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A NUMERICAL APPROXIMATION OF NONFICKIAN FLOWS WITH MIXING LENGTH GROWTH IN POROUS MEDIA

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ABSTRACT. The nonFickian flow of fluid in porous media is complicated by the history effect which characterizes various mixing length growth of the flow, which can be modeled by an integro-differential equation. This paper proposes two mixed finite element methods which are employed to discretize the parabolic integro-differential equation model. An optimal order error estimate is established for one of the discretization schemes.

1. Introduction

The understanding of the behavior of the flow of multi-phase and multicomponent fluids through porous media is influenced by many physical phenomena such as the heterogeneities and the degree of correlation in the permeability fields. The research in reservoir simulation has been mainly focused on dispersion models in the past. Numerical results have illustrated the success of dispersion models for many of the approximation techniques. However, there is a type of channeling of nonFickian flow in reservoirs which may have important history effects in the flow and deserves a thorough study in theory and numerical approximations. Cushman and his colleagues [**6, 7, 8, 20**] have developed a non-local theory and some applications for the flow of fluid in porous media. Furtado, Glimm, Lindquist, and Pereira [**18, 19**], Neuman and Zhang [**25**], and Ewing [**12, 13, 14**] also studied the history effect of various mixing length growth for flow in heterogeneous porous media.

For illustrative purpose, we consider a simple example where a conservative tracer is transported by convection and dispersion under a steady, saturated, incompressible groundwater flow in a nondeformable porous medium of constant porosity [**7**]. The Darcy's scale transport equation is thus

$$\frac{\partial C}{\partial t} + \frac{\partial S}{\partial t} + \nabla \cdot (VC) - \nabla \cdot \mathbf{d} \nabla C = 0, \tag{1.1}$$

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where C is the phase concentration, $V = (v_1, v_2, v_3)^T$ is the Darcy scale velocity, and $\mathbf{d}$ is the local scale dispersion tensor assumed to be constant, and S is the sorbed phase concentration; the adsorption is governed by a non-equilibrium linear rate equation

$$\frac{\partial S}{\partial t} = K_r(K_d C - S), \tag{1.2}$$

where K_r is the reaction rate parameter and K_d is the usual partition coefficient. Because of uncertainty in the data, it is assumed that K_d, S, C and V are random. We decompose them into

$$K_d = \bar{K}_d + k_d, \qquad S = \bar{S} + s, \qquad C = \bar{C} + c, \qquad V = \bar{V} + v, \tag{1.3}$$

where, for example, $\bar{K}$ is the mean and k_d is the fluctuation. The key point here is that there is no "smallness" assumption on any of these fluctuations.

With the decomposition (1.3) and the assumption that $\overline{k_d\, c}$, $\overline{k_d\, k_d}$, and $\overline{k_d v_j}$ are stationary covariances, and that the mean of V is in the x_1 direction, the following equation for $\bar{C}$ has been derived [**7**]:

$$\begin{aligned} &\frac{\partial \bar{C}}{\partial t} + V\frac{\partial \bar{C}}{\partial x_1} - \int_0^t \nabla \cdot D'(t-s)\nabla \bar{C}\, ds = \nabla \int_0^t G'(t-s)\bar{C}\, ds \\ &- K_r \Big\{ \bar{K}_d \bar{C} - e^{K_r t} - K_r \bar{K}_d \int_0^t e^{-K_d(t-s)} \bar{C}\, ds \\ &- \int_0^t [\delta(t-s) - K_r e^{-K_d(t-s)}] \times \Big[\int_0^s G''(s-\tau)\bar{C}\, d\tau + \int_0^s B'(s-\tau)\cdot\nabla\bar{C} d\tau\Big] ds \Big\}, \end{aligned} \tag{1.4}$$

where $\mathbf{D}' = (d'_{ij})$ and $d'_{ij}(t) = d_{ij}\delta(t) + \overline{v_i v_j}(x)B(x,t)$, and $B's$ and $G's$ are functions in time related to K_r, $\bar{K}_d$, $\mathbf{d}$. We note that the correlation between $\overline{v_j v_j}$ is one of the main sources generating the nonlocal effects from microscales to macroscale level. Readers are referred to [**6**], [**7**], [**8**], [**18**], [**20**], [**12**], [**13**], [**14**] and the references therein for the mathematical modeling and other related problems in detail.

The equation (1.4) is a special case of the following general parabolic integro-differential problem: Find $u = u(x,t)$ satisfying

$$\begin{aligned} &u_t = \nabla \cdot \boldsymbol{\sigma} + cu + f \ \text{ in } \ \Omega \times J, \\ &\boldsymbol{\sigma} = A(t)\nabla u - \int_0^t B(t,s)\nabla u(s)\, ds, \ \text{ in } \ \Omega \times J, \\ &u = g \ \text{ on } \ \partial\Omega \times J, \\ &u = u_0(x), \quad x \in \Omega, \quad t = 0, \end{aligned} \tag{1.5}$$

where $\Omega \subset R^d$; $(d = 2, 3)$ is an open bounded domain with smooth boundary $\partial\Omega$, $J = (0, T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 or 3×3 matrices, and A^{-1} exists and is bounded, $c \le 0$, f, g and u_0 are known smooth functions. We remark that even the memroy term in (1.5) does not have the term $\mathbf{b}u$, where $\mathbf{b}$ is a vector, our MFE formulation and analysis are also valid for such case with only a minor modification.

For numerical approximations, several finite difference methods were studied for the approximate solution of (1.5). In [**27**], the method of backward Euler and Crank-Nicolson combined with a certain numerical quadrature rule is employed to deal with the time direction which aims at reducing the computational cost and storage spaces due to the memory effect. In the finite element method, there is extensive literature from the last ten years [**2**], [**3**], [**21**], [**22**], [**23**], in which optimal and super convergence can be found for the corresponding finite element approximations in various norms, such as L^p with $2 \leq p \leq \infty$. In particular, the method of using a Ritz-Volterra projection, discovered by Cannon and Lin [**2**], proved to be a powerful technique behind the analysis.

To the best of our knowledge, there were no rigorous mathematical formulations and analysis for the mixed finite element method for this type of transport flow equations although the standard Galerkin method has been well studied and understood in the last decade.

We are concerned with approximate solutions of (1.5) by mixed finite element methods in this paper. The mathematical difficulty associated with the analysis of numerical approximations to the solution of (1.5) lies on the integral term added to the standard parabolic equation. Two mixed methods are introduced to tackle this obstacle in Section 2. Our methods are in general applicable to the fully non-localized version of reactive transport equations in porous media.

The paper is organized as follows. In Section 2, we propose two numerical schemes for the general parabolic integro-differential equation based on the variables u and $\boldsymbol{\sigma}$. In Section 3, we derive an optimal order error estimate for the mixed finite element approximations in the L^2 norm.

2. Discretizations by Mixed Finite Elements

In this section, we propose two numerical schemes for the parabolic integro-differential equation (1.5). For simplicity, the method will be presented on plane domains.

Let $W = L^2(\Omega)$ be the standard L^2 space on Ω with norm $\|\cdot\|_0$. Denote by

$$\mathbf{V} = H(\mathrm{div}, \Omega) = \left\{\boldsymbol{\sigma} \in (L^2(\Omega))^2 \mid \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega)\right\},$$

the Hilbert space equipped with the following norm:

$$\|\boldsymbol{\sigma}\|_{\mathbf{V}} = \left(\|\boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2\right)^{\frac{1}{2}}.$$

There are several ways to discretize the problem (1.5) based on the variables $\boldsymbol{\sigma}$ and u; each method corresponds to a particular variational form of (1.5).

2.1. A mixed finite element method

Let T_h be a finite element partition of Ω into triangles or quadrilaterals which is quasi-regular. Let $\mathbf{V}_h \times W_h$ denote a pair of finite element spaces satisfying the Brezzi-Babuska condition. For example, the elements of Raviart and Thomas [**26**] would be a good choice for $\mathbf{V}_h$ and W_h. Let us illustrate the scheme using only the Raviart-Thomas elements of the lowest order. The result can be extended to other elements without any difficulty.

Multiplying both sides of the second equation of (1.5) by $A^{-1}(t)$ yields

$$A^{-1}(t)\boldsymbol{\sigma} = \nabla u - \int_0^t A^{-1}(t)B(t,s)\nabla u. \tag{2.1}$$

Solving ∇u from the integral equation (2.1) in term of $\boldsymbol{\sigma}$, we obtain

$$\nabla u = A^{-1}(t)\boldsymbol{\sigma} + \int_0^t R(t,s)A^{-1}(s)\boldsymbol{\sigma}(s)\,ds, \tag{2.2}$$

where $R(t,s)$ is the resolvent of the matrix $A^{-1}(t)B(t,s)$ and is given by

$$R(t,s) = A^{-1}(t)B(t,s) + \int_s^t A^{-1}(t)B(t,\tau)\; R(\tau,s)\,ds, \qquad t > s \geq 0. \tag{2.3}$$

Observe that the resolvent is smooth and bounded since $A^{-1}B$ is.

Test the equation (2.2) against vector-valued functions in $H(div,\Omega)$,

$$(A^{-1}\boldsymbol{\sigma}, \mathbf{v}) + \int_0^t (M(t,s)\boldsymbol{\sigma}(s), \mathbf{v})\,ds = (\nabla u, \mathbf{v}),$$

where $M(t,s) = R(t,s)A^{-1}(s)$. Using the Green's formula and the boundary condition $u = g$, one obtains

$$(\nabla u, \mathbf{v}) = -(\nabla\cdot\mathbf{v}, u) + \langle g, \mathbf{v}\cdot\mathbf{n}\rangle,$$

where, and in what follows in this paper $\langle\cdot,\cdot\rangle$ indicates the L^2-inner product on $\partial\Omega$. Thus,

$$(A^{-1}\boldsymbol{\sigma}, \mathbf{v}) + \int_0^t (M(t,s)\boldsymbol{\sigma}(s), \mathbf{v})\,ds + (\nabla\cdot\mathbf{v}, u) = \langle g, \mathbf{v}\cdot\mathbf{n}\rangle \tag{2.4}$$

for all $\mathbf{v} \in H(\mathrm{div},\Omega)$.

Next, test the first equation of (1.5) against functions in W, yielding

$$(u_t, w) - (\nabla\cdot\boldsymbol{\sigma}, w) - (cu, w) = (f, w), \qquad \forall w \in W. \tag{2.5}$$

The equations (2.4) and (2.5) give immediately a variational form for problem (1.5). Note that the initial condition $u(0,x) = u_0(x)$ should be added to the variational form as well. The corresponding discrete version seeks a pair $(u_h, \boldsymbol{\sigma}_h) \in W_h \times \mathbf{V}_h$ such that

$$\begin{aligned} &(u_{h,t}, w_h) - (\nabla\cdot\boldsymbol{\sigma}_h, w_h) - (cu_h, w_h) = (f, w_h), \\ &(A^{-1}\boldsymbol{\sigma}_h, \mathbf{v}_h) + \int_0^t (M(t,s)\boldsymbol{\sigma}_h(s), \mathbf{v}_h)\,ds + (u_h, \nabla\cdot\mathbf{v}_h) = \langle g, \mathbf{n}\cdot\mathbf{v}_h\rangle, \end{aligned} \tag{2.6}$$

for all $w_h \in W_h$ and $\mathbf{v}_h \in \mathbf{V}_h$. The discrete initial condition $u_h(0,x) = u_{0,h}$, where $u_{0,h} \in W_h$ is some appropriately chosen approximation of the initial data $u_0(x)$, should be added to (2.6) for starting. The pair $(u_h, \boldsymbol{\sigma}_h)$ is an approximation of the true solution of (1.5) in the finite element space $W_h \times \mathbf{V}_h$. For sake of notation, we shall assume that $\boldsymbol{\sigma}_h(0)$ satisfies the equation (2.6) with $t = 0$; namely, it is related to $u_{0,h}$ as follows:

$$(A^{-1}\boldsymbol{\sigma}_h(0), \mathbf{v}_h) + (u_{0,h}, \nabla \cdot \mathbf{v}_h) = \langle g_0, \mathbf{n} \cdot \mathbf{v}_h \rangle, \tag{2.7}$$

where $g_0 = g(0, x)$ is the initial value of the boundary data.

2.2. A hybridized mixed method

The method presented in the previous section is based on the equation (2.2) in which the kernel $R(t, s)$ is the solution of a Volttera integral equation. In other words, one would have to solve $R(t, s)$ from (2.3) before discretizing the integro-differential equation (1.5). Our objective here is to outline a hybridized method which does not require the solution of Volttera equations.

In the hybridized method, the continuity requirement of the flux component $\boldsymbol{\sigma}$ in the normal direction of interior edges is compensated by the use of a set of new variables defined on the interior edges of the finite element partition T_h. The set of new variables are known as Lagrange multipliers and approximate the variable u on the interior edges.

The finite element space for the flux $\boldsymbol{\sigma}$ is given by

$$\mathbf{V}_h = \{\mathbf{v} : \quad \mathbf{v}|_K \in \mathbf{V}(K, j)\},$$

where $\mathbf{V}(K, j)$ is the local finite element space of order j on the triangle or quadrilateral K. Again, we shall illustrate the case of $j = 0$ corresponding to the lowest order Raviart-Thomas element. The finite element space for the variable u is the same as the one described in the previous section. The space of Lagrange multipliers consists of piecewise constant functions on the set of interior edges:

$$\Lambda_h = \{\lambda : \quad \lambda|_e \in P_0(e), \quad \text{on any interior edge } e\}.$$

We are now ready to derive a hybridized discretization scheme for (1.5). First, we test (2.1) against functions $\mathbf{v}_h$ in $\mathbf{V}_h$. Let $G(t, s) = B^T A^{-T}$ where M^T stands for the transpose of the matrix M. Since there is no continuity for $\mathbf{v}_h$ in the normal direction of the element boundary, then we have

$$\begin{aligned}(A^{-1}\boldsymbol{\sigma}, \mathbf{v}_h) = &-\sum_K (\nabla \cdot \mathbf{v}_h, u)_K + \sum_K \int_{\partial K} u\mathbf{v}_h \cdot \mathbf{n}_K \\ &+ \sum_K \int_0^t (u, \nabla \cdot (G(t,s)\mathbf{v}_h))_K ds - \sum_K \int_0^t \left(\int_{\partial K} uG(t,s)\mathbf{v}_h \cdot \mathbf{n}_K \right) ds,\end{aligned}$$

where $(\cdot, \cdot)_K$ stands for the L^2-inner product on the element $K \in \mathsf{T}_h$. Note that u has a given value g on the boundary of Ω. Thus, u shall be replaced by g for boundary integrals on $\partial\Omega$. The interior boundary integrals will substitute u by its approximation, called λ_h, in the finite element space Λ_h. Therefore, the hybridized mixed finite element method seeks $u_h \in W_h$, $\boldsymbol{\sigma}_h \in \mathbf{V}_h$, and $\lambda_h \in \Lambda_h$ satisfying

$$\begin{aligned}
(u_{h,t}, w_h) - (\nabla \cdot \boldsymbol{\sigma}_h, w_h) - (cu_h, w_h) &= (f, w_h), \qquad w_h \in W_h,\\
(A^{-1}\boldsymbol{\sigma}_h, \mathbf{v}_h) + \sum_K (\nabla \cdot \mathbf{v}_h, u_h)_K - \sum_K \int_{\partial K \cap \Omega} \lambda_h \mathbf{v}_h \cdot \mathbf{n}_K &\\
- \sum_K \int_0^t (u_h, \nabla \cdot (G(t,s)\mathbf{v}_h))_K ds + \sum_K \int_0^t \left(\int_{\partial K \cap \Omega} \lambda_h G(t,s) \mathbf{v}_h \cdot \mathbf{n}_K \right) ds &\\
= \langle g, \mathbf{v}_h \cdot \mathbf{n} \rangle - \int_0^t \langle g, G(t,s)\mathbf{v}_h \cdot \mathbf{n} \rangle, \quad \mathbf{v}_h \in \mathbf{V}_h,&\\
\sum_K \int_{\partial K \cap \Omega} \boldsymbol{\sigma}_h \cdot \mathbf{n}_K \mu = 0, \qquad \mu \in \Lambda_h.&
\end{aligned} \tag{2.8}$$

The last equation in (2.8) ensures a continuity of the flux in the normal direction of interior edges.

3. An Error Estimate in L^2

In this section, we derive an optimal order L^2 error estimate for the mixed finite element approximation resulted from the scheme (2.6). The result is similar to those of the mixed method applied to parabolic problems without memory effects.

In the finite element analysis for parabolic problems, it is often convenient to consider projections of the true solution with respect to the elliptic part of the differential operator. For example, the standard Ritz projection is a good candidate in the Galerkin method for parabolic equations [**28, 31**]. For integro-differential equations, the Ritz-Volttera projection [**2**] plays a good role in the error analysis. For the mixed method, we shall consider a projection defined by using the mixed formula of the elliptic operator.

Let $(u, \boldsymbol{\sigma})$ be the solution of (1.5). Consider a pair $(P_h u, F_h \boldsymbol{\sigma})\colon [0,T] \to W_h \times \mathbf{V}_h$ which is defined as the solution of

$$(A^{-1}(\boldsymbol{\sigma} - F_h\boldsymbol{\sigma}), \mathbf{v}_h) + (\nabla \cdot \mathbf{v}_h, u - P_h u) = 0, \qquad \mathbf{v}_h \in \mathbf{V}_h, \tag{3.1}$$

$$(c(u - P_h u), w_h) + (\nabla \cdot (\boldsymbol{\sigma} - F_h\boldsymbol{\sigma}), w_h) = 0, \qquad w_h \in W_h. \tag{3.2}$$

The pair $(P_h u, F_h \boldsymbol{\sigma})$ is called the mixed finite element projection of $(u, \boldsymbol{\sigma})$. The error between $(P_h u, F_h \boldsymbol{\sigma})$ and $(u, \boldsymbol{\sigma})$ has been well studied in many existing papers [**1, 10, 26**]. The following error estimate is standard for the mixed finite element projection [**10**]:

$$\|\boldsymbol{\sigma} - F_h\boldsymbol{\sigma}\| \le Ch\|\boldsymbol{\sigma}\|_1, \qquad \|u - P_h u\| \le Ch\|u\|_1, \tag{3.3}$$

where, and in what follows in this section, we denote by $\|\cdot\|_m$ the norm in the Sobolev space $H^m(\Omega)$. The subscript m will be omitted when $m = 0$. In other words, the standard L^2 norm will also be denoted by $\|\cdot\|_0 = \|\cdot\|$.

Theorem 3.1. *Let* $(u, \boldsymbol{\sigma})$ *and* $(u_h, \boldsymbol{\sigma}_h)$ *be the solutions of* (1.5) *and* (2.6), *respectively. Assume that the initial data for* (2.6) *satisfies*

$$\|P_h u_0 - u_h(0)\|_0 + \|F_h \boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_h(0)\|_0 \le Ch(\|u_0\|_1 + \|\boldsymbol{\sigma}_0\|_1). \tag{3.4}$$

Then, there exists a constant C *such that*

$$\|u - u_h\|_0 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \le Ch \left(\|u_0\|_1 + \|\boldsymbol{\sigma}_0\|_1 + \int_0^t (\|\boldsymbol{\sigma}\|_1 + \|u_s\|_1)\, ds \right). \tag{3.5}$$

Proof. First we decompose the error as follows:

$$\begin{aligned} u - u_h &= (u - P_h u) + (P_h u - u_h) = \rho + \rho_h, \\ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h &= (\boldsymbol{\sigma} - F_h \boldsymbol{\sigma}) + (F_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) = \boldsymbol{\theta} + \boldsymbol{\theta}_h, \end{aligned}$$

where F_h and P_h are described as above. It follows from (1.5) and (2.6) that

$$(\rho_{h,t}, w_h) - (\nabla \cdot \boldsymbol{\theta}_h, w_h) - (c\rho_h, w_h) = -(\rho_t, w_h), \qquad w_h \in W_h, \tag{3.6}$$

$$(A^{-1}\boldsymbol{\theta}_h, \mathbf{v}_h) + (\rho_h, \nabla \cdot \mathbf{v}_h) = -\int_0^t M(t,s)(\boldsymbol{\theta}(s) + \boldsymbol{\theta}_h(s), \mathbf{v}_h)\, ds, \quad \mathbf{v}_h \in \mathbf{V}_h. \tag{3.7}$$

By letting $w_h = \rho_h$ in (3.6) and $\mathbf{v}_h = \boldsymbol{\theta}_h$ in (3.7) we obtain from their sum

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\rho_h\|^2 + \|\boldsymbol{\theta}_h\|_{A^{-1}}^2 - (c\rho_h, \rho_h) &= -(\rho_t, \rho_h) - \left(\int_0^t M(t,s)(\boldsymbol{\theta}(s) + \boldsymbol{\theta}_h(s)) ds, \boldsymbol{\theta}_h \right) \\ &\le \|\rho_t\|\, \|\rho_h\| + \frac{1}{2}\|\boldsymbol{\theta}_h\|_{A^{-1}}^2 + C\int_0^t (\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}_h\|_{A^{-1}}^2)\, ds, \end{aligned}$$

which in turn implies via an integration from 0 to t that

$$\begin{aligned} \|\rho_h\|^2 + 2\int_0^t \|\boldsymbol{\theta}_h\|_{A^{-1}}^2\, ds &\le \|\rho_h(0)\|^2 + \int_0^t \|\rho_s\|\, \|\rho_h\|\, ds \\ &\quad + C\int_0^t \left(\int_0^s (\|\boldsymbol{\theta}(\tau)\|^2 + \|\boldsymbol{\theta}_h(\tau)\|_{A^{-1}}^2) d\tau \right) ds \\ &\le \frac{1}{2}\sup_{0<s<t} \|\rho_h(s)\|^2 + \frac{1}{2}\left(\int_0^t \|\rho_s\| ds \right)^2 + \|\rho_h(0)\|^2 \\ &\quad + C\int_0^t \left(\int_0^s (\|\boldsymbol{\theta}(\tau)\|^2 + \|\boldsymbol{\theta}_h(\tau)\|_{A^{-1}}^2)\, d\tau \right) ds. \end{aligned}$$

Hence, taking the supremum first and then using the Gronwall's inequality, we find

$$\begin{aligned} \|\rho_h\|^2 + 2\int_0^t \|\boldsymbol{\theta}_h\|_{A^{-1}}^2\, ds &\le C\|\rho_h(0)\|^2 + C\int_0^t \|\boldsymbol{\theta}(s)\|^2\, ds + \left(\int_0^t \|\rho_s\|\, ds \right)^2 \\ &\le Ch^2 \left(\|u_0\|_1^2 + \int_0^t \|\boldsymbol{\sigma}(s)\|_1^2\, ds + \left(\int_0^t \|u_s\|_1\, ds \right)^2 \right), \end{aligned}$$

where we have used the standard error estimate (3.3).

In order to estimate $\boldsymbol{\theta}_h(t)$, we differentiate (3.7) to obtain

$$(A^{-1}\boldsymbol{\theta}_{h,t}, \mathbf{v}_h) + (\rho_{h,t}, \nabla\cdot\mathbf{v}_h) = -(M(t,t)(\boldsymbol{\theta}(t)+\boldsymbol{\theta}_h(t)), \mathbf{v}_h) - \int_0^t (M_t(t,s)(\boldsymbol{\theta}(s)+\boldsymbol{\theta}_h(s)), \mathbf{v}_h)\, ds.$$

Thus, we see from letting $w_h = \rho_{h,t}$ in (3.6) and $\mathbf{v} = \boldsymbol{\theta}_h$ in the above identity that

$$\begin{aligned}\|\rho_{h,t}\|^2 &+ \frac{1}{2}\frac{d}{dt}\|\boldsymbol{\theta}_h\|_{A^{-1}}^2 \\ &\le \|\rho_t\|\,\|\rho_h\| - \left(M(\boldsymbol{\theta}+\boldsymbol{\theta}_h) + \int_0^t M_t(t,s)(\boldsymbol{\theta}(s)+\boldsymbol{\theta}_h(s))\,ds, \boldsymbol{\theta}_h\right) \\ &\le \frac{1}{2}\|\rho_h\|^2 + \frac{1}{2}\|\rho_t\|^2 + C\left(\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}_h\|_{A^{-1}}^2 + \int_0^t (\|\boldsymbol{\theta}\|^2 + \|\boldsymbol{\theta}_h\|_{A^{-1}}^2)\,ds\right).\end{aligned}$$

Integrating the above from 0 to t yields

$$\begin{aligned}\int_0^t \|\rho_{h,t}\|^2\,ds + \|\boldsymbol{\theta}_h\|_{A^{-1}}^2 &\le C\left(\|\boldsymbol{\theta}_h(0)\|^2 + \int_0^t (\|\boldsymbol{\theta}(s)\|^2 + \|\rho_s(s)\|^2)\,ds\right) \\ &\le C\|\boldsymbol{\theta}_h(0)\|^2 + Ch^2\left(\int_0^t \|\boldsymbol{\sigma}\|_1^2 ds + \int_0^t \|u_s\|_1^2\,ds\right).\end{aligned}$$

Thus, the estimate (3.5) follows from the above inequality and the assumption (3.4). □

Finally, we comment briefly on the inequality (3.4). Recall that $(P_h u_0, F_h\boldsymbol{\sigma}(0))$ is the mixed finite element projection of $(u_0, \boldsymbol{\sigma}(0))$ in $W_h \times \mathbf{V}_h$. The function $u_h(0)$ is the initial data in the discretization scheme, which is often chosen as the L^2 projection of u_0 in the mixed finite element space W_h. Thus, it is clear that

$$\|P_h u_0 - u_h(0)\|_0 \le Ch(\|u_0\|_1 + \|\boldsymbol{\sigma}_0\|_1).$$

Regarding the error between $F_h\boldsymbol{\sigma}(0)$ and $\boldsymbol{\sigma}_h(0)$, notice that $\boldsymbol{\sigma}_h(0)$ is defined as the solution of (2.7) in which $u_{0,h} = u_h(0)$ is the L^2 projection of the initial data u_0. It is not hard to show that the following holds true:

$$\|F_h\boldsymbol{\sigma}(0) - \boldsymbol{\sigma}_h(0)\|_0 \le Ch(\|u_0\|_1 + \|\boldsymbol{\sigma}_0\|_1).$$

In other words, the assumption (3.4) is satisfied if the initial discrete data $u_{0,h}$ is chosen as the L^2 projection.

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