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STRONG ESTIMATE FOR SQAURE FUNCTIONS IN HIGHER DIMENSIONS

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Abstract. In this paper, we give a generalization to multidimensional case of strong estimate for square functions obtained by Jones Rosenblatt and Ostrovskii [**3**].

We assume that the averages are taken over squares and the operators are commuting and contraction in L^2. For the non-commutative case we need a supplementary condition. Some weak type inequalities are proved.

Introduction

Let (Ω, β, μ) be a σ-finite measure space, and let $\tau \colon \Omega \to \Omega$ be an invertible β-measurable transformation preserving μ. For $Tf = fo\tau$, R. Jones, I. Ostrovskii and J. Rosenblatt [**3**] proved strong and weak L^1estimates for square functions and sqaure maximal functions. In this paper, we give a generalization to multidimensional case of some strong and weak L^1 estimates. In the first section, we prove strong estimates for linear contractions and for power bounded operators in L^2. In the second section, strong estimates for square maximal functions are obtained. In the third section, some weak estimates for linear contractions are proved. In [**3**] it was shown the following result:

Theorem 1. *Given the usual averages $A_n f = \frac{1}{n}\sum_{k=1}^{n} fo\tau^k$ in ergodic theory, let $n_1 \le n_2 \le \dots$ and $Sf = \left(\sum_{k=1}^{\infty} \left|A_{n_k}f - A_{n_{k-1}}f\right|^2\right)^{\frac{1}{2}}$*

1. *For all $f \in L^2$ we have $\|Sf\|_2 \le 25\|f\|_2$.*
2. *For all $f \in L^1$we have $m\{Sf > \lambda\} \le 7000\|f\|_1$.*

Theorem 2. *Let T be a contraction on a Hilbert space H. Let (n_k) be a sequence in Z^+ with $n_k \le n_{k+1}$ for all $k \ge 1$. Let $A_n f = \frac{1}{n}\sum_{k=1}^{n} T^k f$ for all $f \in H$. Then $\left(\sum_{k=1}^{\infty} \|A_{n_k}f - A_{n_{k-1}}f\|_H^2\right)^{\frac{1}{2}} \le 25\|f\|_H$ for all $f \in H$.*

For $H = L^2$, we extend Theorem 2 to multidimensional case, such that the averages are taken over squares and the operators are commuting contractions in L^2.

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The non commuting case for power-bounded operators needs a supplementary condition.

I. Square Functions

a) Commuting case for contractions

We study the two cases:

1. The operators are commuting contractions in L^2 and the averages are taken over squares.
2. The power-bounded operators (may be not commuting). In this case we need a supplementary condition.

We study the case where the averages are taken over squares. Let $T_1, \ldots, T_d$ be linear contractions on L^2 and let

$$A_n(T_1, \ldots, T_d)f = A_n(T_1) \ldots A_n(T_d)f = \frac{1}{n^d} \sum_{k_1 < n} \cdots \sum_{k_d < n} T_1^{k_1} \ldots T_d^{k_d} f$$

and

$$S_d f = \left(\sum_{k=1}^{\infty} \left| A_{n_{k+1}}(T_1, \ldots, T_d)f - A_{n_k}(T_1, \ldots, T_d)f \right|^2 \right)^{\frac{1}{2}}.$$

From Theorem 2, we can easily deduce the following:

Theorem 3. *Let T be a contraction L^2. Let (n_k) be a sequence in Z^+ with $n_k \leq n_{k+1}$ for all $k \geq 1$. Let $A_n f = \frac{1}{n} \sum_{k=1}^{n} T^k f$for all $f \in L^2$. Then*

$$\|Sf\|_2 = \left\| \left(\sum_{k=1}^{\infty} \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 \right)^{\frac{1}{2}} \right\|_2 \leq 25 \|f\|_2$$

for all $f \in L^2$.

Proof. It suffices to remark that

$$\begin{aligned} \|Sf\|_2 &= \int \sum_{k=1}^{\infty} \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 \, d\mu \leq \sum_{k=1}^{\infty} \int \left| A_{n_k} f - A_{n_{k-1}} f \right|^2 \, d\mu \\ &= \sum_{k=1}^{\infty} \|A_{n_k} f - A_{n_{k-1}} f\|_{L^2}^2 \leq 25 \|f\|_2. \end{aligned}$$ ☐

Theorem 4 is our main result in this section, that is an extension of Theorem 3 to higher dimensions.

Theorem 4. *Let $T_1, \ldots, T_d$ be linear commuting contractions on L^2. For $f \in L^2$ we have the following inequality*

$$\|S_d f\|_2 \leq (26^d - 1)\|f\|_2.$$

Proof. First, we study the case $d = 2$. Let $T_1 = T$ and $T_2 = S$ we can write

$$\begin{aligned}
&A_{n_{k+1}}(T)A_{n_{k+1}}(S)f - A_{n_k}(T)A_{n_k}(S)f \\
&\quad = \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right)f \\
&\qquad + \left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)A_{n_k}(S)f + A_{n_k}(T)\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)f
\end{aligned}$$

using the triangle inequality we see that

$$\begin{aligned}
S_2 f &= \left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T)A_{n_{k+1}}(S)f - A_{n_k}(T)A_{n_k}(S)f\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty}\left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right)f\right|^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k=1}^{\infty}\left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)A_{n_k}(S)f\right|^2\right)^{\frac{1}{2}} \\
&\quad + \left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T)\left(A_{n_{k+1}}(S) - A_{n_k}(S)\right)f\right|^2\right)^{\frac{1}{2}} \\
&= Bf + Cf + Df.
\end{aligned}$$

For any $N \geq 1$, the partial sum $\sum_{k=1}^{N}\left|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right|^2$ is in L^1 if $f \in L^2$ and

$$\left\|\sum_{k=1}^{N}\left|A_{n_{k+1}}(T)f - A_{n_k}(T)f\right|^2\right\|_1 = \sum_{k=1}^{N}\|A_{n_{k+1}}(T)f - A_{n_k}(T)f\|_2^2.$$

Now, let $f_{n_k} = A_{n_{k+1}}(S)f - A_{n_k}(S)f$. We first show that $\|Bf\|_2 \leq 25^2\|f\|_2$. To see this we write

$$\begin{aligned}
\|B_N f\|_2^2 &= \int \sum_{k=1}^{N}\left|\left(A_{n_{k+1}}(T) - A_{n_k}(T)\right)f_{n_k}\right|^2 d\mu \\
&\leq \int \sum_{k_1=1}^{N}\sum_{k_2=1}^{N}\left|\left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T)\right)f_{n_{k_2}}\right|^2 d\mu
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k_2=1}^{N} \left\| \sum_{k_1=1}^{N} \left| \left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T) \right) f_{n_{k_2}} \right|^2 \right\|_1 \\
&= \sum_{k_2=1}^{N} \left\| \left(\sum_{k_1=1}^{N} \left| \left(A_{n_{k_1+1}}(T) - A_{n_{k_1}}(T) \right) f_{n_{k_2}} \right|^2 \right)^{\frac{1}{2}} \right\|_2^2 \\
&= \sum_{k_2=1}^{N} \|S_2(f_{n_{k_2}})\|_2^2 \le 25^2 \sum_{k_2=1}^{N} \|f_{n_{k_2}}\|_2^2 \qquad \text{(by Theorem 1 on } T\text{)} \\
&= 25^2 \sum_{k_2=1}^{N} \|A_{n_{k+1}}(S)f - A_{n_k}(S)f\|_2^2 \le 25^4 \|f\|_2^2 \qquad \text{(by Theorem 1 on } S\text{).}
\end{aligned}
$$

Let $N \to \infty$, the monotone convergence Theorem says $\|Bf\|_2 \le 25^2 \|f\|_2$.
To find a majorization for C we shall use the commutation of T and S:

$$
\begin{aligned}
\|C_N f\|_2^2 &= \int \sum_{k=1}^{N} \left| \left(A_{n_{k+1}}(T) - A_{n_k}(T) \right) A_{n_k}(S) f \right|^2 d\mu \\
&= \int \sum_{k=1}^{N} \left| A_{n_k}(S) \left(A_{n_{k+1}}(T) - A_{n_k}(T) \right) f \right|^2 d\mu \qquad \text{(since } TS = ST\text{)} \\
&\le \int \sum_{k=1}^{N} \left| \left(A_{n_{k+1}}(T) - A_{n_k}(T) \right) f \right|^2 d\mu \qquad \text{(since } \|S\|_2 \le 1\text{)} \\
&\le \sum_{k=1}^{N} \left\| \left(A_{n_{k+1}}(T) - A_{n_k}(T) \right) f \right\|_2^2 \le 25^2 \|f\|_2 \qquad \text{(by Theorem 1 on } S\text{)}
\end{aligned}
$$

and then $\|Cf\|_2 \le 25\|f\|_2$.

By the same argument as in C and without the commutation of T and S we can write $\|Df\|_2 \le 25\|f\|_2$. Finally, we have

$$
\|S_2 f\|_2 \le (25^2 + 25 + 25)\|f\|_2 = 675\|f\|_2 = (26^2 - 1)\|f\|_2. \qquad \square
$$

For the general case, when $d > 2$ we need the following technical lemma where the proof can be done by induction on d:

Lemma 5. *Let $a_1, \dots, a_d$ and $b_1, \dots, b_d$ be real numbers. Then we have the following equality*

$$
a_1 \dots a_d - b_1 \dots b_d = \prod_{i=1}^{d} (a_i - b_i) + \sum_{s=1}^{d-1} \left[\sum_{1 = i_1 < \cdots < i_s} b_{i_1} b_{i_2} \dots b_{i_s} \right] \prod_{\substack{j<d \\ j \notin \{i_1, \dots, i_s\}}} (a_j - b_j).
$$

By this lemma we can write

$$\begin{aligned}
&A_{n_{k+1}}(T_1)A_{n_{k+1}}(T_2)f\dots A_{n_{k+1}}(T_d)f - A_{n_k}(T_1)A_{n_k}(T_2)f\dots A_{n_k}(T_d)f\\
&= \prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right) f\\
&\quad + \sum_{s=1}^{d-1}\left[\sum_{1=i_1<\dots<i_s} A_{n_k}(T_{i_1})\dots A_{n_k}(T_{i_s})\right] \prod_{\substack{j<d\\ j\notin\{i_1,\dots,i_s\}}} \left(A_{n_{k+1}}(T_j)f - A_{n_k}(T_j)\right) f.
\end{aligned}$$

Using the triangle inequality we see that

$$\begin{aligned}
Sf &= \left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\dots A_{n_{k+1}}(T_{i_d})f - A_{n_k}(T_{i_1})\dots A_{n_k}(T_{i_d})f\right|^2\right)^{\frac{1}{2}}\\
&\le \left(\sum_{k=1}^{\infty}\left|\left[\prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right)\right]\right|^2 f\right)^{\frac{1}{2}}\\
&\quad + \sum_{i_1=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\prod_{j\ne i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}\\
&\quad + \sum_{1=i_1<i_2}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})A_{n_{k+1}}(T_{i_2})\prod_{j\notin\{i_1,i_2\}}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}\\
&\quad + \dots + \sum_{1=i_1<\dots<i_{d-1}}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\dots A_{n_{k+1}}\right.\right.\\
&\quad \times (T_{i_{d-1}})\left.\left.\prod_{j\notin\{i_1,\dots,i_{d-1}\}}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Let

$$\begin{aligned}
Mf &= \left(\sum_{k=1}^{\infty}\left|\left[\prod_{i=1}^{d}\left(A_{n_{k+1}}(T_i) - A_{n_k}(T_i)\right)\right] f\right|^2\right)^{\frac{1}{2}},\\
M_{i_1}f &= \sum_{i_1=1}^{\infty}\left(\sum_{k=1}^{\infty}\left|A_{n_{k+1}}(T_{i_1})\prod_{j\ne i_1}\left(A_{n_{k+1}}(T_j) - A_{n_k}(T_j)\right) f\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

and let

$$M_{i_1,\dots,i_s} f = \Bigg(\sum_{k=1}^{\infty} \Big| A_{n_{k+1}}(T_{i_1}) \dots A_{n_{k+1}}(T_{i_{d-1}}) \\ \times \prod_{j<d j\notin\{i_1,\dots,i_{s-1}\}} \big(A_{n_{k+1}}(T_j) - A_{n_k}(T_j) \big) f \Big|^2 \Bigg)^{\frac{1}{2}}$$

with these notations, we have

$$Sf \leq Mf + \sum_{i_1=1}^{d} M_{i_1} f + \sum_{1=i_1<i_2}^{d} M_{i_1,i_2} f + \dots + \sum_{1=i_1<\dots<i_{d-1}}^{d} M_{i_1,\dots,i_d} f. \tag{1}$$

We shall majorize each Mf and $M_{i_1,\dots,i_s} f$ in L^2 for $s = 1, \dots, d-1$.

$$\begin{aligned}
\|Mf\|_2^2 &= \sum_{k=1}^{\infty} \int \left| \left[\prod_{i=1}^{d} \big(A_{n_{k+1}}(T_i) - A_{n_k}(T_i) \big) \right] \right|^2 f \, d\mu \\
&\leq \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \int \Big| \big(A_{n_{k_1+1}}(T_1) - A_{n_{k_1}}(T_1) \big) \dots \\
&\quad \Big(A_{n_{d}k+1}(T_d) - A_{n_{k_d}}(T_d) \Big) f \Big|^2 \, d\mu \\
&\leq 25^2 \sum_{k_1=1}^{\infty} \cdots \sum_{k_{d-1}=1}^{\infty} \int \Big| \big(A_{n_{k_1+1}}(T_1) - A_{n_{k_1}}(T_1) \big) \dots \\
&\quad \Big(A_{n_{k_{d-1}+1}}(T_{d-1}) - A_{n_{k_{d-1}}}(T_{d-1}) \Big) f \Big|^2 \, d\mu \\
&\leq \dots \leq 25^{2d} \|f\|_2^2 \quad \text{(by applying successively Theorem 3 on } T_d, \dots, T_1).
\end{aligned}$$

Then

$$\|Mf\|_2 \leq 25^d \|f\|_2.$$

To control each $M_{i_1,\dots,i_s} f$ we write

$$\begin{aligned}
\|M_{i_1} f\| &\leq \sum_{k=1}^{\infty} \int \Big| A_{n_{k+1}}(T_{i_1}) \prod_{j\neq i_1} \big(A_{n_{k+1}}(T_j) - A_{n_k}(T_j) \big) f \Big|^2 \, d\mu \\
&\leq \sum_{k=1}^{\infty} \int \Big| \prod_{j\neq i_1} \big(A_{n_{k+1}}(T_j) - A_{n_k}(T_j) \big) f \Big|^2 \, d\mu \quad \text{(since } \|T_1\|_2 \leq 1) \\
&\leq 25^{d-1} \|f\|_2.
\end{aligned}$$

By the same argument we obtain $\|M_{i_1,\dots,i_s} f\|_2 \leq 25^{d-s} \|f\|_2$.

Applying norm both sides of (1) and use the triangle inequality

$$\begin{aligned}\|S_d f\|_2 &\le \|Mf\|_2 + \sum_{i_1=1}^{d} \|M_{i_1} f\|_2 \\ &\quad + \sum_{1=i_1<i_2}^{\infty} \|M_{i_1,i_2} f\|_2 + \cdots + \sum_{1=i_1<\cdots<i_{d-1}}^{\infty} \|M_{i_1,\ldots,i_{d-1}} f\|_2 \\ &\le (25^d + C_d^1 25^{d-1} + \cdots + C_d^s 25^{d-s} + \cdots + C_d^{d-1} 25)\|f\|_2 \\ &= (26^d - 1)\|f\|_2.\end{aligned}$$

Remark that in the case $d = 2$ we have obtained the constant $675 = 26^2 - 1$.

b) Non-commuting case for power-bounded operators:

We now study the multidimensional averages over rectangles

$$A_{n_1,\ldots,n_d}(T_1,\ldots,T_d)f = A_{n_1}(T_1)\ldots A_{n_d}(T_d)f.$$

Let (n_{k_j}), $j = 1,\ldots,d$; be increasing sequences of integers. In this case, instead of using the strong estimate for square functions, we shall use the dominated ergodic theorem of Akcoglu for positive contraction 1975 [**5**, p. 186], (or positive power-bounded operators) in L^p.

Theorem 6. *Let $T_1,\ldots,T_d$ be linear power-bounded operators in L^q, i.e. $\sup_j \|T_k^j\| \le M_k$, $k = 1,\ldots,d$ with $1 < q < \infty$. Assume that $\sum_{k_j=1,j=1,\ldots,d}^{\infty}\big(1 - \prod_{i=1}^{j} \frac{n_{k_i-1}}{n_{k_i}}\big) < \infty$. Then the q-variation operator*

$$S_d f = \left(\sum_{k_1=2}^{\infty}\cdots\sum_{k_d=2}^{\infty} \left|A_{n_{k_1},\ldots,n_{k_d}} f - A_{n_{k_1-1},\ldots,n_{k_d-1}} f\right|^q\right)^{\frac{1}{q}}$$

is finite a.e. for all bounded f. In fact, $S_d\, f$ verifies a strong estimate in L^q

$$\|S_d f\|_q \le C_{q,d}\|f\|_q.$$

Proof. First we study the case $d = 2$. The general case can be done by a similar argument

$$\begin{aligned}&A_{n_{k_1},n_{k_2}} f - A_{n_{k_1-1},n_{k_2-1}} f \\ &= \frac{1}{n_{k_1}n_{k_2}} \sum_{i=0}^{n_{k_1}}\sum_{j=0}^{n_{k_2}} T_1^i T_2^i f - \frac{1}{n_{k_1-1}n_{k_2-1}} \sum_{i=0}^{n_{k_1}-1}\sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \\ &= \left(\frac{1}{n_{k_1}n_{k_2}} - \frac{1}{n_{k_1-1}n_{k_2-1}}\right) \sum_{i=0}^{n_{k_1}-1}\sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \\ &\quad - \frac{1}{n_{k_1}n_{k_2}} \left[\sum_{i=n_{k_1}-1}^{n_{k_1}}\sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f + \sum_{i=0}^{n_{k_1}}\sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f + \sum_{i=n_{k_1}-1}^{n_{k_1}}\sum_{j=0}^{n_{k_2}} T_1^i T_2^i f\right]\end{aligned}$$

Using the triangle inequality we see that

$$
\begin{aligned}
S_2 f \leq & \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right) \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=0}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
& + \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f \right|^q \right)^{\frac{1}{q}} \\
= & Af + Bf + Cf + Df.
\end{aligned}
$$

We first show that $\|Af\|_q \leq C_q \|f\|_q$. To see this we just write

$$
\begin{aligned}
\|Af\|_q^q \leq & \int \left(\sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right) \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q \right) d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(\frac{1}{n_{k_1} n_{k_2}} - \frac{1}{n_{k_1-1} n_{k_2-1}} \right)^q (n_{k_1-1} n_{k_2-1})^q \\
& \times \int \left| \frac{1}{n_{k_1-1} n_{k_2-1}} \sum_{i=0}^{n_{k_1}-1} \sum_{j=0}^{n_{k_2}-1} T_1^i T_2^i f \right|^q d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(1 - \frac{n_{k_1-1} n_{k_2-1}}{n_{k_1} n_{k_2}} \right)^q \int \left| A_{n_{k_1-1}, n_{k_2-1}} f \right|^q d\mu \\
\leq & M_1 M_2 \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(1 - \frac{n_{k_1-1} n_{k_2-1}}{n_{k_1} n_{k_2}} \right)^q \int |f|^q d\mu \\
= & C_q^q \|f\|_q^q
\end{aligned}
$$

For B, we write

$$
\begin{aligned}
\|Bf\|_q^q \leq & \int \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left| \frac{1}{n_{k_1} n_{k_2}} \sum_{i=n_{k_1}-1}^{n_{k_1}} \sum_{j=n_{k_2}-1}^{n_{k_2}} T_1^i T_2^i f \right|^q d\mu \\
\leq & \sum_{k_1=2}^{\infty} \sum_{k_2=2}^{\infty} \left(\frac{1}{n_{k_1} n_{k_2}} \right)^q \int \left| T_1^{n_{k_1}-1} T_2^{n_{k_2}-1} \sum_{i=0}^{n_{k_1}-n_{k_{1-1}}} \sum_{j=0}^{n_{k_2}-n_{k_{2-1}}} T_1^i T_2^i f \right|^q d\mu
\end{aligned}
$$

$$\begin{aligned}
&\leq \sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(\frac{1}{n_{k_1}n_{k_2}}\right)^q (n_{k_1}-n_{k_1-1})(n_{k_2}-n_{k_2-1}) \\
&\quad\times \int \left|T_1^{n_{k_1}-1}T_2^{n_{k_2}-1}A_{n_{k_1}-n_{k_1-1},n_{k_2}-n_{k_2-1}}f\right|^q d\mu \\
&\leq M_1M_2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(1-\frac{n_{k_1-1}n_{k_2-1}}{n_{k_1}n_{k_2}}\right)^q \int \left|A_{n_{k_1}-n_{k_1-1},n_{k_2}-n_{k_2-1}}f\right|^q d\mu \\
&\leq M_1^2M_2^2\sum_{k_1=2}^{\infty}\sum_{k_2=2}^{\infty}\left(1-\frac{n_{k_1-1}n_{k_2-1}}{n_{k_1}n_{k_2}}\right)^q \int |f|^q\, d\mu \\
&= C_q'^q\|f\|f_q^q.
\end{aligned}$$

By the same argument we can prove that $\|Cf\|_q \leq C_q''\|f\|_q$, and $\|Df\|_q \leq C_q'''\|f\|_q$. Finally we obtain $\|S_{q,d}f\|_q \leq C_{q,d}\|f\|_q$.

II. Square Maximal Functions

a) One dimensional case for power bounded operators

We now study for $1 < q \leq \infty$, the following maximal q-variation operator

$$S_q^*f = \left(\sum_{k=1}^{\infty}\sup_{n_{k-1}\leq n\leq n_k}\left|A_nf - A_{n_k}f\right|^q\right)^{\frac{1}{q}}.$$

In [**3**] it was shown the following result:

Theorem 7. *Let (n_k) denote an increasing sequence of integers. If $n_k = p(k)$ for some polynomial p of degree $s > 0$, then there is a constant C such that*

$$\|S_q^*f\|_2 \leq C\|f\|_2.$$

We shall extend Theorem 7: first to power bounded operator with a class of sequences (n_k) increasing and satisfy the following hypothesis: for $1 < q < \infty$ $\sum_{k=2}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q < \infty$.

We notice that the sequences of the form $n_k = p(k)$ for some polynomial satisfy this condition. Since if $p(k)$ is a polynomial of degree s then $n_k - n_{k-1} = p(k) - p(k-1)$ is a polynomial of degree $s-1$ and then the series has the same nature as

$$\sum_{k=2}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q = \sum_{k=2}^{\infty}\left(\frac{p(k)-p(k-1)}{p(k)}\right)^q = C\sum_{k=2}^{\infty}\frac{1}{k^q} < \infty.$$

Theorem 8. *Let T be a linear power-bounded operator on $1<q<\infty$. Assume that $\rho=\sum_{k=2}^{\infty}\left(1-\frac{n_{k-1}}{n_k}\right)^q<\infty$. Then for $f\in L^q(\Omega,R)$*

$$\|S_q^*f\|_q\le 8\sqrt{\rho}\|f\|_q.$$

Remark 1. In [**1**] M. Akcoglu, R. Jones and P. Schwartz proved that for $q<2$, there is a function $f\in L^\infty$ such that $S_qf=\left(\sum_{k=1}^{\infty}\left|A_{n_k}f-A_{n_{k-1}}f\right|^q\right)^{\frac{1}{q}}=\infty$. So that there is no strong estimate for S_qf and hence for S_q^*f $(S_qf\le S_q^*f)$.

In the proof of Theorem 1.5 we shall use the dominated ergodic theorem to obtain a strong estimate for mutidimensional square functions.

Proof. We write

$$\left|A_nf-A_{n_{k-1}}f\right|=\left|A_{n_{k-1}}f-A_nf\right|=\left|\left(\frac{1}{n_{k-1}}-\frac{1}{n}\right)\sum_{i=0}^{n_{k-1}}T^if-\frac{1}{n}\sum_{i=n_{k-1}+1}^{n}T^if\right|$$

then

$$\begin{aligned}
\sup_{n_{k-1}\le n\le n_k}|A_nf-A_{n_k}f|&\le\sup_{n_{k-1}\le n\le n_k}\left|\left(\frac{1}{n_{k-1}}-\frac{1}{n}\right)\sum_{i=0}^{n_{k-1}}T^if-\frac{1}{n}\sum_{i=n_{k-1}+1}^{n}T^if\right|\\
&\le\sup_{n_{k-1}\le n\le n_k}\left|\left(\frac{1}{n_{k-1}}-\frac{1}{n}\right)\sum_{i=0}^{n_{k-1}}T^if\right|+\sup_{n_{k-1}\le n\le n_k}\left|\frac{1}{n}\sum_{i=n_{k-1}+1}^{n}T^if\right|\\
&=\sup_{n_{k-1}\le n\le n_k}\left|\left(1-\frac{n_{k-1}}{n}\right)A_{n_{k-1}}f\right|\\
&\quad+\sup_{n_{k-1}\le n\le n_k}\left|\left(1-\frac{n_{k-1}}{n}\right)T^{n_{k-1}+1}A_{n-n_{k-1}}f\right|\\
&\le\sup_{n_{k-1}\le n\le n_k}\left|\left(1-\frac{n_{k-1}}{n}\right)\sup_{n_{k-1}\le n\le n_k}\left|A_{n_{k-1}}f\right|\right|\\
&\quad+\sup_{n_{k-1}\le n\le n_k}\left|\left(1-\frac{n_{k-1}}{n}\right)\right|\sup_{n_{k-1}\le n\le n_k}\left|T^{n_{k-1}+1}A_{n-n_{k-1}}f\right|\\
&\le\left(1-\frac{n_{k-1}}{n_k}\right)\left|A_{n_{k-1}}f\right|\\
&\quad+\left(1-\frac{n_{k-1}}{n_k}\right)\sup_{n_{k-1}\le n\le n_k}\left|T^{n_{k-1}+1}A_{n-n_{k-1}}f\right|.
\end{aligned}$$

Using the triangle inequality we see that

$$\begin{aligned}
\left|S_q^*f\right|&\le\left(\sum_{k=1}^{\infty}\left(\left(1-\frac{n_{k-1}}{n_k}\right)\left|A_{n_{k-1}}f\right|\right)^q\right)\\
&\quad+\left(\sum_{k=1}^{\infty}\left(\left(1-\frac{n_{k-1}}{n_k}\right)\sup_{n_{k-1}\le n\le n_k}\left|T^{n_{k-1}+1}A_{n-n_{k-1}}f\right|\right)^q\right)^{\frac{1}{q}}\\
&=Af+Bf
\end{aligned}$$

by integration we have

$$
\begin{aligned}
\|Af\|_q &\le \int \sum_{k=1}^{\infty} \Big(\Big(1 - \frac{n_{k-1}}{n_k}\Big) \left|A_{n_{k-1}} f\right| \Big)^q \, d\mu \\
&\le \sum_{k=1}^{\infty} \Big(1 - \frac{n_{k-1}}{n_k}\Big)^q \int \left|A_{n_{k-1}} f\right|^q \, d\mu \\
&\le M \sum_{k=1}^{\infty} \Big(1 - \frac{n_{k-1}}{n_k}\Big)^q \int |f|^q \, d\mu \\
&= C_q^q \|f\|_q^q \qquad (T \text{ is power-bounded in } L^q).
\end{aligned}
$$

For Bf we have a similar argument.

$$
\begin{aligned}
\|Bf\|_q &\le \int \sum_{k=1}^{\infty} \left((1 - \frac{n_{k-1}}{n_k}) \sup_{n_{k-1} \le n \le n_k} \left|T^{n_{k-1}+1} A_{n-n_{k-1}} f\right| \right)^q \, d\mu \\
&= \int \sum_{k=1}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q \left(\sup_{n_{k-1} \le n \le n_k} \left|T^{n_{k-1}+1} A_{n-n_{k-1}} f\right| \right)^q \, d\mu \\
&\le \int \sum_{k=1}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q \left(T^{n_{k-1}+1} \sup_{n_{k-1} \le n \le n_k} \left|A_{n-n_{k-1}} f\right| \right)^q \, d\mu \quad (T \ge 0) \\
&\le M \int \sum_{k=1}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q \left(\sup_{n_{k-1} \le n \le n_k} \left|A_{n-n_{k-1}} f\right| \right)^q \, d\mu \\
&\quad (\sup_j \|T\|_q \le M) \\
&\le M \int \sum_{k=1}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q \left(\sup_m |A_m f| \right)^q \, d\mu.
\end{aligned}
$$

(By Brunel' Theorem [**2**] or the dominated ergodic theorem for power-bounded operators on T)

$$
\begin{aligned}
&\le K k_q \sum_{k=1}^{\infty} \left(1 - \frac{n_{k-1}}{n_k}\right)^q \int |f|^q \, d\mu \\
&= C'_q \|f\|_q^q.
\end{aligned}
$$

To obtain Theorem 7 it suffices to take $q = 2$ and $Tf = fo\tau$ where τ is a measure preserving transformations on Ω.

b) Multidimensional case

We now study by a similar argument the multidimensional version of Theorem 7.

Let for (n_{k_j}), $j = 1, \dots, d$, be increasing sequences of integers. Let

$$S^*_{q,d}f =$$
$$\left(\sum_{k_1=1}^{\infty} \dots \sum_{k_d=1}^{\infty} \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \dots \sup_{n_{k_d} \le m_2 \le n_{k_d+1}} \left| A_{m_1,\dots,m_d} f - A_{n_{k_1},\dots,n_{k_d}} f \right|^q \right)^{\frac{1}{q}}.$$

We shall prove the following result:

Theorem 9. *Let $T_1, \dots, T_d$ be linear positive power-bounded operators in L^q, $\sup_j \|T_k^j\|_q \le M_k$, $k = 1, \dots, d$, $1 < q < \infty$. Assume that $\sum_{k=2,j=1,\dots,d}^{\infty} \big(1 - \prod_{i=0}^{j} \frac{n_{k_i-1}}{n_{k_i}}\big)^q < \infty$. Then the q-variation operator satisfies the strong estimate: for all $f \in L^q(\Omega, R)$*

$$\|S^*_{q,d} f\|_q \le C_{q,d} \|f\|_q.$$

Proof. It suffices to prove the case where $d = 2$: we can write as above

$$A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f = \frac{1}{m_1 m_2} \sum_{i=0}^{m_1} \sum_{j=0}^{m_2} T_1^i T_2^i f - \frac{1}{n_{k_1} n_{k_2}} \sum_{i=0}^{n_{k_1}} \sum_{j=0}^{n_{k_2}} T_1^i T_2^i f$$
$$= \left(\frac{1}{m_1 m_2} - \frac{1}{n_{k_1} n_{k_2}} \right) \sum_{i=n_{k_1}}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f$$
$$- \frac{1}{n_{k_1} n_{k_2}} \left[\sum_{i=n_{k_1}}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f + \sum_{i=0}^{m_1} \sum_{j=n_{k_2}}^{m_2} T_1^i T_2^i f + \sum_{i=n_{k_1}}^{m_1} \sum_{j=0}^{m_2} T_1^i T_2^i f \right]$$
$$= \left(1 - \frac{m_1 m_2}{n_{k_1} n_{k_2}} \right) \{ A_{n_{k_1},n_{k_2}} f + T_1^{n_{k_1}} T_2^{n_{k_2}} A_{m_1 - n_{k_1}, m_2 - n_{k_2}} f$$
$$+ T_2^{n_{k_2}} A_{m_1, m_2 - n_{k_2}} f + T_1^{n_{k_1}} A_{m_1 - n_{k_1}, m_2} f \}.$$

Applying sup on both sides we obtain

$$\sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right|$$
$$\le \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}} \right) \Big\{ \left| A_{n_{k_1},n_{k_2}} f \right|$$
$$+ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_1^{n_{k_1}} T_2^{n_{k_2}} A_{m_1 - n_{k_1}, m_2 - n_{k_2}} f \right|$$
$$+ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_2^{n_{k_2}} A_{m_1, m_2 - n_{k_2}} f \right|$$
$$+ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| T_1^{n_{k_1}} A_{m_1 - n_{k_1}, m_2} f \right| \Big\}.$$

Using the triangle inequality we see that

$$\begin{aligned}
Sf &\leq \left(\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q \left|A_{n_{k_1},n_{k_2}}f\right|^q\right)^{\frac{1}{q}} \\
&\quad + \Bigg[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q \\
&\qquad \times \left|T_1^{n_{k_1}}T_2^{n_{k_2}}\left(\sup_{n_{k_1}\leq m_1\leq n_{k_1+1}}\sup_{n_{k_2}\leq m_2\leq n_{k_2+1}}\left|A_{m_1-n_{k_1},m_2-n_{k_2}}f\right|\right)\right|^q\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q \\
&\qquad \times \left|T_2^{n_{k_2}}\left(\sup_{n_{k_1}\leq m_1\leq n_{k_1+1}}\sup_{n_{k_2}\leq m_2\leq n_{k_2+1}}\left|A_{m_1,m_2-n_{k_2}}f\right|\right)\right|^q\Bigg]^{\frac{1}{q}} \\
&\quad + \Bigg[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q \\
&\qquad \times \left|T_1^{n_{k_1}}\left(\sup_{n_{k_1}\leq m_1\leq n_{k_1+1}}\sup_{n_{k_2}\leq m_2\leq n_{k_2+1}}\left|A_{m_1-n_{k_1}++,m_2}f\right|\right)\right|^q\Bigg] \\
&= Af + Bf + Cf + Df.
\end{aligned}$$

We first show that $\|Af\|_q \leq C_q\|f\|_q$. We can write

$$\begin{aligned}
\|Af\|_q^q &\leq \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q\int\left|A_{n_{k_1},n_{k_2}}f\right|^q\,d\mu \\
&\leq M_1M_2\sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q\int|f|^q\,d\mu \\
&= C_{q,1}^q\|f\|_q^q.
\end{aligned}$$

For Bf, Cf, and Df we shall use the dominated ergodic theorem of Brunel [**2**]. Cesaro bounded operators (or the dominated ergodic theorem power-bounded positive operator) which was extended by Olsen to higher dimension.

We see that

$$\begin{aligned}
\|Bf\|_q^q \leq \sum_{k_1=1}^{\infty}\cdots\sum_{k_d=1}^{\infty}&\left(1-\frac{n_{k_1}n_{k_2}}{n_{k_1+1}n_{k_2+1}}\right)^q \\
&\times\int\left|T_1^{n_{k_1}}T_2^{n_{k_2}}\left(\sup_{m_1}\sup_{m_2}\left|A_{m_1,m_2}f\right|\right)\right|^q\,d\mu
\end{aligned}$$

$$\le M_1 M_2 \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q$$

$$\times \int \left| \left(\sup_{m_1} \sup_{m_2} |A_{m_1,m_2} f| \right) \right|^q d\mu$$

$$\le M_1 M_2 k_q^q \sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} \left(1 - \frac{n_{k_1} n_{k_2}}{n_{k_1+1} n_{k_2+1}}\right)^q \int |f|^q \, d\mu = C_{q,2}^q \|f\|_q^q.$$

By the same argument we can majorize Cf and Df in L^q. □

Remark 2. Since $S_d \le S_{q,d}^*$ then the result of Theorem 6 can be obtained from Theorem 9. But we have another multiple constant.

III. Weak Estimates

From Theorem 5 we can deduce the following result:

Theorem 10. *Let $T_1, \dots, T_d$ be linear positive power-bounded operators in L^q, $\sup_j \|T_k^j\|_q \le M_k$, $k = 1, \dots, d$, $1 < q < \infty$. Assume that $\sum_{k=2,j=1,\dots,d}^{\infty} \big(1 - \prod_{i=0}^{j} \frac{n_{k_i - 1}}{n_{k_i}}\big)^q < \infty$. Then the q-variation operator satisfies the strong estimate: for all $f \in L^q(\Omega, R)$*

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} m \left\{ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \cdots \sup_{n_{k_d} \le m_d \le n_{k_d+1}} \left| A_{m_1,\dots,m_d} f - A_{n_{k_1},\dots,n_{k_d}} f \right| > \lambda \right\}$$
$$\le \frac{C_q^q}{\lambda^q} \|f\|_q^q.$$

Proof. Study the case $d = 2$. The general case can be done similarly.

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} m \left\{ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right| > \lambda \right\}$$
$$= \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} m \left\{ \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right|^q > \lambda^q \right\}$$
$$\le \frac{1}{\lambda^q} \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \left\| \sup_{n_{k_1} \le m_1 \le n_{k_1+1}} \sup_{n_{k_2} \le m_2 \le n_{k_2+1}} \left| A_{m_1,m_2} f - A_{n_{k_1},n_{k_2}} f \right| \right\|_q^q$$
$$\le \frac{C_q^q}{\lambda^q} \|f\|_q^q \qquad \text{(by Theorem 5).}$$
□

Corollary 11. *Under the same hypothesis of Theorem* 10 *we have: for all* $f \in L^q(\Omega, R)$

$$\sum_{k_1=1}^{\infty} \cdots \sum_{k_d=1}^{\infty} m\left\{\left|A_{n_{k_1}+1,\ldots,n_{k_d}+1}f - A_{n_{k_1},\ldots,n_{k_d}}f\right| > \lambda\right\} \leq \frac{C_q^q}{\lambda^q}\|f\|_q^q.$$

From Theorem 2 and Theorem 6 we can deduce the following:

Corollary 12. *Let* $T_1, \ldots, T_d$ *be linear contracting commuting operators on* L^2. *For* $f \in L^q$ *we have the following weak type inequality*

$$\sum_{k=1}^{\infty} m\left\{|A_{n_k+1}(T_1, \ldots, T_d)f - A_{n_k}(T_1, \ldots, T_d)f| > \lambda\right\} \leq \frac{(26^d - 1)}{\lambda^2}\|f\|_2^2.$$

In [**4**] R. Jones proved that if $Tf = fo\theta$, then the square functions

$$Sf = \left(\sum |A_{n_k+1}f - A_{n_k+1}f|^2\right)^{\frac{1}{2}}$$

there is a weak estimate, $m\{Sf > l\} \leq \frac{C}{\lambda}\|f\|_1$ valid for some constant$C < \infty$ and $f \in L^1$. We shall prove that for a linear positive contraction T on L^1 such that

$$m\left\{\sup_n |T^n f| > \lambda\right\} \leq \frac{C}{\lambda}\|f\|_1$$

the result of Jones remains true.

Theorem 13. *Let* T *be a linear positive on* L^1*:*

(i) *If* T *is a self-adjoint positive contraction on* L^2, *then*

$$m\left\{\left(\sum_{k=1}^{\infty} |A_{n_k+1}(T)f - A_{n_k}(T)f|^2\right)^{\frac{1}{2}} > \lambda\right\} \leq \frac{124}{\lambda^2}\|f\|_2^2.$$

(ii) *If* T *is contraction on* L^1 *and on* L^∞ *and satisfies that*

$$(*) \qquad m\left\{\sup_n |T^n f| > \lambda\right\} \leq \frac{C'}{\lambda}\|f\|_1.$$

Then there is a constant $C < \infty$ *such that*

$$m\left\{\left(\sum_{k=1}^{\infty} |A_{n_k+1}(T)f - A_{n_k}(T)f|^2\right)^{\frac{1}{2}} > \lambda\right\} \leq \frac{C'}{\lambda}\|f\|_1.$$

Proof. In [**5**, pp. 190] Stein proved that if T is a self-adjoint positive operator in L^2 then $\| \sup_n T^n\, |f|\, \|_2 \leq 6\|f\|_2$. We shall use this estimate to prove the inequality (i). we can write

$$A_k(T)f - A_{k+1}(T)f = \left(\frac{1}{k} - \frac{1}{k+1}\right)\sum_{j=0}^{k} T^j f - \frac{1}{k+1}T^{k+1}f$$

so

$$|A_k(T)f - A_{k+1}(T)f| \leq \left(\frac{1}{k} - \frac{1}{k+1}\right)\left|\sum_{j=0}^{k} T^j f\right| + \frac{1}{k+1}\left|T^{k+1}f\right|$$

using the triangle inequality we see

$$\begin{aligned}
Sf &\leq \left[\sum_{k=1}^{\infty}\left(1 - \frac{k}{k+1}\right)^2 \left|\frac{1}{k}\sum_{j=0}^{k} T^j f\right|^2\right]^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\left|T^{k+1}f\right|^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\sup_n |A_n(T)f|^2\right)^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sup_n \left|T^{n+1}f\right|^2\right)\right)^{\frac{1}{2}} \\
&\leq \left[\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sup_n |A_n(T)f|\right)^2\right]^{\frac{1}{2}} + \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\sup_n \left|T^{n+1}f\right|\right)^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1}f\right|\right\} \\
&\leq \left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^{\frac{1}{2}}\left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1}f\right|\right\} \\
&\leq \sqrt{2}\left\{\sup_n |A_n(T)f| + \sup_n \left|T^{n+1}f\right|\right\}.
\end{aligned}$$

Using the dominated ergodic theorem of Akcoglu for positive contraction and that of Stein we have

$$\begin{aligned}
\|Sf\|_2 &\leq \sqrt{2}\left\{\|\sup_n\left|A_n(T)f\right|\|_2 + \|\sup_n\left|T^{n+1}f\right|\|_2\right\} \\
&\leq \sqrt{2}(2+6)\|f\|_2 = 8\sqrt{2}\|f\|_2.
\end{aligned}$$

Clearly,

$$m\{Sf > \lambda\} \leq \frac{1}{\lambda^2}\|f\|_2^2 \leq \frac{124}{\lambda^2}\|f\|_2^2.$$

For (ii), the set

$$\{Sf > \lambda\} \subseteq \left\{\sup_n\left|A_n(T)f\right| > \frac{\lambda}{2\sqrt{2}}\right\} \bigcup \left\{\sup_n\left|T^{n+1}f\right| > \frac{\lambda}{2\sqrt{2}}\right\}.$$

But by the Dunford-schwartz theorem we have

$$m\left\{\sup_n \left|A_n(T)f\right| > \frac{\lambda}{2\sqrt{2}}\right\} \leq \frac{2\sqrt{2}}{\lambda}\|f\|_1$$

and by the condition on T

$$m\{Sf > \lambda\} \leq \frac{2\sqrt{2}+C}{\lambda}\|f\|_1.$$

Remark 3. The condition $(*)$ in Theorem 13 can be replaced by an operator T for which $T^n f$ already converges a.e. at least for $f \in L^2$.

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