Acta Math. Univ. Comenianae
Vol. LXX, 2(2001), pp. 197–205

EXISTENCE THEOREMS FOR A CLASS OF FIRST ORDER IMPULSIVE DIFFERENTIAL INCLUSIONS

M. BENCHOHRA AND S. K. NTOUYAS

Abstract. A fixed point theorem for condensing maps is used to investigate the existence of solutions for a class of first order initial value problems for impulsive differential inclusions.

1. Introduction

The theory of impulsive differential equations appears as a natural description of several real processes subject to certain perturbations whose duration is negligible in comparison with the duration of the process. Differential equations involving impulse effects occurs in many applications: physics, population dynamics, ecology, biological systems, biotechnology, industrial robotic, pharmacokinetics, optimal control, etc. The reader can see for instance the book of Bainov and Simeonov [**2**], Lakshmikantham, Bainov and Simeonov [**14**], Samoilenko and Perestyuk [**19**], the thesis of Pierson Gorez [**18**] and the papers of Frigon and O'Regan [**9**], Liz and Nieto [**16**], Vatsala and Sun [**22**] and Yujun and Erxin [**23**]. However very few results are available for impulsive differential inclusions or related topics (see for example the paper of Benchohra and Boucherif [**3**], [**4**], Erbe and Krawcewicz [**7**], Frigon and O'Regan [**10**], Silva and R. B. Vinter [**20**] and Stewart [**21**]).

The fundamental tools used in the existence proofs of all above mentioned works are essentially fixed point arguments, Nonlinear alternative of Leray-Schauder type, Degree theory, Topological transversality theorem or the monotone iterative technique combined with upper and lower solutions.

In this paper, we shall be concerned with the existence of solutions of the first order initial value problem for the impulsive differential inclusion:

$$y' \in F(t,y), \quad t \in J, \ \ t \neq t_k, \ k = 1, \dots, m, \tag{1.1}$$

$$y(t_k^+) = I_k(y(t_k^-)), \ k = 1, \dots, m, \tag{1.2}$$

$$y(0) = y_0, \tag{1.3}$$

where $F\colon J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a compact convex valued multivalued map defined from a single-valued function, $J = [0,T]$ $(0 < T < \infty)$, $y_0 \in \mathbb{R}$, $0 = t_0 < t_1 < \cdots <$

Received June 19, 2000.
2000 *Mathematics Subject Classification.* Primary 34A37, 34A60.

$t_m < t_{m+1} = T$; and $I_k \in C(\mathbb{R}, \mathbb{R})$ $(k = 1, 2, \dots, m)$. $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively.

The multivalued map considered in this paper has been used by Chang [**5**], Erbe and Krawcewicz [**8**], Frigon [**11**] and Klein-Thompson [**13**] for the study of differential inclusions of second order.

In this paper we shall extend the above results to the impulsive case. We shall give two existence results to (1.1)-(1.3). In our results we do not assume any type of monotonicity condition on I_k, $k = 1, \dots, m$, which is usually the situation in the literature.

We use a fixed point approach to establish our existence results. In particular we use a fixed point theorem for condensing maps as used by Martelli ([**17**]).

2. Preliminaries

In this section, we introduce notations, definitions, and results which are used throughout the paper.

$AC(J, \mathbb{R})$ is the space of all absolutely continuous functions $y\colon J \longrightarrow \mathbb{R}$.

Condition

$$y \le z \quad \text{if and only if} \quad y(t) \le z(t) \quad \text{for all} \quad t \in J$$

defines a partial ordering in $AC(J, \mathbb{R})$. If $\alpha, \beta \in AC(J, \mathbb{R})$ and $\alpha \le \beta$, we denote

$$[\alpha, \beta] = \{y \in AC(J, \mathbb{R}) : \alpha \le y \le \beta\}.$$

$C(J, \mathbb{R})$ is the Banach space of continuous functions $y\colon J \longrightarrow \mathbb{R}$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : t \in J\} \quad \text{for all} \quad y \in C(J, \mathbb{R}).$$

$L^2(J, \mathbb{R})$ denotes the Banach space of Lebesgue measurable functions $y\colon J \longrightarrow \mathbb{R}$ for which $\int_0^T |y(t)|^2\, dt < +\infty$, with the norm

$$\|y\|_{L^2} = \Big(\int_0^T |y(t)|^2\, dt\Big)^{1/2} \quad \text{for all} \quad y \in L^2(J, \mathbb{R}).$$

Finally $H^1(J, \mathbb{R})$ denotes the Banach space of functions $y\colon J \longrightarrow \mathbb{R}$ which are absolutely continuous and whose derivative y' (which exists almost everywhere) is an element of $L^2(J, \mathbb{R})$ with the norm

$$\|y\|_{H^1} = \|y\|_{L^2} + \|y'\|_{L^2} \quad \text{for all } y \in H^1(J, \mathbb{R}).$$

In order to define the solution to (1.1)-(1.3) we shall consider the following spaces.

$$\begin{aligned}\Omega = &\{y : [0, T] \longrightarrow \mathbb{R} : y \text{ is continuous for } \ t \ne t_k,\ y(t_k^+) \text{ and}\\ &y(t_k^-) \text{ exist and } y(t_k) = y(t_k^-),\ k = 1, \dots, m\ \}.\end{aligned}$$

Evidently, Ω is a Banach space with the norm

$$\|y\|_\Omega = \sup_{t \in J} |y(t)|.$$

$\Omega^1 := \Omega \cap \cup_{k=0}^m H^1(t_k, t_{k+1})$. For $y \in \Omega^1$ we let $\|y\|_{\Omega^1} = \|y\|_{H^1}$. Hence Ω^1 is a Banach space.

Definition 2.1. By a solution to (1.1)-(1.3), we mean a function $y \in \Omega_0^1 := \{y \in \Omega^1 : y(0) = y_0\}$ that satisfies the differential inclusion

$$y'(t) \in F(t, y(t)) \text{ almost everywhere on } J \backslash \{t_k\},\ k = 1, \dots, m,$$

and for each $k = 1, \dots, m$ the function y satisfies the equations $y(t_k^+) = I_k(y(t_k^-))$.

Let $(X, \|\cdot\|)$ be a normed space. A multivalued map $G\colon X \longrightarrow 2^X$ has convex (closed) values if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B)$ is bounded in X for any bounded subset B of X (i.e. $\sup\limits_{x\in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completly continuous if $G(B) = \cup_{x \in B} G(x)$ is relatively compact for every bounded subset $B \subseteq X$. G has a fixed point if there is $x \in X$ such that $x \in G(x)$.

In the following $CC(X)$ denotes the set of all nonempty compact convex subsets of X.

An upper semi-continuous map $G : X \longrightarrow 2^X$ is said to be condensing [**17**] if for any bounded subset $N \subseteq X$, we have $\alpha(G(N)) < \alpha(N)$, with $\alpha(N) \neq 0$, where α denotes the Kuratowski measure of noncompacteness (see [**1**], [**17**]).

We remark that a compact map is the simplest example of a condensing map. For more details on multivalued functions see the books of Deimling [**6**] and Hu and Papagerogiou [**12**].

Definition 2.2. A function $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be Carathéodory if

(i) $t \longmapsto f(t, y)$ is measurable for each $y \in \mathbb{R}$;
(ii) $y \longmapsto f(t, y)$ is continuous for almost all $t \in J$.

Definition 2.3. A function $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ is said to be of type $\mathcal{M}$ if for each measurable function $y\colon J \longrightarrow \mathbb{R}$, the function $t \longmapsto f(t, y(t))$ is measurable.

Notice that a Carathéodory map is of type $\mathcal{M}$.

Let $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Define

$$\underline{f}(t, y) = \lim_{u \to y} \inf f(t, u) \quad \text{and} \quad \overline{f}(t, y) = \lim_{u \to y} \sup f(t, u).$$

Notice that for all $t \in J$, $\underline{f}$ is lower semi-continuous (l.s.c.) i.e. (for all $t \in J$, $\{y \in \mathbb{R}\colon \underline{f}(t, y) > \alpha\}$ is open for each $\alpha \in \mathbb{R}$) and $\overline{f}$ is upper semi-continuous (u.s.c.) i.e. (for all $t \in J$, $\{y \in \mathbb{R} : \overline{f}(t, y) < \alpha\}$ is open for each $\alpha \in \mathbb{R}$).

Let $f\colon J \times \mathbb{R} \longrightarrow \mathbb{R}$. We define the multivalued map $F : J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ by

$$F(t, y) = [\underline{f}(t, y), \overline{f}(t, y)].$$

We say that F is of type $\mathcal{M}$ if $\underline{f}$ and $\overline{f}$ are of type $\mathcal{M}$.

The following result is crucial in the proof of our main results:

Theorem 2.4 (**[17]**). *Let $G\colon X \longrightarrow CC(X)$ be an u.s.c. and condensing map. If the set*

$$M := \{v \in X : \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

We need also the following result

Theorem 2.5 (**[11]** Prop. (VI. 1), p. 40). *Assume that F is of type $\mathcal{M}$ and for each $k \geq 0$, there exists $\phi_k \in L^2(J, \mathbb{R})$ such that*

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq \phi_k(t) \quad \text{for} \quad |y| \leq k.$$

Then the operator $\mathcal{F}\colon C(J,\mathbb{R}) \longrightarrow 2^{L^2(J,\mathbb{R})}$ defined by

$$\mathcal{F}y := \{h : J \longrightarrow \mathbb{R} \text{ measurable: } h(t) \in F(t, y(t)) \text{ a.e. } t \in J\}$$

is well defined, u.s.c., bounded on bounded sets in $C(J,\mathbb{R})$ and has convex values.

3. Main Result

We are now in a position to state and prove our first existence result for the impulsive IVP (1.1)-(1.3).

Theorem 3.1. *Let $t_0 = 0$, $t_{m+1} = T$, and assume that $F\colon J \times \mathbb{R} \longrightarrow CC(\mathbb{R})$ is of type $\mathcal{M}$. Suppose that the following hypotheses hold:*

(H1) *there exist $\{r_i\}_{i=0}^m$ and $\{s_i\}_{i=0}^m$ with $s_0 \leq y_0 \leq r_0$ and*

$$s_{i+1} \leq \min_{[s_i,r_i]} I_{i+1}(y) \leq \max_{[s_i,r_i]} I_{i+1}(y) \leq r_{i+1};$$

(H2)

$$\overline{f}(t, r_i) \leq 0, \ \underline{f}(t, s_i) \geq 0 \quad \text{for} \quad t \in [t_i, t_{i+1}], \ i = 1, \ldots, m.$$

(H3) *there exists $\psi\colon [0,\infty) \to (0,\infty)$ continuous such that $\psi \in L^2_{loc}([0,\infty))$ and*

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \leq \psi(|y|) \quad \text{for all} \quad t \in J.$$

Then the impulsive initial value problem (1.1)-(1.3) has at least one solution.

Proof. This proof will be given in several steps.

Step 1: We restrict our attention to the problem on $[0, t_1]$, that is the initial value problem

$$y'(t) \in F(t, y(t)), \ t \in (0, t_1), \tag{3.1}$$

$$y(0) = y_0. \tag{3.2}$$

Define the modified function $f_1\colon [0,t_1] \times \mathbb{R} \longrightarrow \mathbb{R}$ relative to r_0 and s_0 by:

$$f_1(t,y) = \begin{cases} f(t, r_0), & \text{if } y > r_0; \\ f(t,y), & \text{if } s_0 \leq y \leq r_0; \\ f(t, s_0), & \text{if } y < s_0 \end{cases}$$

and the correponding multivalued map

$$F_1(t,y) = \begin{cases} [\underline{f}(t,r_0), \overline{f}(t,r_0)], & \text{if } y > r_0; \\ [\underline{f}(t,y), \overline{f}(t,y)], & \text{if } s_0 \le y \le r_0; \\ [\underline{f}(t,s_0), \overline{f}(t,s_0)], & \text{if } y < s_0 \end{cases}$$

Consider the modified problem:

$$y' \in F_1(t,y), \quad t \in [0,t_1), \tag{3.3}$$

$$y(0) = y_0. \tag{3.4}$$

We transform the problem into a fixed point problem. For this, consider the operators $L\colon H^1([0,t_1],\mathbb{R}) \longrightarrow L^2([0,t_1],\mathbb{R})$ defined by $L(y) = y'$, $j\colon H^1([0,t_1],\mathbb{R}) \longrightarrow C([0,t_1],\mathbb{R})$, the completely continuous imbedding, and

$$\mathcal{F}\colon C([0,t_1],\mathbb{R}) \longrightarrow 2^{L^2([0,t_1],\mathbb{R})}$$

defined by:

$$\mathcal{F}(y) = \Big\{ v : [0,t_1] \longrightarrow \mathbb{R} \text{ measurable} : \ v(t) \in F_1(t,y(t)) \text{ for a.e. } t \in [0,t_1] \Big\}.$$

Clearly, L is linear, continuous and invertible. It follows from the open map theorem that L^{-1} is a linear bounded operator. $\mathcal{F}$ is by Theorem 2.5 well defined, bounded on bounded subsets of $C([0,t_1],\mathbb{R})$, u.s.c. and has convex values. Thus, the problem (3.3)-(3.4) is equivalent to $y \in L^{-1}\mathcal{F}j(y) := G_1(y)$. Consequently, G_1 is compact, u.s.c., and has convex closed values. Therefore, G_1 is a condensing map.

Now, we show that the set

$$M_1 := \{y \in C([0,t_1],\mathbb{R}) : \lambda y \in G_1(y) \ \text{ for some } \ \lambda > 1\}$$

is bounded.

Let $\lambda y \in G_1(y)$ for some $\lambda > 1$. Then $y \in \lambda^{-1}G_1(y)$, where

$$G_1(y) := \Big\{ g \in C([0,t_1],\mathbb{R}) : g(t) = y_0 + \int_0^t h(s)\,ds : h \in \mathcal{F}(y) \Big\}.$$

Let $y \in \lambda^{-1}G_1(y)$, then there exists $h \in \mathcal{F}(y)$ such that for each $t \in J$

$$y(t) = \lambda^{-1}y_0 + \lambda^{-1}\int_0^t h(s)\,ds.$$

Thus

$$|y(t)| \le |y_0| + \|h\|_{L^2} \quad \text{for each} \quad t \in [0,t_1].$$

Now, since $h(t) \in F_1(t,y(t))$, it follows from the definition of $F_1(t,y)$ and assumption (H3) that there exists a positive constant h_0 such that $\|h\|_{L^2} \le h_0$. In fact

$$h_0 = \max\left\{ |r_0|, |s_0|, \sup_{s_0 \le y \le r_0} |\psi(y)| \right\}.$$

We then have

$$\|y\|_\infty \le |y_0| + h_0 < +\infty.$$

Hence, Theorem 2.4 applies and so G_1 has at least one fixed point which is a solution on $[0, t_1]$ to problem (3.3)-(3.4).

We shall show that the solution y of (3.1)-(3.2) satisfies

$$s_0 \le y(t) \le r_0 \text{ for all } t \in [0, t_1].$$

Let y be a solution to (3.3)-(3.4). We prove that

$$s_0 \le y(t) \text{ for all } t \in [0, t_1].$$

Suppose not. Then there exist $\sigma_1, \sigma_2 \in [0, t_1]$, $\sigma_1 < \sigma_2$ such that $y(\sigma_1) = s_0$ and

$$s_0 > y(t) \text{ for all } t \in (\sigma_1, \sigma_2).$$

This implies that

$$f_1(t, y(t)) = f(t, s_0) \text{ for all } t \in (\sigma_1, \sigma_2),$$

and

$$y'(t) \in [\underline{f}(t, s_0), \overline{f}(t, s_0)],$$

then,

$$y'(t) \ge \underline{f}(t, s_0) \text{ for all } t \in (\sigma_1, \sigma_2).$$

This implies that

$$y(t) \ge y(t_1) + \int_{t_1}^{t} \underline{f}(s, s_0) ds \text{ for all } t \in (\sigma_1, \sigma_2).$$

Since $\underline{f}(t, s_0) \ge 0$ for $t \in [0, t_1]$ we get

$$0 > y(t) - y(\sigma_1) \ge \int_{\sigma_1}^{t} \underline{f}(s, s_0) ds \ge 0 \text{ for all } t \in (\sigma_1, \sigma_2)$$

which is a contradiction. Thus $s_0 \le y(t)$ for $t \in [0, t_1]$.

Similarly, we can show that $y(t) \le r_0$ for $t \in [0, t_1]$. This shows that the problem (3.3)-(3.4) has a solution y on the interval $[0, t_1]$, which we denote by y_1. Then y_1 is a solution of (3.1)-(3.2).

Step 2: Consider now the problem:

$$y' \in F_2(t, y), \quad t \in (t_1, t_2), \tag{3.5}$$

$$y(t_1^+) = I_1(y_1(t_1^-)), \tag{3.6}$$

where

$$F_2(t, y) = \begin{cases} [\underline{f}(t, r_1), \overline{f}(t, r_1)], & \text{if } y > r_1; \\ [\underline{f}(t, y), \overline{f}(t, y)], & \text{if } s_1 \le y \le r_1; \\ [\underline{f}(t, s_1), \overline{f}(t, s_1)], & \text{if } y < s_1. \end{cases}$$

Analogously, we can show that set

$$M_2 := \{y \in C([t_1, t_2], \mathbb{R}) : \lambda y \in G_2(y) \text{ for some } \lambda > 1\}$$

is bounded. Here the operator G_2 is defined by $G_2 := L^{-1}\mathcal{F}j$ where

$$L^{-1}\colon L^2([t_1,t_2],\mathbb{R}) \longrightarrow H^1([t_1,t_2],\mathbb{R}),$$
$$j\colon H^1([t_1,t_2],\mathbb{R}) \longrightarrow C([t_1,t_2],\mathbb{R})$$

the completely continuous imbedding, and $\mathcal{F}\colon C([t_1,t_2],\mathbb{R}) \longrightarrow 2^{L^2([t_1,t_2],\mathbb{R})}$ defined by:

$$\mathcal{F}(y) = \Big\{v : [t_1,t_2] \longrightarrow \mathbb{R} \text{ measurable : } v(t) \in F_2(t,y(t)) \text{ for a.e. } t \in [t_1,t_2]\Big\}.$$

We again apply the theorem of Martelli to show that G_2 has a fixed point, which we denote by y_2, and so is a solution of problem (3.5)-(3.6) on the interval $(t_1,t_2]$.

We now show that

$$s_1 \leq y_2(t) \leq r_1 \quad \text{for all} \quad t \in [t_1,t_2].$$

Since $y_1(t_1^-) \in [s_0,r_0]$ then (H1) implies that

$$s_1 \leq I_1(y(t_1^-)) \leq r_1, \quad \text{i.e.} \quad s_1 \leq y(t_1^+) \leq r_1.$$

Since $\overline{f}(t,r_1) \leq 0$ and $\underline{f}(t,s_1) \geq 0$ we can show that

$$s_1 \leq y_2(t) \leq r_1 \quad \text{for} \quad t \in [t_1,t_2],$$

and hence y_2 is a solution to

$$y' \in F(t,y), \quad t \in (t_1,t_2), y(t_1^+) = I_1(y_1(t_1^-)).$$

Step 3: We continue this process and we construct solutions y_k on $[t_{k-1},t_k]$, with $k = 3,\dots,m+1$ to

$$y' \in F(t,y), \quad t \in (t_{k-1},t_k), \tag{3.7}$$

$$y(t_{k-1}^+) = I_{k-1}(y_{k-1}(t_{k-1}^-)), \tag{3.8}$$

with $s_{k-1} \leq y_k(t) \leq r_{k-1}$ for $t \in [t_{k-1},t_k]$. Then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0,t_1)]; \\ y_2(t), & \text{if } t \in (t_1,t_2]; \\ \vdots & \\ y_{m+1}(t), & \text{if } t \in (t_m,T] \end{cases}$$

is a solution to (1.1)-(1.3). □

Using the same reasoning as that used in the proof of Theorem 3.1 we can obtain the following result.

Theorem 3.2. *Let $t_0 = 0, t_{m+1} = T$, and suppose that $F\colon J \times \mathbb{R} \longrightarrow CC(\mathbb{R})$ is of type $\mathcal{M}$. Suppose the following hypotheses hold.*

(H4) *There are functions* $\{r_i\}_{i=0}^m$ *and* $\{s_i\}_{i=0}^m$ *with* $r_i,\ s_i \in C([t_i, t_{i+1}])$ *and* $s_i(t) \le r_i(t)$ *for* $t \in [t_i, t_{i+1}]$, $\quad i = 0, \dots, m$. *Also,* $s_0 \le y_0 \le r_0$ *and*

$$\begin{aligned} s_{i+1}(t_{i+1}^+) &\le \min_{[s_i(t_{i+1}^-), r_i(t_{i+1}^-)]} I_{i+1}(y) \\ &\le \max_{[s_i(t_{i+1}^-), r_i(t_{i+1}^-)]} I_{i+1}(y) \\ &\le r_{i+1}(t_{i+1}^+), \quad i = 0, \dots, m-1; \end{aligned}$$

(H5)

$$\begin{aligned} &\int_{z_i}^{w_i} \underline{f}(t, s_i(t))\,dt \ge s_i(w_i) - s_i(z_i), \\ &\int_{z_i}^{w_i} \overline{f}(t, r_i(t))\,dt \le r_i(w_i) - r_i(z_i), \quad i = 0, \dots, m \end{aligned}$$

with

$$z_i < w_i \text{ and } z_i,\ w_i \in [t_i, t_{i+1}].$$

Then the impulsive initial value problem (1.1)-(1.3) *has at least one solution.*

References

1. Banas J. and Goebel K., *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, New York, 1980.
2. Bainov D.D , and Simeonov P.S., *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.
3. Benchohra M. and Boucherif A., *On first order initial value problems for impulsive differential inclusions in Banach Spaces*, Dynam. Systems Appl. **8**(1) (1999), 119–126.
4. Benchohra M. and Boucherif A., *Initial value problems for impulsive differential inclusions of first order*, Differ. Equ. Dyn. Syst. **8** (2000), 51–66.
5. Chang K. C., *The obstacle problem and partial differential equations with discontinuous nonlinearities*, Comm. Pure Appl. Math. **33** (1980), 117–146.
6. Deimling K., *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.
7. Erbe L. and Krawcewicz W., *Existence of solutions to boundary value problems for impulsive second order differential inclusions*, Rockey Mountain J. Math. **22** (1992), 519–539.
8. Erbe L. and Krawcewicz W., *Nonlinear boundary value problems for differential inclusions* $y'' \in F(t, y, y')$, Ann. Pol. Math. **LIV**(3) (1991), 195–226.
9. Frigon M. and O'Regan D., *Existence results for first order impulsive differential equations*, J. Math. Anal. Appl. **193**, (1995), 96–113.
10. Frigon M. and O'Regan D., *Boundary value problems for second order impulsive differential equations using set-valued maps*, Appl. Anal. **58** (1995), 325–333.
11. Frigon M., *Application de la théorie de la transversalité topologique à des problèmes non linéaires pour des équations différentielles ordinaires*, Diss. Math. **296** (1990), 1–79.
12. Hu Sh. and Papageorgiou N., *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, Boston, London, 1997.
13. Klein E. and Thompson A., *Theory of Correspondences*, Wiley, New York, 1984.
14. Lakshmikantham V., Bainov D. D. and Simeonov P. S., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.
15. Liu X., *Nonlinear boundary value problems for first order impulsive differential equations*, Appl. Anal. **36** (1990), 119–130.
16. Liz E. and Nieto J. J., *Positive solutions of linear impulsive differential equations*, Commun. Appl. Anal. **2**(4) (1998), 565–571.

17. Martelli M., *A Rothe's type theorem for non compact acyclic-valued maps*, Boll. Un. Mat. Ital. **11** (1975), 70–76.
18. Pierson-Gorez C., *Problèmes aux Limites Pour des Equations Différentielles avec Impulsions*, Ph.D. Thesis, Univ. Louvain-la-Neuve, 1993 (in French).
19. Samoilenko A. M. and Perestyuk N. A., *Impulsive Differential Equations*, World Scientific, Singapore, 1995.
20. Silva G. N. and Vinter R. B., *Measure driven differential inclusions*, J. Math. Anal. Appl. **202** (1996), 727–746.
21. Stewart D. E., *Existence of solutions to rigid body dynamics and the Painleve paradoxes*, C.R. Acad. Sci. Paris Ser. I Math. **325** (1997), 689–693.
22. Vatsala A. S., and Sun Y., *Periodic boundary value problems of impulsive differential equations*, Appl. Anal. **44** (1992), 145–158.
23. Yujun D. and Erxin Z., *An application of coincidence degree continuation theorem in existence of solutions of impulsive differential equations*, J. Math. Anal. Appl. **197**, (1996), 875–889.

M. Benchohra, Département de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie, *e-mail*: benchohra@yahoo.com

S. K. Ntouyas, Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece, *e-mail*: sntouyas@cc.uoi.gr