analyze (for $N=2$) a finite element approximation of the problem (1.2)-(1.4). In [**5**], the motion of two-dimensional nonparametric surface by its mean curvature, governed by the equation

$$\frac{1}{\sqrt{1+|\nabla u|^2}}u_t - \nabla.(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) = 0 \text{ in } I\times\Omega,$$

is considered, provided $u=0$ on $\partial\Omega$ and starting with smooth initial graph. We adapt their convergence and error estimates results to our situation - equation (1.2) with zero Neumann boundary conditions.

The semidiscrete scheme (Galerkin approximation) for solving (1.2)-(1.4) then reads as follows

$$\int_\Omega \frac{u_{h,t}\varphi_h}{g(|\nabla u_h|)\sqrt{\varepsilon+|\nabla u_h|^2}} + \int_\Omega \frac{\nabla u_h.\nabla\varphi_h}{\sqrt{\varepsilon+|\nabla u_h|^2}} = 0,\ \forall\varphi_h\in X_h, t\in I, \tag{1.5}$$

$$u_h(0,.) = \overline{u}_{h0}, \tag{1.6}$$

where $u_h(t,.)\in X_h$ is the approximation of u, X_h is suitable finite element space with grid size parameter h (see (2.2)) and $\overline{u}_{h0}$ is a modification to our case of the so called minimal surface projection of continuous initial data u_0 (see (4.1)).

Our purpose is to prove the convergence of u_h to u in some functional spaces. After some notations and assumptions given in Section 2, we present the main results- existence and error estimates- in Section 3. Section 4 is devoted to proofs of theorems.

2. Notations and assumptions

We shall denote the usual norm in Sobolev space $H^m(\Omega)$ by $||.||_m$, the norm in $H^{m,p}(\Omega)$ by $||.||_{m,p}$ where $m\geq 0, p\geq 1$; for $m=0$ we write $||.||$ and $||.||_{L_p}$ respectively. In our theoretical analysis we consider a bounded domain

$$\Omega\subset\mathbb{R}^2 \text{ with } \partial\Omega\in C^6. \tag{2.1}$$

Let τ_h be a partition of Ω into generalized isoparametric triangles T, i.e. T is a triangle if $\bar{T}$ and $\partial\Omega$ have at most one point in common, otherwise one of the faces may be curved. The usual regularity condition is fulfilled [**4**, Chapter 2.1]. We define the finite dimensional subspace X_h by

$$X_h := \{v_h\in C(\Omega) | v_h \text{ is linear on each } T\in\tau_h\} \tag{2.2}$$

where the isoparametric modification is used in curved elements ([**22**],[**23**]). Under these hypotheses, for functions $v \in H^{k,p}(\Omega)$, $2 \leq p \leq \infty$, and the corresponding interpolants $I_h v$, $I_h : H^{k,p}(\Omega) \rightarrow X_h$, the usual approximation and inverse properties hold (see [**4**, Theorems 3.2.6, 3.3.6]):

$$||(v - I_h v)||_{j,p} \leq ch^{m-j}||\nabla^m v||_{L_p},\ 0 \leq j \leq 1, m = \min(2,k) \tag{2.3}$$

and for $v_h \in X_h$ we have

$$\begin{aligned} ||\nabla v_h||_{L_p} &\leq ch^{-1}||v_h||_{L_p}, 1 \leq p \leq \infty \\ ||v_h||_{L_\infty} &\leq ch^{-1}||v_h|| \\ ||v_h||_{L_\infty} &\leq c|\log h|^{1/2}||v_h||_1. \end{aligned} \tag{2.4}$$

For the data of (1.2)-(1.4) we assume that

$$g \in C^4(\mathbb{R}), g(0) = 1, 0 < g(s) \leq 1 \text{ (we admit } g(s) \rightarrow 0, \text{ for } s \rightarrow \infty), \tag{2.5}$$
$$\text{with bounded derivatives up to 4-th order.}$$

$$\begin{gathered} u_0(x) \in C^5(\bar{\Omega}) \text{ satisfying the compatibility conditions} \\ \tfrac{\partial^{|\alpha|} u_0(x)}{\partial x_1^{\alpha_1} \ldots \partial x_N^{\alpha_N}}|_{\partial\Omega} = 0, \text{ for } |\alpha| \leq 3. \end{gathered} \tag{2.6}$$

3. Main results

As we have mentioned above for proving the existence of a solution of the continuous problem in adequate function spaces and obtaining some error estimates for discrete solution we use the ideas and results of Deckelnick and Dziuk [**5**]. Let us state an existence and uniqueness of a solution result for problem (1.2) - (1.4).

Theorem 3.1. *Let (2.1), (2.5) and (2.6) be satisfied. Then there exists a time $T > 0$ such that (1.2)-(1.4) has a unique solution $u \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^6(\Omega))$ with $u_t \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^4(\Omega))$ and $u_{tt} \in L_\infty(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega))$.*

For the Galerkin approximation u_h given by (1.5)-(1.6) and its relation to the continuous solution u from Theorem 3.1 we have

Theorem 3.2. *Let (2.1), (2.5) and (2.6) be satisfied. There exists $h_0 > 0$ such that problem (1.5)-(1.6) has a unique solution $u_h \in L_\infty(I, L_2(\Omega)) \cap L_2(I, H^1(\Omega))$ for all $0 < h \leq h_0$. Furthermore, we have the following error estimates:*

$$\sup_{(0,T)} ||u - u_h|| \leq ch^2|\log h|^2, \quad (\int_0^T ||\nabla(u - u_h)||^2)^{1/2} \leq ch,$$

$$\sup_{(0,T)} ||u_t - u_{h,t}|| \leq ch|\log h|, \quad (\int_0^T ||\nabla(u_t - u_{h,t})||^2)^{1/2} \leq ch|\log h|.$$

These statements will be consequences of results obtained by deformation technique introduced by Frehse [**8**] which has been used also in [**5**]. We consider the following family of initial-boundary value problems depending on a parameter $\sigma \in [0,1]$:

$$\begin{aligned} u_t^\sigma - g(\sigma|\nabla u^\sigma|)\frac{\sqrt{\varepsilon+\sigma|\nabla u^\sigma|^2}}{(1-\sigma)\sqrt{\varepsilon}+\sigma}\nabla.(\frac{\nabla u^\sigma}{\sqrt{\varepsilon+|\nabla u^\sigma|^2}}) &= 0 \ \text{ in } I\times\Omega, \qquad (P^\sigma) \\ \partial_\nu u^\sigma &= 0 \ \text{ on } I\times\partial\Omega, \\ u^\sigma(0,.) &= u_0 \ \text{ in } \Omega. \end{aligned}$$

The corresponding Galerkin approximation then reads as

$$\begin{aligned} \int_\Omega \frac{((1-\sigma)\sqrt{\varepsilon}+\sigma)u^\sigma_{h,t}\varphi_h}{g(\sigma|\nabla u^\sigma_h|)\sqrt{\varepsilon+\sigma|\nabla u^\sigma_h|^2}} + \int_\Omega \frac{\nabla u^\sigma_h.\nabla\varphi_h}{\sqrt{\varepsilon+|\nabla u^\sigma_h|^2}} &= 0,\ \forall\varphi_h\in X_h, t\in I, \qquad (P^\sigma_h) \\ u^\sigma_h(0,.) &= \bar{u}_{h,0}\ , \end{aligned}$$

where $\bar{u}_{h,0}$ is defined as in (4.1).

We can prove the existence result for the continuous problem (P^σ)

Theorem 3.3. *Let (2.1), (2.5) and (2.6) be satisfied. Then there exists a unique solution $u^\sigma \in L_\infty(I;H^5(\Omega))\cap L_2(I;H^6(\Omega))$ with $u^\sigma_t \in L_\infty(I;H^3(\Omega))\cap L_2(I;H^4(\Omega))$, $u^\sigma_{tt}\in L_\infty(I;H^1(\Omega))\cap L_2(I;H^2(\Omega))$ to problem (P^σ), provided that $T>0$ is small enough.*

In case $\sigma=1$, (P^σ) is our original problem (1.2)-(1.4), so if we prove the Theorem 3.3, Theorem 3.1 is also proved. In case $\sigma=0$, (P^σ) is deformed into

$$\begin{aligned} u_t - \nabla.(\frac{\nabla u}{\sqrt{\varepsilon+|\nabla u|^2}}) &= 0 \ \text{ in } I\times\Omega \\ \partial_\nu u &= 0 \ \text{ on } I\times\partial\Omega \\ u(0,.) &= u_0 \ \text{ in } \Omega. \end{aligned} \tag{3.1}$$

This equation is still nonlinear but its elliptic part is in the divergence form. Therefore we first investigate problem (3.1) and its Galerkin approximation u_h given by

$$\begin{aligned} \int_\Omega u_{h,t}\varphi_h + \int_\Omega \frac{\nabla u_h.\nabla\varphi_h}{\sqrt{\varepsilon+|\nabla u_h|^2}} &= 0,\ \forall\varphi_h\in X_h,\ t\in I, \\ u_h(0,.) &= \bar{u}_{h,0}. \end{aligned} \tag{3.2}$$

We obtain the following result which itself gives the error estimates for the finite element approximation of widely used regularization of *pure anisotropic diffusion* introduced by Osher & Rudin [**15**].

Theorem 3.4. *Let (2.1), (2.6) be satisfied. Let u be a solution to (3.1) and let u_h be a discrete solution given by (3.2). Then*

$$\sup_{(0,T)} ||\nabla u_h||_{L_\infty} \le c,$$

$$\sup_{(0,T)} ||u-u_h|| \le ch^2|\log h|^2, \quad (\int_0^T ||\nabla(u_t-u_{h,t})||^2) \le ch^2|\log h|^2.$$

Finally, for $h \le 1$ and $\gamma > 0, k_1 > 0$ we define a set $\Theta_h \subseteq [0,1]$ by

$$\Theta_h := \{\sigma \in [0,1] \mid (P_h^\sigma) \text{ has a solution } u_h^\sigma \text{ on } I \text{ and}$$
$$||\nabla u_h^\sigma||_{L_\infty} < 2\gamma, \int_0^T ||\nabla(u_t^\sigma - u_{h,t}^\sigma)||^2 < k_1^2 h^2 |\log h|^2\}$$

where γ is a uniform upper bound on $||\nabla u^\sigma||_{L_\infty}$ for $\sigma \in [0,1]$. We prove the following result.

Theorem 3.5. *For each $h \le h_0$ (it may depend on the data of the problem and k_1) the set Θ_h is nonempty, open and closed with respect to $[0,1]$ and therefore must coincide with $[0,1]$.*

Since $u^1 = u$, Theorem 3.1 is a direct consequence of Theorem 3.3. Theorem 3.5 together with the fact that $u_h^1 = u_h$ will be used in the proof of Theorem 3.2.

4. Proofs of theorems

Proposition 4.1. *For every $u \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^6(\Omega))$ with $u_t \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^4(\Omega))$, $u_{tt} \in L_\infty(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega))$ and for all $0 \le h \le h_0$, h_0 sufficiently small, there exists a unique function $\bar{u}_h$, $\bar{u}_h(t,.) \in X_h$ (for a.e. $t \in I$), such that for every $\varphi_h \in X_h$*

$$\int_\Omega \bar{u}_h \varphi_h + \int_\Omega \frac{\nabla \bar{u}_h . \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = \int_\Omega u \varphi_h + \int_\Omega \frac{\nabla u . \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2}} \tag{4.1}$$

and the error between u and $\bar{u}_h$ can be estimated as follows

$$\sup_{(0,T)} ||u - \bar{u}_h|| + h \sup_{(0,T)} ||\nabla(u - \bar{u}_h)|| \le Ch^2, \tag{4.2}$$

$$\sup_{(0,T)} ||u - \bar{u}_h||_{L_\infty} + h \sup_{(0,T)} ||\nabla(u - \bar{u}_h)||_{L_\infty} \le Ch^2 |\log h|, \tag{4.3}$$

$$\sup_{(0,T)} ||u_t - \bar{u}_{h,t}|| \le Ch^2 |\log h|^2, \quad \sup_{(0,T)} ||\nabla(u_t - \bar{u}_{h,t})|| \le Ch, \tag{4.4}$$

$$(\int_0^T ||\nabla(u_{tt} - \bar{u}_{h,tt})||^2)^{1/2} \le Ch|\log h|, \tag{4.5}$$

$$(\int_0^T ||u_{tt} - \bar{u}_{h,tt}||^2)^{1/2} \le Ch|\log h|. \tag{4.6}$$

Remark: The definition of so-called surface projection $\bar{u}_h$ is different as in [**5**] due to Neumann boundary condition (see also [**20**]).

Proof. From equation (4.1) we immediately have

$$\int_\Omega (u - \bar{u}_h)\varphi_h + \int_\Omega \frac{\nabla(u - \bar{u}_h).\nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = \int_\Omega \left(\frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla u|^2}} \right) \nabla u . \nabla \varphi_h.$$

We take $\varphi_h = I_h u - \bar{u}_h \in X_h$ and after some rearrangement we obtain

$$\begin{aligned}
&\int_\Omega |u-\bar{u}_h|^2 + \int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} \\
&= \int_\Omega \left(\frac{1}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} - \frac{1}{\sqrt{\varepsilon+|\nabla u|^2}}\right)\nabla u.\nabla(I_h u-\bar{u}_h) \\
&+ \int_\Omega (u-\bar{u}_h)(u-I_h u) + \int_\Omega \frac{\nabla(u-\bar{u}_h).\nabla(u-I_h u)}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} = I_1+I_2+I_3.
\end{aligned}$$

We estimate

$$\begin{aligned}
|I_1| &\le \int_\Omega \frac{|\nabla u||\nabla(I_h u-\bar{u}_h)||\nabla(u-\bar{u}_h)|(|\nabla u|+|\nabla\bar{u}_h|)}{\sqrt{\varepsilon+|\nabla u|^2}\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}(\sqrt{\varepsilon+|\nabla u|^2}+\sqrt{\varepsilon+|\nabla\bar{u}_h|^2})} \\
&\le \gamma\int_\Omega \frac{|\nabla(I_h u-u)||\nabla(u-\bar{u}_h)|}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} + \gamma\int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} \\
&\le \gamma(1+\delta_1)\int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} + \bar{C}C_{\delta_1}||\nabla(u-I_h u)||^2, \\
|I_2| &\le \delta_2||u-\bar{u}_h||^2 + C_{\delta_2}||u-I_h u||^2, \\
|I_3| &\le \delta_3\int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} + C_{\delta_3}\int_\Omega \frac{|\nabla(u-I_h u)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} \\
&\le \delta_3\int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} + \bar{C}C_{\delta_3}||\nabla(u-I_h u)||^2
\end{aligned}$$

where $\gamma = \max_\Omega \frac{|\nabla u|}{\sqrt{\varepsilon+|\nabla u|^2}} < 1$. Then, for $\delta_i, i=1,2,3$ sufficiently small, we obtain

$$||u-\bar{u}_h||^2 + \int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} \le C||u-I_h u||_1^2.$$

Using (2.3) and the regularity of u we have

$$||u-\bar{u}_h||^2 + \int_\Omega \frac{|\nabla(u-\bar{u}_h)|^2}{\sqrt{\varepsilon+|\nabla\bar{u}_h|^2}} \le C_1 h^2||u||_2^2 \le Ch^2.$$

Now, one can obtain (see also [**11**]) that

$$||\nabla\bar{u}_h||_{L_\infty} \le C. \tag{4.7}$$

and so we derive the estimate for $||\nabla(u-\bar{u}_h)||$ in (4.2).

The estimate for $||u-\bar{u}_h||$ in (4.2) and estimates in (4.3) can be proved in similar way as in [**18**, Theorem 1] and it's mentioned modification, (see also [**9**]) with respect to the definition of $\bar{u}_h$ see also [**20**, Theorem 1], for linear case with Neumann boundary condition. The proof is rather technical so we omit them here. Next we will use the abbreviation

$$F(p) = \frac{p}{\sqrt{\varepsilon+|p|^2}} \quad (p \in \mathbb{R}^2). \tag{4.8}$$

Let us differentiate (4.1) with respect to t and get

$$\int_\Omega (u-\bar u_h)_t\varphi_h + \int_\Omega F'(\nabla u)\nabla u_t.\nabla\varphi_h - \int_\Omega F^{'}(\nabla \bar u_h)\nabla \bar u_{h,t}.\nabla\varphi_h = 0. \tag{4.9}$$

We take $\varphi_h = I_h u_t - \bar u_{h,t}$ and using the properties of F and u we successively obtain

$$\begin{aligned}
&||u_t-\bar u_{h,t}||^2 + \int_\Omega F'(\nabla\bar u_h)|\nabla(u_t-\bar u_{h,t})|^2\\
&= \int_\Omega (u_t-\bar u_{h,t})(u_t - I_h u_t) + \int_\Omega F'(\nabla \bar u_h)\nabla(u_t-\bar u_{h,t}).\nabla(u_t-I_hu_t)\\
&+\int_\Omega (F'(\nabla\bar u_h)-F'(\nabla u))\nabla u_t.\nabla(I_hu_t-\bar u_{h,t})\\
&\le \delta_1||u_t-\bar u_{h,t}||^2 + C_{\delta_1}||u_t-I_hu_t||^2 + C_1\int_\Omega |\nabla(u_t-\bar u_{h,t})||\nabla(u_t-I_hu_t)|\\
&+C_2||\nabla u_t||_{L_\infty}\int_\Omega|\nabla(u-\bar u_h)||\nabla(I_hu_t-\bar u_{h,t})|\\
&\le \delta_1||u_t-\bar u_{h,t}||^2 + C_{\delta_1}||u_t-I_hu_t||^2 + \delta_2||\nabla(u_t-\bar u_{h,t})||^2\\
&+C_{\delta_2}||\nabla(u_t-I_hu_t)||^2 + C||\nabla u_t||_{L_\infty}(||\nabla(u-\bar u_h)||^2 + C||\nabla(I_hu_t-u_t)||^2\\
&+\delta_3||\nabla(u_t-\bar u_{h,t})||^2 + C_{\delta_3}||\nabla(u-\bar u_h)||^2).
\end{aligned}$$

Finally, using the properties of u and strict positivity of F', then for $\delta_i, i=1,2,3$, sufficiently small, we obtain

$$||u_t-\bar u_{h,t}||^2 + ||\nabla(u_t-\bar u_{h,t})||^2 \le C(||u_t-I_hu_t||_1^2 + ||\nabla(u-\bar u_h)||^2),$$

and using (2.3), (4.2) and the properties of u we derive

$$||u_t-\bar u_{h,t}||^2 + ||\nabla(u_t-\bar u_{h,t})||^2 \le C_1h^2||u_t||_2^2 + Ch^2$$

uniformly for t and the estimate for $||\nabla(u_t-\bar u_{h,t})||$ in (4.4) is completed. The rest of (4.4) can be proved in the similar way as in [**5**]. Let v be the solution of the linear equation

$$v-\nabla.(F'(\nabla u)\nabla v) = u_t-\bar u_{h,t} \text{ in } \Omega$$

with zero Neumann boundary condition. We have

$$||u_t-\bar u_{h,t}||^2 = (v, u_t-\bar u_{h,t}) + (F'(\nabla u)\nabla v, \nabla(u_t-\bar u_{h,t}).$$

Using (4.9), (2.3) and the well know estimates of v (see [**12**]) we derive

$$\begin{aligned}
||u_t-\bar u_{h,t}||^2 \le ch||u_t-\bar u_{h,t}||^2 &+ \int_\Omega F'(\nabla u)\nabla(u_t-\bar u_{h,t}).\nabla(v-I_hv)\\
&+\int_\Omega (F'(\nabla\bar u_h)-F'(\nabla u))\nabla\bar u_{h,t}.\nabla I_hv
\end{aligned}$$

and after some rearrangement, for $h\le h_0$, h_0 sufficiently small, we obtain practically in the same way as in [**5**] with respect to zero Neumann boundary condition and the estimates for v [**12**, Chapter 3]:

$$||u_t-\bar u_{h,t}||^2 \le Ch^2||\nabla v|| + ch^2|\log h|^2||\nabla u_t||||\nabla v||$$

$$+C(||u||_{2,\infty}||u_t||_1||v||_1 + ||u_t||_2||v||_2)||u-\bar{u}_h||_{L_\infty} \le ch^2|\log h|^2||u_t - \bar{u}_{h,t}||$$

where the properties of u, v, (4.3) and the estimate for $||\nabla(u_t - \bar{u}_{h,t})||$ were used. Summarizing these results we obtain the estimate (4.4).
Now we shall treat the second derivative with respect to t. After differentiation of (4.9) we obtain

$$\int_\Omega (u-\bar{u}_h)_{tt}\varphi_h + \int_\Omega (F'(\nabla u)\nabla u_{tt} - F'(\nabla \bar{u}_h)\nabla \bar{u}_{h,tt})\,\nabla\varphi_h$$

$$= \int_\Omega (F''(\nabla \bar{u}_h)\nabla\bar{u}_{h,t}\nabla\bar{u}_{h,t} - F''(\nabla u)\nabla u_t\nabla u_t)\,\nabla\varphi_h. \tag{4.10}$$

Inserting $\varphi_h = \bar{u}_{h,tt}$ into (4.10), in a similar way like above we get

$$\int_0^T ||\bar{u}_{h,tt}||^2 + \int_0^T ||\nabla\bar{u}_{h,tt}||^2$$

$$\le C\int_0^T (||u_{tt}||^2 + ||\nabla u_{tt}||^2 + ||\nabla\bar{u}_h||^2_{L_\infty}||\nabla\bar{u}_{h,t}||^2_{L_\infty} + ||\nabla u||^2_{L_\infty}||\nabla u_t||^2_{L_\infty}) \le C$$

due to the properties of u and $\bar{u}_{h,t}$ and using (4.7) and (4.4).

Now we put $\varphi_h = I_h u_{tt} - \bar{u}_{h,tt}$ in (4.10). We get

$$||u_{tt} - \bar{u}_{h,tt}||^2 + ||\nabla(u_{tt} - \bar{u}_{h,tt})||^2 \le C\int_\Omega (u_{tt} - \bar{u}_{h,tt})(I_h u_{tt} - u_{tt})$$

$$+C\int_\Omega F'(\nabla u)\nabla(u_{tt} - \bar{u}_{h,tt}).\nabla(I_h u_{tt} - u_{tt})$$

$$+C\int_\Omega |\nabla(u-\bar{u}_h)||\nabla\bar{u}_{h,tt}||\nabla(I_h u_{tt} - u_{tt})|$$

$$+C\int_\Omega (F''(\nabla\bar{u}_h)\nabla\bar{u}_{h,t}.\nabla\bar{u}_{h,t} - F''(\nabla u)\nabla u_t.\nabla u_t)\nabla(I_h u_{tt} - \bar{u}_{h,tt}).$$

Using (4.3), (2.3), the properties of u and $\bar{u}_h$ we get

$$||u_{tt} - \bar{u}_{h,tt}||^2 + ||\nabla u_{tt} - \bar{u}_{h,tt}||^2 \le ch^2||u_{tt}||_2^2 + ch^2|\log h|||\nabla\bar{u}_{h,tt}||||u_{tt}||_2$$

$$+Ch^2|\log h|^2||\nabla\bar{u}_{h,tt}||^2 + ch^2|\log h|^2.$$

Integrating this inequality and using the boundedness of $||u_{tt}||_2$ and $||\nabla\bar{u}_{h,tt}||$ we obtain estimates (4.5), (4.6). □

Proof. (**Proof of Theorem 3.4.**) From (3.1) and the definition of $\bar{u}_h$ we have

$$\int_\Omega u_t\varphi_h + \int_\Omega \frac{\nabla\bar{u}_h\nabla\varphi_h}{\sqrt{\varepsilon + |\nabla\bar{u}_h|^2}} = \int_\Omega (u-\bar{u}_h)\varphi_h,\ \varphi_h \in X_h,\ t\in I. \tag{4.11}$$

Taking the difference of (4.11) and (3.2) we obtain

$$\int_\Omega (\bar{u}_{h,t} - u_{h,t})\varphi_h + \int_\Omega \frac{\nabla(\bar{u}_h - u_h)\nabla\varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}}$$

$$= \int_\Omega (\bar{u}_{h,t} - u_t)\varphi_h + \int_\Omega \Big(\frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}}\Big)\nabla \bar{u}_h \nabla \varphi_h + \int_\Omega (u - \bar{u}_h)\varphi_h$$

Now, we choose $\varphi_h = \bar{u}_h - u_h$ to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||\bar{u}_h - u_h||^2 + \int_\Omega \frac{|\nabla(\bar{u}_h - u_h)|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \\
&\leq ||\bar{u}_{h,t} - u_t||||\bar{u}_h - u_h|| + ||u - \bar{u}_h||||\bar{u}_h - u_h|| \\
&+ \int_\Omega ||\frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}}|||\nabla(\bar{u}_h - u_h)||\nabla \bar{u}_h| \\
&\leq ||\bar{u}_h - u_h||^2 + ||u_t - \bar{u}_{h,t}||^2 + ||u - \bar{u}_h||^2 + \alpha \int_\Omega \frac{|\nabla(\bar{u}_h - u_h)|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}},
\end{aligned}$$

with $\alpha < 1$, where (4.7) has been used. Using Gronwall's lemma and results of Proposition 4.1 we obtain

$$\sup_{(0,T)} ||(\bar{u}_h - u_h)||^2 + \int_0^T \int_\Omega \frac{|\nabla(\bar{u}_h - u_h)|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \leq Ch^4 |\log h|^4,$$

which together with (4.2) implies the second inequality of Theorem. We can also conclude

$$||\nabla(\bar{u}_h - u_h)||_{L_\infty} \leq C_1 h^{-1} ||\nabla(\bar{u}_h - u_h)|| \leq C_2 h^{-2} ||(\bar{u}_h - u_h)|| \leq C|\log h|^2$$

uniformly for a.e. $t \in [0, T]$ and therefore

$$\sup_{(0,T)} ||\nabla u_h||_{L_\infty} \leq C|\log h|^2. \tag{4.12}$$

Now differentiating (4.11) and (3.2) with respect to t and taking the difference of the resulting equations we obtain

$$\begin{aligned}
&\int_\Omega (\bar{u}_{h,tt} - u_{h,tt})\varphi_h + \int_\Omega \frac{\nabla(\bar{u}_{h,t} - u_{h,t})\nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \\
&= \int_\Omega (\bar{u}_{h,tt} - u_{tt})\varphi_h + \int_\Omega \Big(\frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}}\Big)\nabla \bar{u}_{h,t} \nabla \varphi_h \\
&+ \int_\Omega \frac{\nabla u_h \nabla \varphi_h}{(\sqrt{\varepsilon + |\nabla u_h|^2})^3} \nabla u_h \nabla(\bar{u}_{h,t} - u_{h,t}) \\
&+ \int_\Omega \left(\frac{\nabla \bar{u}_h \nabla \varphi_h}{(\sqrt{\varepsilon + |\nabla \bar{u}_h|^2})^3} \nabla \bar{u}_h - \frac{\nabla u_h \nabla \varphi_h}{(\sqrt{\varepsilon + |\nabla u_h|^2})^3} \nabla u_h\right) \nabla \bar{u}_{h,t} \\
&+ \int_\Omega (u_t - \bar{u}_{h,t})\nabla \varphi_h.
\end{aligned}$$

We take $\varphi_h = \bar{u}_{h,t} - u_{h,t}$ and similarly as above and as in [**5**] we get

$$\frac{1}{2}\frac{d}{dt}||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{\varepsilon}{2(\varepsilon + \sup_{(0,T)} ||\nabla u_h||^2_{L_\infty})}\int_\Omega \frac{|\nabla(\bar{u}_{h,t} - u_{h,t})|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}}$$
$$\leq \frac{1}{2}||u_{tt} - \bar{u}_{h,tt}||^2 + ||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{1}{2}||u_t - \bar{u}_{h,t}||^2$$
$$+C(\varepsilon + \sup_{(0,T)} ||\nabla u_h||^2_{L_\infty})h^2||\nabla \bar{u}_{h,t}||^2_{L_\infty}.$$

Integrating it with respect to t, estimating $||(\bar{u}_{h,t} - u_{h,t})(0)||$ as in [**5**] and using (4.4)-(4.5) we obtain

$$||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{\varepsilon}{2(\varepsilon + \sup_{(0,T)} ||\nabla u_h||^2_{L_\infty})}\int_0^t \int_\Omega \frac{|\nabla(\bar{u}_{h,t} - u_{h,t})|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}}$$

$$\leq Ch^2|\log h|^8 + \int_0^t ||\bar{u}_{h,t} - u_{h,t}||^2.$$

If we apply Gronwall's lemma, we have

$$\sup_{(0,T)} ||\bar{u}_{h,t} - u_{h,t}||^2 \leq ch^2|\log h|^8,$$

and using (4.12) we get

$$\int_0^T \int_\Omega \frac{|\nabla(\bar{u}_{h,t} - u_{h,t})|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \leq Ch^2|\log h|^{12},$$

from which we have

$$||\nabla(\bar{u}_h - u_h(t))||^2 \leq Ch^3|\log h|^{10}$$

and

$$||\nabla u_h(t)||_{L_\infty} \leq C + Ch^{-1}||\nabla(\bar{u}_h - u_h(t))|| \leq C.$$

Now, using this result in similar way as in [**5**] we can obtain

$$\int_0^T ||\nabla(u_t - u_{h,t})||^2 \leq Ch^2|\log h|^2.$$

Proposition 4.1 gives the estimates for $u^\sigma - \bar{u}_h^\sigma$, the next assertion will gives us some useful relations between $u^\sigma - u_h^\sigma$ and $\bar{u}_h^\sigma - u_h^\sigma$ which we will use in the proof of Theorem 3.5. $\square$

Proposition 4.2. *Let u_h^σ be a solution of (P_h^σ) satisfying $||\nabla u_h^\sigma||_{L_\infty} \le 2\gamma$. Denote $e^\sigma = u^\sigma - u_h^\sigma$ and $e_h^\sigma = \bar u_h^\sigma - u_h^\sigma$. Then*

$$\sup_{(0,T)} ||e_h^\sigma||^2 \le c_1 h^4 |\log h|^4 exp(c_1 \int_0^T ||\nabla e_t^\sigma||^2), \tag{4.13}$$

$$\int_0^T ||\nabla e_h^\sigma||^2 \le c_1 h^4 |\log h|^4 \left(1 + exp(c_1 \int_0^T ||\nabla e_t^\sigma||^2). \int_0^T ||\nabla e_t^\sigma||^2\right), \tag{4.14}$$

$$\begin{aligned} \sup_{(0,T)} ||e_{h,t}^\sigma||^2 \le c_2 \ (\ & h^2|\log h|^2 + \sup_{(0,T)} ||\nabla e^\sigma||^2 + (h^2|\log h|^2 \\ & + \sup_{(0,T)} ||\nabla e^\sigma||^2_{L_\infty}) \int_0^T ||\nabla e_t^\sigma||^2 \ ) \ exp(c_2 \int_0^T ||\nabla e_t^\sigma||^2), \end{aligned} \tag{4.15}$$

$$\begin{aligned} \int_0^T ||\nabla e_{h,t}^\sigma||^2 \le {} & c_2((h^2|\log h|^2 + \sup_{(0,T)} ||\nabla e^\sigma||^2) \int_0^T ||\nabla e_t^\sigma||^2 \\ & + h^2|\log h|^2 + \sup_{(0,T)} ||\nabla e^\sigma||^2) \left(1 + exp(c_2 \int_0^T ||\nabla e_t^\sigma||^2) \int_0^T ||\nabla e_t^\sigma||^2\right). \end{aligned} \tag{4.16}$$

Proof. The proof is similar to the one in [**5**]. In order to simplify the presentation we only look at the case $\sigma = 1$ and we omit this upper index. We can write

$$e = u - u_h = (u - \bar u_h) + (\bar u_h - u_h) =: \bar\varepsilon + e_h.$$

Now, from definition of (P^σ), for $\sigma = 1$, we have

$$\int_\Omega \frac{u_t \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2} g(|\nabla u|)} + \int_\Omega \frac{\nabla u \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2}} = 0, \ \forall \varphi_h \in X_h, t \in I.$$

By definition of $\bar u_h$ we get

$$\begin{aligned} & \int_\Omega \frac{u_t \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2} g(|\nabla u|)} + \int_\Omega \frac{\nabla \bar u_h \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar u_h|^2}} \\ & \qquad = \int_\Omega (u - \bar u_h)\varphi_h, \ \forall \varphi_h \in X_h, t \in I. \end{aligned} \tag{4.17}$$

Taking the difference of (4.17) and (P_h^1) we obtain

$$\begin{aligned} & \int_\Omega \frac{e_{h,t} \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} + \int_\Omega \left(\frac{\nabla \bar u_h}{\sqrt{\varepsilon + |\nabla \bar u_h|^2}} - \frac{\nabla u_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \right) \nabla \varphi_h \\ & = \int_\Omega (u - \bar u_h)\varphi_h - \int_\Omega u_t \varphi_h \left(\frac{1}{\sqrt{\varepsilon + |\nabla u|^2} g(|\nabla u|)} - \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} \right) \\ & \quad - \int_\Omega \frac{\bar\varepsilon_t \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)}. \end{aligned} \tag{4.18}$$

We use the function F defined in the proof of Proposition 4.1. We also define $G : \mathbb{R}^2 \to \mathbb{R}$ by

$$G(p) = \frac{1}{\sqrt{\varepsilon + |p|^2 g(|p|)}}.$$

In the same way as in [**5**], using the mean value theorem, we have

$$\frac{\nabla \bar{u}_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} - \frac{\nabla u_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} = \int_0^1 F^{'}(s\nabla \bar{u}_h + (1-s)\nabla u_h) ds \nabla e_h$$

and we can define the bilinear form

$$a^h(v, w) = \int_\Omega \left(\int_0^1 F^{'}(s\nabla \bar{u}_h + (1-s)\nabla u_h) ds \nabla v \right) . \nabla w.$$

Due to the properties of F, a^h is symmetric and from the fact that $||\nabla \bar{u}_h||_{L_\infty} \leq 2\gamma$, $||\nabla u_h||_{L_\infty} \leq 2\gamma$ we can prove

$$a^h(v, v) \geq c_0(\gamma) ||\nabla v||^2. \tag{4.19}$$

Similarly as above, if we denote

$$b^h = \int_0^1 G^{'}(s\nabla u + (1-s)\nabla u_h) ds,$$

we can write

$$\frac{1}{\sqrt{\varepsilon + |\nabla u|^2 g(|\nabla u|)}} - \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2 g(|\nabla u_h|)}} = b^h . \nabla e.$$

Introducing the smooth function $b := G'(\nabla u)$, it is easy to see that

$$|b - b^h| \leq c_1(\gamma) |\nabla e|, \quad |b| \leq c_2(\gamma). \tag{4.20}$$

With these abbreviations (4.18) can be written as

$$\begin{aligned} &\int_\Omega \frac{e_{h,t} \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2 g(|\nabla u_h|)}} + a^h(e_h, \varphi_h) \\ &= \int_\Omega (u - \bar{u}_h) \varphi_h - \int_\Omega u_t b^h . \nabla e \varphi_h - \int_\Omega \frac{\bar{\varepsilon}_t \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2 g(|\nabla u_h|)}}. \end{aligned} \tag{4.21}$$

Now setting $\varphi_h = e_h$ in (4.21) and using (4.19) we get

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{e_h^2}{\sqrt{\varepsilon + |\nabla u_h|^2 g(|\nabla u_h|)}} + c_0 ||\nabla e_h||^2 \\ &\leq -\frac{1}{2} \int_\Omega \frac{e_h^2}{(\sqrt{\varepsilon + |\nabla u_h|^2})^3 g(|\nabla u_h|)} \nabla u_h . \nabla u_{h,t} \\ &-\frac{1}{2} \int_\Omega \frac{e_h^2 g(|\nabla u_h|)_t}{\sqrt{\varepsilon + |\nabla u_h|^2} g^2(|\nabla u_h|)} - \int_\Omega (\bar{u}_h - u) e_h \qquad (4.22) \\ &- \int_\Omega u_t e_h b^h . \nabla e - \int_\Omega \frac{\bar{\varepsilon}_t e_h}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} \qquad (4.23) \\ &= I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned}$$

The term I_1 we estimate in similar way as in [**5**], but the inequality

$$||\varphi||_{L_4} \leq c(||\varphi||_{H^1})^{1/2}(||\varphi||_{L_2})^{1/2} \tag{4.24}$$

is used for $\varphi \in H^1(\Omega)$. We get

$$|I_1| \leq C\int_\Omega |e_h|^2|\nabla u_{h,t}| \leq C||e_h||^2_{L_4}||\nabla u_{h,t}||$$

$$\leq C||e_h||||\nabla e_h||(||\nabla u_t|| + ||\nabla e_t||) \leq \delta||\nabla e_h||^2 + C_\delta(||\nabla u_t||^2 + ||\nabla e_t||^2)||e_h||^2.$$

Using the properties of u and g, for the term I_2 we also obtain

$$|I_2| \leq C\int_\Omega |e_h|^2|\nabla u_{h,t}|$$

and then we continue as above. Employing Proposition 4.1, we have

$$|I_3| \leq Ch^2 + \frac{1}{2}||e_h||^2.$$

We rewrite I_4 into the form

$$I_4 = \int_\Omega u_t e_h(b - b^h).\nabla e - \int_\Omega u_t e_h b.\nabla e = I_{41} + I_{42}.$$

To estimate the term I_{41} we can proceed similarly as in [**5**], and using continuous embedding, (4.20) and Proposition 4.1 we have

$$|I_{41}| \leq Ch^4|\log h|^2 + \delta||\nabla e_h||^2 + C_\delta||e_h||^2.$$

I_{42} we estimate using the properties of b, u and Preposition 1

$$|I_{42}| \leq c_2(\gamma)||u_t||_{L_\infty}\int_\Omega |e_h||\nabla e|$$

$$\leq C(||e_h||||\nabla\bar{\varepsilon}|| + ||e_h||||\nabla e_h||) \leq Ch^2 + C_\delta||e_h||^2 + \delta||\nabla e_h||^2.$$

Finally, I_5 yields

$$|I_5| \leq C||\bar{\varepsilon}_t||^2 + C||e_h||^2 \leq ch^4|\log h|^4 + C||e_h||^2.$$

Now, integrating (4.22) from 0 to t, taking into account the estimates of terms $I_1, \ldots, I_5$, and the fact that $e_h(0) = 0$ we get

$$||e_h||^2 + \int_0^t ||\nabla e_h||^2 \leq Ch^4|\log h|^4 + C\int_0^t (1 + ||\nabla e_t||^2)||e_h||^2.$$

Then Gronwall's lemma gives

$$\sup_{(0,T)} ||e_h(t)||^2 \leq Ch^4|\log h|^4 exp(c\int_0^T ||\nabla e_t||^2)$$

and the proofs of (4.13) and (4.14) are complete.

In order to prove (4.15) and (4.16) we differentiate (4.21) with respect to t. Then we have

$$\int_\Omega \frac{e_{h,tt}\varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}g(|\nabla u_h|)} + a^h(e_{h,t}, \varphi_h)$$

$$= \int_\Omega e_{h,t}\varphi_h \left(\frac{\nabla u_h \nabla u_{h,t}}{(\sqrt{\varepsilon+|\nabla u_h|^2})^3 g(|\nabla u_h|)} + \frac{g(|\nabla u_h|)_t}{\sqrt{\varepsilon+|\nabla u_h|^2} g^2(|\nabla u_h|)} \right)$$
$$-a_t^h(e_h,\varphi_h) - \int_\Omega (\bar u_h - u)_t \varphi_h - \int_\Omega u_{tt} b^h \nabla e \varphi_h - \int_\Omega u_t b_t^h \nabla e \varphi_h$$
$$-\int_\Omega u_t b^h \nabla e_t \varphi_h - \int_\Omega \frac{\bar\varepsilon_{tt}\varphi_h}{\sqrt{\varepsilon+|\nabla u_h|^2} g(|\nabla u_h|)} -$$
$$\int_\Omega \bar\varepsilon_t \varphi_h \left(\frac{\nabla u_h \nabla u_{h,t}}{(\sqrt{\varepsilon+|\nabla u_h|^2})^3 g(|\nabla u_h|)} + \frac{g(|\nabla u_h|)_t}{\sqrt{\varepsilon+|\nabla u_h|^2} g^2(|\nabla u_h|)} \right)$$

Now we take $\varphi_h = e_{h,t}$ and similarly as above we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \frac{e_{h,t}^2}{\sqrt{\varepsilon+|\nabla u_h|^2} g(|\nabla u_h|)} + c_0 ||\nabla e_{h,t}||^2$$
$$\le -\frac{1}{2}\int_\Omega \frac{e_{h,t}^2}{(\sqrt{\varepsilon+|\nabla u_h|^2})^3 g(|\nabla u_h|)} \nabla u_h . \nabla u_{h,t} - \frac{1}{2}\int_\Omega \frac{e_{h,t}^2 g(|\nabla u_h|)_t}{\sqrt{\varepsilon+|\nabla u_h|^2} g^2(|\nabla u_h|)}$$
$$-a_t^h(e_h, e_{h,t}) - \int_\Omega (\bar u_h - u)_t e_{h,t} - \int_\Omega u_{tt} e_{h,t} b^h . \nabla e \tag{4.25}$$
$$-\int_\Omega u_t b_t^h \nabla e \; e_{h,t} - \int_\Omega u_t b^h \nabla e_t \; e_{h,t} - \int_\Omega \frac{\bar\varepsilon_{tt} e_{h,t}}{\sqrt{\varepsilon+|\nabla u_h|^2} g(|\nabla u_h|)}$$
$$-\int_\Omega \bar\varepsilon_t e_{h,t} \left(\frac{\nabla u_h \nabla u_{h,t}}{(\sqrt{\varepsilon+|\nabla u_h|^2})^3 g(|\nabla u_h|)} + \frac{g(|\nabla u_h|)_t}{\sqrt{\varepsilon+|\nabla u_h|^2} g^2(|\nabla u_h|)} \right) \equiv \sum_{i=1}^{9} I_i$$

We estimate terms I_1 and I_2 as above and obtain

$$|I_1| + |I_2| \le \delta ||\nabla e_{h,t}||^2 + C_\delta (||\nabla e_t||^2 + 1) ||e_{h,t}||^2.$$

For the term I_3 we realize

$$\|\frac{\partial}{\partial t}\int_0^1 F'(s\nabla \bar u_h + (1-s)\nabla u_h) ds \| \le c(|\nabla u_t| + |\nabla \bar\varepsilon_t| + |\nabla e_t|)$$

and again as in [**5**] we get

$$|I_3| \le \delta ||\nabla e_{h,t}||^2 + C_\delta ||u_t||_3^2 ||\nabla e_h||^2 + C_\delta ||\nabla \bar\varepsilon_t||^2 + C_\delta ||\nabla e_h||_{L_\infty}^2 ||\nabla e_t||^2.$$

The term I_4 is easy to estimate because

$$|I_4| \le C(||\bar u_{h,t} - u_t||^2 + ||e_{h,t}||^2),$$

and

$$I_5 = -\int_\Omega u_{tt}(b^h - b).\nabla e \; e_{h,t} - \int_\Omega u_{tt} b.\nabla e \; e_{h,t} = I_{51} + I_{52}.$$

Using (4.20) and (4.24) we get

$$|I_{51}| \le \delta ||\nabla e_{h,t}||^2 + C_\delta ||u_{tt}||_1^2 ||e_{h,t}||^2 + C_\delta ||\nabla e||^2 ||u_{tt}||_1,$$

$$|I_{52}| \le c_2(\gamma) ||u_{tt}||_{L_\infty} \int_\Omega |e_{h,t}||\nabla e| \le C||e_{h,t}||^2 + C||\nabla \bar\varepsilon||^2 + C||\nabla e_h||^2.$$

From the inequality
$$|b_t^h| \leq C(|\nabla u_t| + |\nabla e_t|)$$
we obtain as in [**5**]
$$|I_6| \leq C||\nabla e||^2 + C||u_t||_3^2||e_{h,t}||^2 + C||\nabla e||_{L_\infty}^2||\nabla e_t||^2 + C||e_{h,t}||^2.$$
From the properties of b we get
$$|I_7| + |I_8| \leq C||u_t||_{L_\infty}||\nabla e||||\nabla e_{h,t}|| + ||\bar{\varepsilon}_{tt}||||e_{h,t}||$$
$$\leq \delta||\nabla e_{h,t}||^2 + C_\delta||e_{h,t}||^2 + C(||\nabla\bar{\varepsilon}_t||^2 + ||\bar{\varepsilon}_{tt}||^2).$$
Finally, in the last term we use the properties of g and as we get
$$|I_9| \leq C\int_\Omega |\bar{\varepsilon}_t||e_{h,t}||\nabla u_{h,t}| + C\int_\Omega |\bar{\varepsilon}_t||e_{h,t}|.$$
Using the results of Proposition 4.1 we obtain
$$|I_9| \leq Ch^4|\log h|^4 + C_\delta(1 + ||u_t||_3^2)||e_{h,t}||^2 + \delta||\nabla e_{h,t}||^2 + C_\delta h^4|\log h|^4||\nabla e_t||^2.$$
Now integrating (4.25) from 0 to t and using the estimates for $I_1, \dots I_9$ with δ sufficiently small we obtain
$$||e_{h,t}||^2 + \int_0^t ||\nabla e_{h,t}||^2 \leq c||e_{h,t}(0)||^2 + c\int_0^t (||\bar{\varepsilon}_{tt}||^2 + ||\nabla e_t||^2)$$
$$+C\int_0^t (1 + ||u_t||_3^2 + ||u_{tt}||_1^2 + ||\nabla e_t||^2)||e_{h,t}||^2$$
$$+C\int_0^t (||u_t||_3^2 + ||u_{tt}||_1^2 + 1)(h^2 + \sup_{(0,T)}||\nabla e||^2)$$
$$+C(\sup_{(0,T)}||\nabla e||_{L_\infty}^2 + h^2|\log h|^2)\int_0^T ||\nabla e_t||^2 + Ch^4|\log h|^4.$$
Because (see also [**5**])
$$||e_{h,t}(0)|| \leq Ch,$$
we get
$$||e_{h,t}||^2 + \int_0^t ||\nabla e_{h,t}||^2 \leq Ch + C\sup_{(0,T)}||\nabla e||^2 + C(h^2|\log h|^2$$
$$+ \sup_{(0,T)}||\nabla e||_{L_\infty}^2)\int_0^T ||\nabla e_t||^2 + C\int_0^t (1 + ||u_t||_3^2 + ||u_{tt}||_1^2 + ||\nabla e_t||^2)||e_{h,t}||^2.$$
Applying Gronwall's lemma we prove (4.15) and similarly (4.16). □

Proof. (**Proof of Theorem 3.5.**) First, Θ_h is not empty, because $0 \in \Theta_h$ by Theorem 3.4. We prove that Θ_h is open. As in [**5**], let $\sigma \in \Theta_h$, i.e (P_h^σ) is solvable. Using the implicit function theorem it can be shown that (P_h^μ) has a solution for all μ in a neighborhood of σ. Because the same is true for u^σ we obtain the strict inequalities

$$||\nabla u_h^\mu||_{L_\infty} < 2\gamma, \quad \int_0^T ||\nabla(u_t^\mu - u_{h,t}^\mu)||^2 < k_1^2 h^2 |\log h|,$$

provided μ lies in a neighborhood of σ. Finally we prove that Θ_h is closed. Let $\{\sigma_n\}_{n\in N} \subset \Theta_h, \sigma_n \to \sigma, n \to \infty$. Because of continuous dependence of u_h^σ, u^σ on σ we immediately get

$$||\nabla u_h^\sigma||_{L_\infty} \le 2\gamma, \quad \int_0^T ||\nabla(u_t^\sigma - u_{h,t}^\sigma)||^2 \le k_1^2 h^2 |\log h|. \tag{4.26}$$

Furthermore, u_h^σ is the unique solution of (P_h^σ). It remains to show the strict inequalities in (4.26). For this purpose we use results of Proposition 4.2. We infer from (4.14) and (4.26) that

$$\begin{aligned}\int_0^T ||\nabla e_h^\sigma||^2 &\le c_1 h^4 |\log h|^4 \left(1 + exp(c_1 k_1^2 h^2 |\log h|^2)\right) k_1^2 h^2 |\log h|^2 \\ &\le c h^4 |\log h|^4,\end{aligned} \tag{4.27}$$

provided $h \le h_0$ and $h_0^2 |\log h_0| \le c_1^{-1} k_1^{-2}$. With the help of (4.26), (4.27) and Proposition 4.1, since $e_h^\sigma(0) = 0$ we have

$$||\nabla e_h^\sigma||^2 \le 2 \left(\int_0^T ||\nabla e_h^\sigma||^2\right)^{1/2} \left(\int_0^T ||\nabla e_{h,t}^\sigma||^2\right)^{1/2} \le C h^3 |\log h|^3 (k_1 + 1).$$

Then using (2.4) and Proposition 4.1 we also have

$$||\nabla e_h^\sigma||_{L_\infty}^2 \le C(1 + k_1) h |\log h|^3.$$

Combining these results with Proposition 4.1 we get

$$||\nabla e^\sigma||^2 \le C h^2 + c k_1 h^3 |\log h|^3 \tag{4.28}$$

$$||\nabla e^\sigma||_{L_\infty}^2 \le C h^2 |\log h|^2 |c(1 + k_1) h |\log h|^3 \le C(1 + k_1) h |\log h|^3 \tag{4.29}$$

for $h \le h_0$. So we immediately obtain

$$||\nabla u_h^\sigma||_{L_\infty} \le ||\nabla u^\sigma||_{L_\infty} + ||\nabla e^\sigma||_{L_\infty} \le \gamma + c\sqrt{1 + k_1} h^{1/2} |\log h|^{3/2} < 2\gamma,$$

for $h \le h_1 \le h_0$ and $c\sqrt{1 + k_1} h_1^{1/2} |\log h_1|^{3/2} < \gamma$. Combining (4.16), (4.26), (4.28) and (4.29) we have

$$\begin{aligned}&\int_0^T ||\nabla e_{h,t}^\sigma||^2 \\ &\le c(h^2 |\log h|^2 + k_1 h^3 |\log h|^3 + (h^2 |\log h|^2| + (1 + k_1) h |\log h|^3) k_1^2 h^2 |\log h|^2) \\ &\le c h^2 |\log h|^2 (1 + (1 + k_1)^3 h |\log h|^3).\end{aligned}$$

Now, we use (4.4) to obtain

$$\int_0^T ||\nabla e_t^\sigma||^2 \le 2 \left(\int_0^T ||\nabla e_{h,t}^\sigma||^2 + \int_0^T ||\nabla(u_t^\sigma - \bar{u}_{h,t}^\sigma)||^2\right)$$

$$\le ch^2|\log h|^2(1+(k_1+1)^3 h|\log h|^3).$$

Let us fix $k_1 > 2c$ and choose $h_2 \le h_1$ so small that $(1+k_1)^3 h_2|\log h_2|^3 \le 1$. Then

$$\int_0^T ||\nabla e_t^\sigma||^2 < k_1 h^2|\log h|^2,$$

which is the second inequality we have had to prove. So $\sigma \in \Theta_h$ and the set is closed. □

Proof. (**Proof of Theorem 3.2.**) The existence of a solution u_h is a consequence of Theorem 3.5, existence of this discrete solution and its properties we can obtain also due the properties of Galerkin approximation of elliptic operator (see also [**18**]). The fourth error estimate is fulfilled due to Theorem 3.5, since $\Theta_h = [0,1]$. To obtain the others we can use the results of Propositions 4.1 and 4.2. So

$$\sup_{(0,T)} ||u-u_h|| \le \sup_{(0,T)} ||\bar{\varepsilon}|| + \sup_{(0,T)} ||e_h|| \le Ch^2 + Ch^2(e^{C\int_0^T ||\nabla e_t||^2})^{1/2} \le Ch^2,$$

due to Theorem 3.5, and in a similar way we obtain the rest. □

Proof. (**Proof of Theorem 3.3.**) Here, we briefly describe only the main ideas of the proof. First we denote

$$a_{ij}^\sigma(p) := \frac{g(\sigma|p|)}{(\varepsilon+|p|^2)^{\frac{3}{2}}}\frac{\sqrt{\varepsilon+\sigma|p|^2}}{(1-\sigma)\sqrt{\varepsilon}+\sigma}\left(\delta_{ij}(\varepsilon+|p|^2)-p_ip_j\right), \quad p \in \mathbb{R}^2$$

where δ_{ij} denotes Kronecker's symbol. We can write the differential equation of problem (P^σ) in the form

$$u_t - a_{ij}^\sigma(\nabla u)u_{x_ix_j} = 0.$$

First, we linearize (P^σ) expanding a_{ij}^σ around ∇u_0 and after that we change variable $v = u - u_0$ to obtain

$$v_t - a_{ij}^\sigma(\nabla u_0)v_{x_ix_j} - a_{ij,p_k}^\sigma(\nabla u_0)u_{0,x_ix_j}v_{x_k} =$$

$$a_{ij}^\sigma(\nabla u_0)u_{0,x_ix_j} + a_{ij,p_k}^\sigma(\nabla u_0)v_{x_k}v_{x_ix_j} + r_{ij}^\sigma(\nabla u_0,\nabla v)(v+u_0)_{x_ix_j} \equiv F^\sigma(v),$$

where

$$r_{ij}^\sigma(\nabla u_0,\nabla v) = \int_0^1 (1-s)a_{ij,p_k,p_l}^\sigma(\nabla u_0 + s\nabla v)ds\, v_{x_k}v_{x_l}.$$

Setting $a_{ij}^\sigma(x) := a_{ij}^\sigma(\nabla u_0)$ and $b_i^\sigma := -a_{ij,p_k}^\sigma(\nabla u_0)u_{0,x_ix_j}$ we have

$$\begin{aligned} v_t - a_{ij}^\sigma v_{x_ix_j} + b_i^\sigma v_{x_i} &= F^\sigma(v) \text{ in } I\times\Omega, \qquad (L^\sigma)\\ \partial_\nu v &= 0 \text{ on } I\times\partial\Omega,\\ v(0) &= v_0 \text{ in } \Omega. \end{aligned}$$

It is clear that u is a solution of (P^σ) if and only if $v = u - u_0$ solves (L^σ). Now we analyze the following linear problem

$$
\begin{aligned}
v_t - a_{ij} v_{x_i,x_j} + b_i v_{x_i} &= f \text{ in } I \times \Omega,\\
\partial_\nu v &= 0 \text{ on } I \times \partial\Omega,\\
v(0) &= v_0 \text{ in } \Omega.
\end{aligned}
\tag{4.30}
$$

For this problem we use the results of [**12**, Chapter 4]. We obtain that under the assumptions on the data, the linear problem (4.30) has a unique solution $v \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^6(\Omega))$ with $v_t \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^4(\Omega))$, $v_{tt} \in L_\infty(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega))$ and moreover

$$||v||_2^{(6)} \leq c(||v_0||_2^{(5)} + ||f||_2^{(4)})$$

where norms are denoted as in [**12**]: $v \in W_2^{2l,l}(Q_T)$ is a function $v \in L_2(Q_T)$ such that v has generalized derivative of $D_t^r D_x^s$ for all $r, s; 2r + s \leq 2l$ with the norm

$$||v||_2^{(2l)} = \sum_{j=0}^{2l} (\sum_{2r+s=j} ||D_t^r D_x^s v||_{L_2(Q_T)}).$$

Now, similarly as in [**5**], we use the Banach fixed point theorem for existence the solution of (L_σ). We will consider the Banach space $X = C^0(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega))$ with the norm

$$||v||_X^2 = \sup_{[0,T]} ||v(t)||_1^2 + \int_0^T ||v(s)||_2^2 ds.$$

For $0 < T \leq 1,\ M > 0$ we define

$$R_{T,M} := \{v \in X | v(0) = 0, \partial_\nu v(t,.)|_{\partial\Omega} = 0, 0 \leq t \leq T,$$

$$v \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^6(\Omega)), v_t \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^4(\Omega)),$$

$$v_{tt} \in L_\infty(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega)), ||v||_2^{(6)} \leq M^2\}.$$

Let us introduce the map $S : R_{T,M} \to X$ which assigns to a function $u \in R_{T,M}$ the unique solution v of the linear problem

$$
\begin{aligned}
v_t - a_{ij}^\sigma v_{x_i,x_j} + b_i^\sigma v_{x_i} &= F^\sigma(u) \text{ in } I \times \Omega,\\
\partial_\nu v &= 0 \text{ on } I \times \partial\Omega,\\
v(0) &= 0 \text{ in } \Omega.
\end{aligned}
$$

Now the aim is to prove that S has a fixed point, provided T is sufficiently small. This proof is rather technical and long and is practically the same as in [**5**] so we omit it here. First, it was shown, that for arbitrary $u \in R_{T,M}$ its image $S(u)$ is in $R_{T,M}$ too, so $S(R_{T,M}) \subset R_(T, M)$. Then the proof that S is a contraction is presented. This fixed point is a solution of (L^σ) so we have the solution of (P^σ) as well. $\square$

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