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ON THE MULTIPLICITY OF $(X^a - Y^b, X^c - Y^d)$

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ABSTRACT. In this paper an explicite formula for the computation of the multiplicity of ideal $(X^a - Y^b, X^c - Y^d)$ is given.

Let $K[X,Y]$ be a polynomial ring over a field K, $A = K[X,Y]_{(X,Y)}$ be a local ring with the maximal ideal $M = (X,Y) \cdot A$. For an M-primary ideal Q in A we denote by $P(n) := \ell(A/Q^{n+1})$ the Hilbert-Samuel function, where $\ell(A/Q^{n+1})$ is the length of A-module A/Q^{n+1}. The function $P(n)$ is for $n \gg 0$ a polynomial in n of degree 2 which can be written as $P(n) = e_0(Q)\frac{n^2}{2} + e_1(Q)n + e_2(Q)$. The coefficient $e_0(Q)$ is called the multiplicity of Q. It is well-known, that $e_0(Q)$ is a positive integer (for more details see [**3**]).

In this short note we give a formula for the computation of the multiplicity for certain class of M-primary ideals in A. It is a third article of the series beginning with [**1**], [**2**]. Our main result is the following theorem.

Theorem. *Let $Q = (X^a - Y^b, X^c - Y^d) \cdot A$ be a M-primary ideal in $A = K[X,Y]_{(X,Y)}$ (a, b, c, d are positive integers). Then*

$$e_0(Q) = \min\{ad, bc\}.$$

To prove the Theorem we need the following lemma.

Lemma. *Let $Q = (X^a - Y^b, X^c - Y^d) \cdot A$ be a M-primary ideal in the local ring $A = K[X,Y]_{(X,Y)}$ (a, b, c, d are positive integers).*

(a) *if $b \le a$ and $c \le d$ then $e_0(Q,A) = bc$.*
(b) *if $a \le b$ and $d \le c$ then $e_0(Q,A) = ad$.*

Proof. See [**4**, Lemma 3.1]. □

Proof of the Theorem. On the ground of Lemma the only case to prove is $b < a$ and $d < c$. Let $bc = \min\{ad, bc\}$, t.m. $\frac{b}{d} < \frac{a}{c}$. Note, that the conditions of Theorem imply $ad \ne bc$. Let $\left[\frac{b}{d}\right]$ indicates the integer part of $\frac{b}{d}$.

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Let now $k := \left[\frac{b}{d}\right] < \left[\frac{a}{c}\right]$. Then we have

$$k \le \frac{b}{d} < k+1 < \cdots < k+\rho \le \frac{a}{c} < k+\rho+1, \quad \rho \in N, \quad \rho > 0\,.$$

For $kc < a$ and $kd \le b$, we can write

$$\begin{aligned} Q &= \left(X^{kc} \cdot X^{a-kc} - Y^{kd} \cdot Y^{b-kd}, X^c - Y^d\right) \\ &= \left(X^{kc} \cdot X^{a-kc} - X^{kc} \cdot Y^{b-kd}, X^c - Y^d\right) \\ &\quad \text{because } X^{kc} \equiv Y^{kd} \pmod{Q} \\ &= \left(X^{kc} \cdot \left(X^{a-kc} - X^{b-kd}\right), X^c - Y^d\right) \end{aligned}$$

and therefore

$$\begin{aligned} e_0(Q) &= e_0\left(X^{kc}, X^c - Y^d\right) + e_0\left(X^c - Y^d, X^{a-kc} - Y^{b-kd}\right) \\ &= kcd + e_0\left(X^c - X^{a-kc} \cdot Y^{d-(b-kd)}, X^{a-kc} - Y^{b-kd}\right) \\ &= kcd + e_0\left(X^c, Y^{b-kd}\right) \end{aligned}$$

since $\left(X^c \cdot (1 - X^{a-kc-c} \cdot Y^{d-(b-kd)}), X^{a-kc} - Y^{b-kd}\right) = \left(X^c, X^{a-kc} - Y^{b-kd}\right) = \left(X^c, Y^{b-kd}\right)$ in A. So we have $e_0(Q) = kcd + c(b-kd) = bc$. This completes the proof if $\left[\frac{b}{d}\right] < \left[\frac{a}{c}\right]$.

Let now $k := \left[\frac{b}{d}\right] = \left[\frac{a}{c}\right]$. Then we have

$$k \le \frac{b}{d} < \frac{a}{c} < k+1$$

and from this follows

$$\begin{aligned} a &= kc + p, \quad 0 < p < c, \\ b &= kd + q, \quad 0 \le q < d. \end{aligned}$$

Then as above $e_0(Q) = e_0(X^{kc}(X^{a-kc} - Y^{b-kd}), X^c - Y^d) = e_0(X^{kc}, Y^d) = bc$ if $q = 0$. Let $q \ne 0$. Then $e_0(Q) = kcd + e_0(Q_1)$, where $Q_1 = (X^p - Y^q, X^c - Y^d)$. We denote the integer part of $\frac{c}{p}$ as k_1. From $k_1 = \left[\frac{c}{p}\right]$ follows $k_1 \le \frac{c}{p} < \frac{d}{q}$ so there exist p_1, q_1 such that

$$\begin{aligned} c &= k_1 p + p_1, \quad 0 \le p_1 < p, \\ d &= k_1 q + q_1, \quad 0 < q_1\,. \end{aligned}$$

If $(k_1 + 1) \cdot q \le d$, then

$$\begin{aligned} Q_1 &= \left(X^p - Y^q, X^c - Y^{(k_1+1)q} \cdot Y^{d-(k_1+1)q}\right) \\ &= \left(X^p - Y^q, X^c\left(1 - X^{(k_1+1)p-c} \cdot Y^{d-(k_1+1)q}\right)\right) \end{aligned}$$

while $X^{(k_1+1)p} \equiv Y^{(k_1+1)q} \pmod{Q}_1$

$$= (X^c, X^p - Y^q) \text{ in } A.$$

Then $e_0(Q) = kcd + cq = bc$.

Let now $(k_1 + 1)q > d$. Then we have

$$c = k_1 p + p_1, \quad 0 \le p_1 < p,$$
$$d = k_1 q + q_1, \quad 0 < q_1 < q.$$

Then

$$\begin{aligned} Q_1 &= \left(X^p - Y^q, X^{k_1 p + p_1} - Y^{k_1 q + q_1}\right) \\ &= \left(X^p - Y^q, X^{k_1 p} \cdot X^{p_1} - Y^{k_1 q} \cdot Y^{q_1}\right) \end{aligned}$$

because $X^{k_1 p} \equiv Y^{k_1 q} \pmod{Q_1}$

$$= \left(X^p - Y^q, X^{k_1 p}\left(X^{p_1} - Y^{q_1}\right)\right)$$

and hence $e_0(Q) = kcd + e_0(X^p - Y^q, X^{k_1 p}) + e_0(X^p - Y^q, X^{p_1} - Y^{q_1}) = kcd + k_1 pq + e_0(Q_2)$ with $Q_2 = (X^p - Y^q, X^{p_1} - Y^{q_1})$.

We continue our algorithm.

Let k_2 denotes the integer part of $\frac{q}{q_1}$, i.e. $k_2 q_1 \le q$, but $(k_2 + 1)q_1 > q$. Then there are integers p_2, q_2 such that

$$p = k_2 p_1 + p_2, \quad 0 < p_2$$
$$q = k_2 q_1 + q_2, \quad 0 \le q_2 < q_1.$$

If $(k_2 + 1)p_1 \le p$, then

$$\begin{aligned} Q_2 &= \left(X^{(k_2+1)p_1} \cdot X^{p-(k_2+1)p_1} - Y^q, X^{p_1} - Y^{q_1}\right) \\ &= \left(Y^{(k_2+1)q_1} \cdot X^{p-(k_2+1)p_1} - Y^q, X^{p_1} - Y^{q_1}\right) \\ &= \left(Y^q\left(Y^{(k_2+1)q_1 - q} \cdot X^{p-(k_2+1)p_1} - 1\right), X^{p_1} - Y^{q_1}\right) \\ &= \left(X^q, X^{p_1} - Y^{q_1}\right) \text{ in } A. \end{aligned}$$

Then $e_0(Q) = kcd + k_1 pq + qp_1 = bc$.

Let now $(k_2 + 1)p_1 > p$. Then we have $p_2 < p_1$,

$$\begin{aligned} Q_2 &= \left(X^{k_2 p_1} \cdot X^{p_2} - Y^{k_2 q_1} \cdot Y^{q_2}, X^{p_1} - Y^{q_1}\right) \\ &= \left(X^{k_2 p_1} \cdot \left(X^{p_2} - Y^{q_2}\right), X^{p_1} - Y^{q_1}\right) \end{aligned}$$

and within

$$e_0(Q) = kcd + k_1pq + k_2p_1q_1 + e_0(Q_3), \quad 0 \le p_2 < p_1.$$

with $Q_3 = (X^{p_2} - Y^{q_2}, X^{p_1} - Y^{q_1})$.

There are two descending chains of nonnegatives integers

$$\begin{aligned} &p > p_1 > p_2 > \dots \\ &q > q_1 > q_2 > \dots \end{aligned}$$

which have to stop after n steps. Note that $p_{2n} \neq 0$ and $q_{2n-1} \neq 0$ for all n. Let $q_{2n} = 0$ is the first zero and for all $k < 2n$ $q_k \neq 0$, $p_k \neq 0$. Then

$$\begin{aligned} e_0(Q) &= kcd + k_1pq + k_2p_1q_1 + k_3p_2q_2 + \dots + k_{2n} \cdot p_{2n-1} \cdot q_{2n-1} \\ &= kcd + q(c - p_1) + p_1(q - q_2) + q_2(p_1 - p_3) + \dots + p_{2n-1} \cdot (q_{2n-2} - q_{2n}) \\ &= kcd + qc = kcd + c(b - kd) = bc\,. \end{aligned}$$

Let consequently $p_{2n-1} = 0$ ($p_k \neq 0$, $q_k \neq 0$ for all $k < 2n - 1$).

Then it holds

$$\begin{aligned} e_0(Q) &= kcd + k_1pq + k_2p_1q_1 + k_3p_2q_2 + \dots + k_{2n-1} \cdot p_{2n-2}q_{2n-2}) \\ &= kcd + q(c - p_1) + p_1(q - q_2) + q_2(p_1 - p_3) + \dots + q_{2n-2}(p_{2n-3} - p_{2n-1}) \\ &= kcd + c(b - kd) = bc, \end{aligned}$$

which completes the proof for bc as a minimum of $\{bc, ad\}$. The proof for the second case ($ad = \min\{ad, bc\}$) is the same as the first one. $\square$

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