Acta Math. Univ. Comenianae
Vol. LXX, 1(2001), pp. 153–165
Proceedings of Algoritmy 2000

ON PRICING AMERICAN AND ASIAN OPTIONS WITH PDE METHODS

G. H. MEYER

Abstract. The influence of the analytical properties of the Black-Scholes PDE formulation for American and Asian options on the quality of the numerical solution is discussed. It appears that numerical methods for PDEs are quite robust even when the mathematical formulation is not well posed.

1. Introduction

It is now well known that time continuous models for pricing financial options with early exercise lead to optimal stopping problems which are equivalent to parabolic free boundary problems. For example, the standard formulation for an American put (the right to sell an asset between now and T for a specified price K) leads to the Black Scholes equation for the value $P(S,t)$ of the option

$$\frac{1}{2}\sigma^2 S^2 P_{SS} + rSP_S - rP + P_t = 0 \tag{1}$$

subject to the boundary data

$$\begin{aligned}\lim_{S\to\infty} P(S,t) &= 0\\ P(S(t),t) &= K - S(t)\\ P_S(S(t),t) &= -1\end{aligned}$$

Here S is the value of the underlying asset and t is time. The coefficients in (1) are assumed known from market data. Not known a priori is the early exercise boundary $S(t)$ which needs to be found together with the option value $P(S,t)$. However, at maturity T the value of the option and the exercise boundary are known with certainty

$$\begin{aligned}P(S,T) &= \max\{(K-S),0\},\\ S(T) &= K.\end{aligned}$$

This is a final time condition for the backward diffusion equation (1) and it is readily seen that this free boundary problem is a standard parabolic obstacle problem and thus amenable to analysis in this framework [**7**]. There is, however,

Received January 10, 2001; revised June 21, 2001.
2000 *Mathematics Subject Classification.* Primary 65M06, 91B24.

no known analytical solution of this nonlinear problem so one has to resort to numerical techniques to price options.

The numerical solution of one-dimensional parabolic free boundary problems was a topic of considerable interest twenty and more years ago when the one-dimensional phase change (Stefan) problem was still considered a "difficult" problem. Many of the algorithms developed then for Stefan like problems and variational inequalities are, in principle, immediately applicable to the Black Scholes equation of American options. However, in view of the demands of this application for speed of computation and accuracy not all feasible methods are practical and research into efficient numerical and approximate methods continues unabated (see, e.g. [**5**]). Our comments below are meant to identify some points which indicate why the numerical solution of the Black Scholes equation remains a challenging problem.

2. The American Put

One of the numerical problems hidden in the above Black Scholes formulation is the infinite speed of propagation of the free boundary at maturity T. In fact, it has been shown [**2**] that the analytic solution $s(t)$ satisfies

$$K - s(t) = O\left(\sqrt{(T-t)|\ln(T-t)|}\right) \tag{2}$$

as $t \to T$. However, it appears difficult to recover this speed of propagation with standard numerical methods. For example, if the time continuous problem is approximated at $t_{N-1} = T - \Delta t$ with an implicit Euler method then the free boundary problem

$$\frac{1}{2}\sigma^2 x^2 u'' + rxu' - (r + 1/\Delta t)u = -\max\{1-x, 0\}/\Delta t \tag{3}$$

$$\lim_{x\to\infty} u(x) = 0$$

$$u(s_{N-1}) = 1 - s_{N-1}$$

$$u'(s_{N-1}) = -1 \tag{4}$$

has to be solved where, for convenience, we have set $x = S/K$, $u = P/K$. Because the source term in (3) is piecewise linear we can solve this problem analytically. The solution $u_{N-1}(x)$ has the representation

$$u_{N-1}(x) = \begin{cases} c_1 x^{\gamma_1} + c_2 x^{\gamma_2} + 1/(1+r\Delta t) - x, & s_{N-1} < x < 1 \\ d_1 x^{\gamma_1}, & x > 1 \end{cases}$$

where γ_1 and γ_2 are the negative and positive roots of

$$\frac{1}{2}\sigma^2 \gamma(\gamma - 1) + r\gamma - (r + 1/\Delta t) = 0 \tag{5}$$

and where the coefficients c_1, c_2 and d_1 and the free boundary s_{N-1} are determined such that u and u', and hence u'', are continuous at $x = 1$ and such that $u(s_{N-1}) =$

$1-s_{N-1}$ and $u'(s_{N-1}) = -1$. The early exercise boundary s_{N-1} can be computed explicitly as

$$s_{N-1} = \left(\frac{-\gamma_1 r\Delta t}{1 + r\Delta t - \gamma_1 r\Delta t}\right)^{1/\gamma_2} \tag{6}$$

From (5) follows that $\gamma_{1,2} = \mp\sqrt{2}/(\sigma\sqrt{\Delta t}) + B(\Delta t)$ where $B(\Delta t) = O(1)$ as $\Delta t \to 0$ which implies that the free boundary of the time discrete problem satisfies

$$1 - s_{N-1} = \frac{\sigma\sqrt{\Delta t}}{\sqrt{2}} \left|\ln\sqrt{\Delta t}\right| + C(\Delta t)\sqrt{\Delta t}$$

where $C(\Delta t) = O(1)$ as $\Delta t \to 0$. Since $\lim_{t\to 0}\sqrt{t}\ln\sqrt{t} = o(t^{.5-\epsilon})$ and $\lim_{t\to 0}\sqrt{t|\ln t|} = o(t^{.5-\epsilon})$ for $\epsilon > 0$ it follows that for arbitrary $\epsilon > 0$

$$|s_{N-1} - s(T - \Delta t)| = o(\Delta t^{.5-\epsilon})$$

On the other hand,

$$\frac{1 - s(T - \Delta t)}{1 - s_{N-1}} = \frac{O\left(\sqrt{|\ln\Delta t|}\right)}{\frac{\sigma}{\sqrt{2}}\left|\ln\sqrt{\Delta t}\right|} = o(1)$$

as $\Delta t \to 0$. Hence the time discrete approximation yields a reasonable approximation to the location, but not the speed, of the free boundary near expiry. We remark that any numerical solution of the problem at t_{N-1} would also introduce a spatial discretization error, but if the method of [**11**] is used then a comparison of the numerical and the above analytic solution (5) show that both free boundary locations are virtually identical at the first time step.

Since the initial condition for (1) has a discontinuous u_x at $x = 1$ it is not clear how one can devise a purely numerical method which tracks the free boundary at the first time step with sufficient accuracy to give the correct speed. Higher order difference schemes do not apply because of lack of smoothness while an extrapolation of results for different Δt is doubtful because of the strong nonlinearity of the problem near $t = T$. It is probable that the correct asymptotic results as $t \to T$ will have to come from an analytic approximation.

Additional numerical complications arise throughout the life of the option because the early exercise boundary does not necessarily depend continuously on the data of the problem. To see this problem in context we recall that front tracking methods for the classical one phase Stefan problem are stable in the sense that if the free boundary has moved off its correct position because the temperature was found incorrectly then the fluxes and hence the boundary velocity are slowed or increased to move it back toward the right position. In contrast the free boundary $s(t)$ of the option problem is defined implicitly and encoded only in the solution $u(x, t)$. As a consequence $s(t)$ may change discontinuously with arbitrarily small changes of $u(x, t)$. For example, suppose the asset pays a one-time dividend D at time t_n, then the Black Scholes equation must be solved over $[0, t_n]$ subject to the initial condition

$$u(x,t_n) = u\left(x - \frac{D}{K}, t_n+\right)$$

where $u(x,t_n+)$ is the solution found over $[t_n,T]$ (see, e.g. [**9**]). For any $D > 0$, no matter how small, the initial value $u(x,t_n)$ lifts off the obstacle and the free boundary jumps from $s(t_n+) \gg 0$ to (essentially) $s(t_n) = 0$ and then fails to exist over a time interval $[t_*, t_n]$ given by

$$t_n - t_* = 1/r \ln(1 + D/K)$$

before it (spontaneously?) reappears. Hence the free boundary moves discontinuously with respect to perturbations of the data at the preceding time level. (For a numerical simulation of American options with discrete dividends and the behavior of the early exercise boundary see [**12**]).

An additional demand is placed on numerical methods for the Black Scholes equation by the requirement that numerical values for u_x and u_{xx} are expected for hedging purposes. Moreover, a recent asymptotic stochastic volatility model for European and American options [**6**] requires the solution of an additional one-dimensional Black Scholes equation whose source term depends on u_{xxx} which again increases the demand on the order of approximation and accuracy of the numerical Black Scholes solver.

Nonetheless, in spite of these complications the numerical solutions for a given problem appear to be quite stable with respect to refinements of the space and time steps so that in practice any American option problem described by a one-dimensional diffusion equation is relatively straightforward to solve numerically. To illustrate the robustness of a time implicit tracking of the early exercise boundary near expiry we show in Table 1 the computed scaled free boundary and option value at time $t = T - .001$ found for three different time steps when u_t is approximated by

$$u_t(x, T - \Delta t) = \frac{u(x,T) - u(x, T - \Delta t)}{\Delta t}$$

for the first time step (yielding the free boundary (6) and by the second order formula

$$u_t(x, n\Delta t)$$
$$= -\left[\frac{3}{2}\frac{u(x, T - n\Delta t) - u(x,(n+1)\Delta t)}{\Delta t} - \frac{1}{2}\frac{u(x,(n+1)\Delta t) - u(x,(n+2)\Delta t)}{\Delta t}\right]$$

for all subsequent time steps.

The results were obtained with the front tracking algorithm of [**11**]. They are highly resolved with respect to x so that Table 1 may be assumed to show the influence of Δt only.

To show the reliability of computations for a discontinuous early exercise boundary we show in Fig. 1 the computed free boundary for an American put on an option where the underlying pays dividends of known size at two dates t_a and t_b. Fig. 2 contains the normalized price and its first three derivatives at $t = 0$. The curves for u_0, u_0' and u_0'' are the linear interpolants between computed values on

t	$s(T-.001)$	$u(1,T-.001)$
.001/10	.98491	.0024846
.001/100	.98486	.0024882
.001/200	.98482	.0024885

$r = .1$, continuous dividend rate $\rho = .02$, $\sigma = .2$

Table 1. Stability of the numerical solution near expiry for an American put with respect to Δt.

a fixed mesh found with the algorithm of [**12**]. The curve for u_0''' is obtained from a central finite difference approximation applied to u_0''.

We note that the quality of the numerical solution can be monitored a posteriori at the free boundary. It is reasonable to infer from the regularity results of [**7**] that for $t \leq t_f < T$ all terms in the Black-Scholes equation have bounded limits as $x \searrow s(t)$. The boundary conditions

$$u(s(t),t) = 1 - s(t)$$

and

$$u_x(s(t),t) = -1$$

imply that $u_t(s(t),t) = 0$ and $u_{xx}(s(t),t)s'(t) + u_{xt}(s(t),t) = 0$. It follows from the Black Scholes equation and its derivative that, at least formally,

$$u_{xx}(s(t),t) = \frac{2r}{\sigma^2 s^2(t)} \tag{7}$$

and

$$u_{xxx}(s(t),t) = \frac{4r(s'(t) - (r+\sigma^2)s)}{(\sigma s(t))^4}\,. \tag{8}$$

For the calculation of Fig. 2 the early exercise boundary and its speed, computed at $t = 0$ from

$$s'(0) = -\left[\frac{3}{2\Delta t}\,(s_0 - s_1) - \frac{1}{2\Delta t}\,(s_1 - s_2)\right],$$

are found to be

$$s(0) = .648113$$
$$s'(0) = 1.050$$

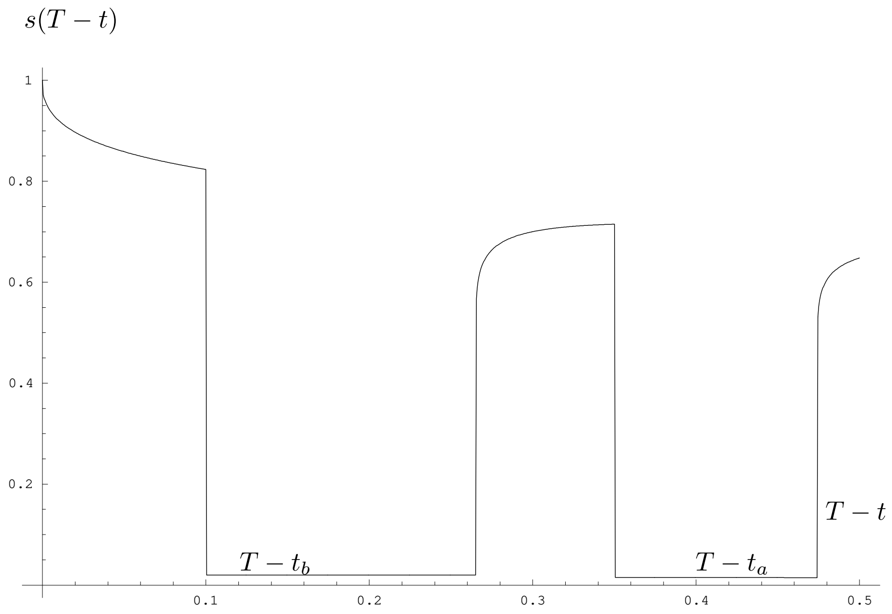

Figure 1. Early exercise boundary for an American put on an asset paying a (scaled) dividend of $D = .015$ at $t_a = .15$ and $D = .02$ at $t_b = .4$.
$r = .12$, $\sigma = .4$, $T = .5$, $\Delta t = .1/200$, $\Delta x = .001$.

The formulas (7) and (8) predict

$$u_{xx}(s(0), 0) = 3.570$$

and

$$u_{xxx}(s(0), 0) = -130.9.$$

The values found numerically for the above parameters are

$$u_0''(s_0) = 3.43$$

and

$$u_0'''(x_*) = -129.5$$

where $x_* = .64875$ is the second regular mesh point beyond s_0 whence u''' is available from a central difference. We need to point out, however, that the results for u'' and u''' for a smaller Δt begin to degrade. We suspect that the discontinuity of u_0'' at s_0 is the cause. u, u', and s do not change noticeably.

In the phase change setting once the one-dimensional problem was understood and could be solved reliably attention turned to multi-dimensional problems. The same development is taking place in quantitative and numerical finance.

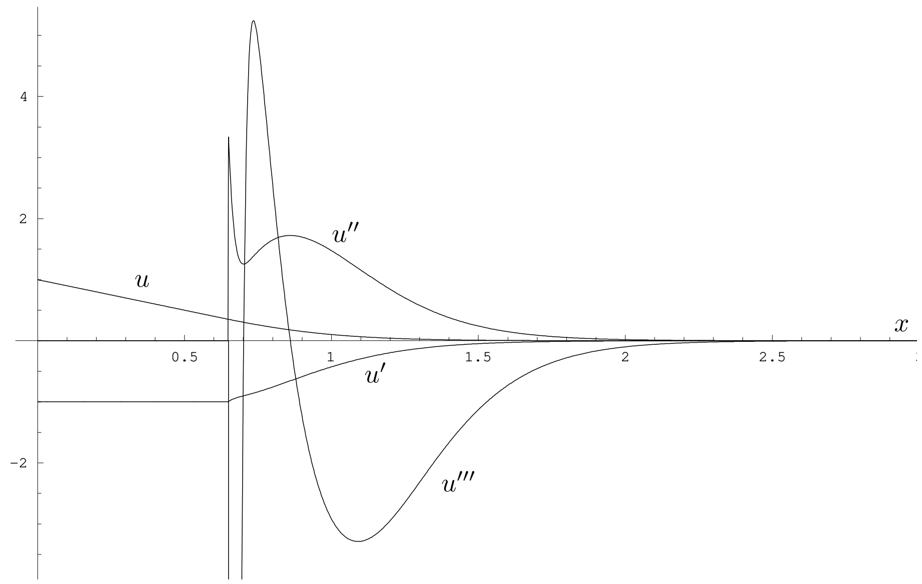

Figure 2. Option and Greeks at $t = 0$ for the put of Fig. 1.

3. Asian Options

One broad area which can be modeled with a higher dimensional diffusion equation is the pricing of certain path dependent options depending on average values of the underlying asset, the so called Asian options. The analog of the Black Scholes equation for this case is a degenerate two dimensional diffusion equation whose structure imposes special conditions on the admissible boundary data in order to have a well posed problem. This aspect appears to have been somewhat ignored in the financial literature. In fact, we found only one source [**3**] addressing the issue of degeneracy but the problems considered there are set on infinite domains while numerical work in general requires finite domains. Yet conditions for the existence, uniqueness and continuous dependence of solutions for degenerate parabolic and elliptic equations can be found in the mathematical literature. We shall examine here what they imply for Asian options.

We shall focus on the fixed strike Asian call with payoff at maturity T. A Black-Scholes type model was introduced in [**8**] together with completely defined boundary conditions. It is known that this model can be transformed to a one-dimensional diffusion equation on the real line [**1**]; however, the model is characteristic for a number of Asian options so that the conclusions below transfer readily to cases where such a transformation is not possible, such as the fixed strike call with early exercise.

For simplicity let us assume that the averaging period is the same as the life of the option. The value of the call $u(x,a,t)$, normalized by the strike price K, is the solution of

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x + \frac{x}{T}u_a - ru + u_t = 0 \tag{9}$$

Here $x = S/K$ and

$$a = \frac{1}{T}\int_0^t x(t)\,dt.$$

We note that if instead the definition

$$a = \frac{1}{t}\int_0^t x(t)\,dt$$

is used then the pricing equation (9) becomes

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x + \frac{1}{t}(x-a)u_a - ru + u_t = 0 \tag{10}$$

which is suggested in [**15**] for a finite element solution of the fixed strike Asian option. In either case payoff at maturity T is

$$u(x,a,T) = \max\{a-1, 0\}.$$

Let us now turn to the boundary conditions for (9). At $x = 0$ we have the condition

$$u(0,a,t) = \max\{(a-1)e^{-r(T-t)}, 0\}$$

because $x(t^*) = 0$ implies $x(t) = 0$ for $t > t^*$.

The boundary condition at X as $X \to \infty$ is not known with certainty but will be approximated by

$$u_x(X,t) = \frac{1 - e^{-r(T-t)}}{rT}$$

which is consistent with the analytic solution of (9) derived in [**8**] for $a > 1$ and which represents an upper bound on the delta of the option [**1**].

Since only a first derivative with respect to a occurs in (9) it is generally agreed that only one additional boundary condition can be imposed. In [**8**] the boundary condition is placed at $a = 1$. The boundary conditions used in [**15**] for the solution of (10) are not specifically identified but implied by the numerical upwinding scheme for $\partial u/\partial a$. In contrast, the finite element code of [**14**] for (10) proceeds without any boundary conditions as $x \to \infty$, $a \to \infty$ and $x = a = 0$. So the question is: What is required to make the boundary value problem for (9) well posed?

In the terminology of [**13**] the above equation is a degenerate second order equation with nonnegative characteristic form (as is the standard Black Scholes equation for vanilla puts and calls). The general form of such an equation is

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)u_{x_i} + c(x)u = 0 \tag{11}$$

where the matrix $A(x) = (a_{ij}(x))$ is positive semi-definite on the closure of an open bounded domain $D \subset R_n$. As developed in [**13**] the well-posedness of the Dirichlet problem for (11) depends essentially on the algebraic sign of the so called Fichera function

$$b(x) \equiv \sum_{i=1}^{n} \left(b_i(x) - \sum_{j=1}^{n} (a_{ij}(x))_{x_j} \right) n_i$$

defined on the boundary ∂D with inward unit normal $\vec{n} = (n_1, \ldots, n_n)$. Specifically, the part of the boundary where

$$\sum_{i,j=1}^{n} a_{ij}(x) n_i n_j = 0$$

is decomposed into disjoint arcs Σ_0, Σ_1 and Σ_2 where

$$\begin{aligned} \Sigma_0 &= \{x : b(x) = 0\} \\ \Sigma_1 &= \{x : b(x) > 0\} \\ \Sigma_2 &= \{x : b(x) < 0\}. \end{aligned}$$

We note that in heat and fluid flow the Fichera function describes the direction of the convective flow on ∂D. For example, on Σ_2 convection into the region D is given. The remainder of the boundary is denoted by

$$\Sigma_3 = \partial D - \bigcup_{i=1}^{3} \partial \Sigma_i.$$

The existence and uniqueness of a suitably defined weak solution is then proved (see [**13**, p. 30]) provided we have Dirichlet data

$$u = g \quad \text{on } \Sigma_2 \cup \Sigma_3.$$

The proof is based on adding $\epsilon \Delta u$ to the degenerate equation, extending the boundary data to ∂D and using standard elliptic theory to obtain a priori estimates on the solution which are independent of ϵ and which allow passage to the limit.

If we agree to solve (9) on the domain $D = \{(x, a, t),\ 0 < x < X,\ 0 < a < 1,\ 0 < t < T\}$ then the hypotheses of the existence theorem require Dirichlet data on $x = X$, $a = 1$, $t = T$, and no data on the remainder of the boundary. However, if the boundary $x = 0$ is replaced by a down and out barrier at $x_0 \ll 1$, as is often suggested for numerical work then the line $x = x_0$ belongs to Σ_3 so that Dirichlet data there are consistent with the theory. If the theory is applied to the equation (10) then Dirichlet data are required only on $x > a$ along the boundary $a =$ constant.

We note that the Neumann condition on $x = X$ does not fit the framework of the theory. We do not know of a source where the results on degenerate equations are extended to mixed Dirichlet/Neumann boundary conditions, but numerical evidence shown below suggests that the problem (9) remains well posed.

The Asian option problem with European pay-off (9) can be readily solved by reducing it to a sequence of coupled Black Scholes type equations and applying a standard numerical Black Scholes pde solver. We discretize time and the averaging variable a by writing

$$\Delta a = 1/M, \quad a_m = m\Delta a, \quad m = 0, \ldots, M$$
$$\Delta t = T/N, \quad t_n = n\Delta t, \quad n = 0, \ldots, N$$

and replace (9) along the line $a = a_m$ with the semi-discrete method of lines approximation

$$\frac{1}{2}\sigma^2 x^2 u''_{mn} + rxu'_{mn} + \frac{x}{T}\nabla_a u_{mn} - ru_{mn} + \nabla_t u_{mn} = 0 \tag{12}$$

where

$$u_{mn} = u(x, a_m, t_n)$$

and where ∇_a and ∇_t denote finite difference approximations to $\partial u(x,a,t)/\partial a$ and $\partial u/\partial t$. To be specific we shall now consider the first order differences

$$\nabla_a u_{mn} = \frac{u_{m+1n} - u_{mn}}{\Delta a} \qquad \nabla_t u_{mn} = \frac{u_{mn+1} - u_{mn}}{\Delta t}. \tag{13}$$

The equation (12) is solved at time t_n for $m = 0, 1, \ldots, M-1$, i.e. at each time level we solve M coupled Black-Scholes type problems. Once $\{u_{mn}(x)\}$ are found then t_n is decreased by Δt to t_{n-1}. The first order difference quotient (13) for ∇_a is attractive because it corresponds to upwinding the convection term which makes the boundary value problem (12) coercive much like a non-degenerate problem. In particular, this structure allows a convergent line iteration by assuming an initial guess $\{u^0_{mn}\}$ and for $k = 1, 2, \ldots$ cycling forward from $m = 0$ to $m = M-1$ until the iterates $\{u^k_{mn}\}$ no longer change. Convergence of this method at a given time level can be proven a priori as in the elliptic case considered in [**10**]. However, numerical simulations of the call price for (9) suggest that a single backward pass from $m = M-1$ to $m = 0$ yields the same result as an iterative forward solution which on average requires six iterations per time step and hence takes six times as long per time step as the backward pass.

A serious drawback of the first order approximation (13) is its slow convergence as Δa and $\Delta t \to 0$. We shall discuss improving the approximation with respect to $\partial u/\partial a$ first. From a practical point of view we need a second order approximation for $\partial u/\partial a$. In view of the success of a three level level scheme for a standard Black Scholes model [**11**], and in view of the fact that an analytic solution is available for $a > 1$ we have opted for a three level backward approximation of $\partial u/\partial a$. Hence it is suggested that at time t_n for $m = 0, \ldots, M$ we solve (12) with

$$\nabla_a u_{mn} = -\left[\frac{3}{2}\frac{u_{mn} - u_{m+1n}}{\Delta a} - \frac{1}{2}\frac{u_{m+1n} - u_{m+2n}}{\Delta a}\right]. \tag{14}$$

Since $u(x,a,t)$ is known for $a > 1$ the source term for $m = M-1$ is known. If $u(x,a,t)$ were not known the two-level scheme (13) can be used to compute u_{M-1n} before switching to (14). The computing speed of the direct forward or iterative backward method was unchanged when using a three level rather than a two level scheme but convergence as $\Delta a \to 0$ was markedly improved.

Even more important is the order of approximation of the time derivative. Again, one can go to a three level scheme as in [**11**], but now the calculation at $t = T - \Delta t$ requires u at $T + \Delta t$ which is not available. Hence in general equation (12) with the first order implicit difference quotient (13) would have to be used at $T - \Delta t$. However, since the European call for Asian options is linear a Richardson extrapolation is an attractive, and often practiced, alternative. The solution of (12) with (13) can be expected to be correct to order Δt. Hence if at a given time we have two numerical solutions $u_{\Delta t}(x, a)$ and $u_{2\Delta t}(x, a)$ obtained for time steps Δt and $2\Delta t$, resp., then the extrapolated solution is

$$u^*(x, a) = 2u_{\Delta t}(x, a) - u_{2\Delta t}(x, a).$$

The solution u^* may be assumed to be second order accurate in Δt. Higher order corrections may also be used but are not considered here.

Table 2 illustrates the influence of the three level scheme for $\partial u/\partial a$ and the Richardson extrapolation with respect to time for a typical Asian call with arithmetic averaging and European pay-off. It may be observed that for a given time step the data obtained with the second order approximation for $\partial u/\partial a$ change little with the size of Δa and that the Richardson extrapolation yields quite consistent values for the call at the money, its delta and its gamma, whereas without extrapolation very (i.e., infeasibly) small time steps must be chosen.

Value of the option

$N \setminus M$	100	100 extrap.*	200	200 extrap.	400	400 extrap.
200	6.4366		6.4358		6.4362	
400	6.0880	5.7394	6.0856	5.7354	6.0857	5.7352
800	5.9057	5.7234	5.9021	5.7186	5.9020	5.7183
1600	5.8125	5.7193	5.8081	5.7141	5.8078	5.7136
3200	5.7653	5.7181	5.7605	5.7129	5.7601	5.7124

Value of the delta $u'_{00}(1)$

$N \setminus M$	100	200	400	400 extrap.
200	0.5577	0.5544	0.5536	
400	0.5605	0.5566	0.5557	0.5578
800	0.5625	0.5583	0.5572	0.5587
1600	0.5637	0.5593	0.5581	0.5590
3200	0.5644	0.5598	0.5586	0.5591

Value of the gamma $u''_{00}(1)$				
$N \backslash M$	100	200	400	400 extrap.
200	2.6656	2.6697	2.6698	
400	2.8368	2.8451	2.8457	3.0216
800	2.9340	2.9456	2.9467	3.0477
1600	2.9857	2.9995	3.0010	3.0553
3200	3.0123	3.0274	3.0292	3.0574

$r = .1$, sigma $= .4$, $X = 3$, $\Delta x = .01$, $\Delta a = 1/M$, $T = .3$, $\Delta t = T/N$
*the extrapolated values are computed from the values for N and $N/2$ in the column immediately to the left

Table 2. Fixed strike Asian call $u_{00}(1)$ with European pay-off.

We remark that for this linear problem one may also use (13) and extrapolation with respect to Δa instead of the second order scheme (14). Indeed, if the entries of Table 2 for, say, $N = 400$ and $M = 200$ and $M = 400$ are computed with (13) and extrapolation is applied, the following values are obtained

M	200	400	extrapolated value
call	6.7537	6.4298	6.1059
delta	0.5555	0.5555	0.5555
gamma	2.5252	2.6711	2.8170

These values are quite consistent with the entries of Table 2 for $M = N = 400$. One may infer from the numerical experiments of [**4**] that extrapolation can remain an effective tool even for the nonlinear problem of American options with its dissimilar convergence rates for the option price and the early exercise boundary. However, the theoretical foundation of this approach requires knowledge of the convergence rates which are generally not known in this setting. Hence we prefer a formally second order discretization (14) over the extrapolation method for a first order discretization.

References

1. Alziary B., Decamps J.-P. and Koehl P-F., *A P.D.E. approach to Asian options: analytical and numerical evidence*, J. Banking Finance **21** (1997), 613–640.

2. Barles G., Burdeau J., Romano M. and Samsoen N., *Estimation de la frontière libre des options américaines au voisinage de l'échéance*, C. R. Acad. Sci. Paris Ser. I Math. **316** (1993), 171–174.

3. Barles G., *Convergence of numerical schemes for degenerate parabolic equations arising in finance theory*, Numerical Methods in Finance (L. C. G. Rogers and D. Talay, eds.), Cambridge U. Press, 1997.

4. Carr P. and Faguet D., *Valuing finite-lived options as perpetual*, Cornell University, working paper, 1996.

5. ______, *Randomization and the American Put*, Review of Financial Studies, Fall 1998.

6. Fouque J.-P., Papanicolaou G. and Sircar K. R., From the implied volatility skew to a robust correction to Black-Scholes American option prices, NCSU working paper, March 2000.

7. Jaillet P., Lamberton D. and Lapeyre B., *Variational inequalities and the pricing of American options*, Acta Appl. Math. **21** (1990), 263–289.

8. Kemna A. G. Z. and Vorst A., *A pricing method for options based on average asset values*, J. Banking Finance **14** (1990), 113–129.

9. Kwok Y. K., *Mathematical Models of Financial Derivatives*, Springer Verlag, 1998.

10. Meyer G. H., *Free boundary problems with nonlinear source terms*, Numer. Math. **43** (1984), 463–482.

11. Meyer G. H. and van der Hoek J., *The valuation of American options with the method of lines*, Advances in Futures and Options Research **9** (1997), 265–285.

12. Meyer G. H., *Pricing American Puts for Assets with Discrete Dividends*, J. Comp. Finance, to appear.

13. Oleinik O. A. and Radkevic E. V., *Second Order Equations with Nonnegative Characteristic Form*, AMS Translation, Plenum Press, 1973.

14. Zhu Z. and Stokes N., *A finite element platform for pricing path dependent exotic options*, CSIRO working paper, 1998.

15. Zvan R., Forsyth P. A. and Vetzal K. R., *Robust numerical methods for PDE models of Asian options*, Journal of Computational Finance **1** (1997/1998), 39–78.

G. H. Meyer, School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332-0160