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KERNELS OF TOLERANCE RELATIONS

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Abstract. Algebras with 0 and their ideals in Gumm and Ursini's sense [**11**], [**12**] are considered. A variety $\mathcal{K}$ is called **0-tolerance regular** if each tolerance relation α of any $A \in \mathcal{K}$ is uniquely determined by its kernel $[0]_\alpha = \{x \in A\colon \langle 0, x\rangle \in \alpha\}$. The main result, strengthening Agliano and Ursini [**1**], asserts that every 0-tolerance regular variety is congruence permutable. Tolerance kernels of single algebras are also considered.

1. Introduction and Basic Definitions

Ideals of universal algebras were introduced by Ursini [**12**], cf. also Fichtner [**9**]. For definition, let $\mathcal{K}$ be a variety of algebras with a distinguished nullary operation 0 (or an equationally defined term 0) in its type. We say that $\mathcal{K}$ is a **variety with** 0; its members are called **algebras with** 0. In the sequel, $\mathcal{K}$ will always denote a variety with 0. Even without explicit mentioning all varieties and algebras in this paper are assumed to be with 0. A term $p(x_1, \dots, x_m, y_1, \dots, y_n)$ of $\mathcal{K}$ is called a $\mathcal{K}$**-ideal term** in the variables $y_1, \dots, y_n$ if $\mathcal{K}$ satisfies the identity $p(x_1, \dots, x_m, 0, \dots, 0) \approx 0$. A nonvoid subset I of an algebra $A \in \mathcal{K}$ is called a $\mathcal{K}$**-ideal** of A if for every $\mathcal{K}$-ideal term $p(x_1, \dots, x_m, y_1, \dots, y_n)$ in the last n variables and for all $a_1, \dots, a_m \in A$ and $b_1, \dots, b_n \in I$ we have $p(a_1, \dots, a_m, b_1, \dots, b_n) \in I$. When A does not belong to any specified variety, by a $\mathcal{K}$**-ideal term** resp. a $\mathcal{K}$**-ideal** we mean an $\mathbf{HSP}\{A\}$-ideal term resp. an $\mathbf{HSP}\{A\}$-ideal. $\mathbf{HSP}\{A\}$-ideal terms and $\mathbf{HSP}\{A\}$-ideals of A will also be called **ideal terms** and **ideals** even when A belongs to some variety $\mathcal{K}$. Note that, for $A \in \mathcal{K}$, every ideal of A is a $\mathcal{K}$-ideal of A. Notice that 0 belongs to every $\mathcal{K}$-ideal since the constant unary operation $c_0(y)$ with value 0 is a $\mathcal{K}$-ideal term in y. If $\mathcal{K}$ is the variety of all rings or lattices with zero, then $\mathcal{K}$-ideals are exactly the ideals in the usual sense.

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Following Agliano and Ursini [**1**], a nonempty subset C of an algebra A is called a **clot** if $0 \in C$ and for every term $q(x_1 \dots , x_n)$ with $q(0, \dots , 0) = 0$ and for all $c_1, \dots , c_n \in C$ we have $q(c_1, \dots , c_n) \in C$. For example, every ideal is a clot.

Given a compatible reflexive binary relation α of $A \in \mathcal{K}$ (i.e., a subalgebra of A^2 that includes the diagonal), the subset

$$[0]_\alpha = \{x \in A \colon \langle 0, x\rangle \in \alpha\}$$

is called the **kernel** of α. It is easy to see that $[0]_\alpha$ is an ideal of A. Kernels of congruences have been studied, e.g., in [**1**], [**6**], [**11**] and [**12**].

Recall that an algebra A is said to be **congruence permutable** if $\alpha \circ \beta = \beta \circ \alpha$ for all $\alpha, \beta \in \mathrm{Con}\,(A)$. As usual, a variety $\mathcal{K}$ is said to have a property if all of its members have this property. If a property of single algebras includes ideals, then (even without explicit mentioning) the corresponding property for $\mathcal{K}$ includes $\mathcal{K}$-ideals instead of ideals, of course. A classical theorem of A. I. Mal'cev asserts that a variety $\mathcal{K}$ is congruence permutable iff there is a **Mal'cev term** in $\mathcal{K}$, i.e. a ternary term p such that the identities $p(x, x, y) \approx y$ and $p(x, y, y) \approx x$ hold in $\mathcal{K}$. If $[0]_{\alpha\circ\beta} = [0]_{\beta\circ\alpha}$ holds for all $\alpha, \beta \in \mathrm{Con}\,(A)$, then A is called **(congruence) permutable at** 0. When $[0]_\alpha = [0]_\beta$ implies $\alpha = \beta$ for any $\alpha, \beta \in \mathrm{Con}\,(A)$, A is called 0-**regular**. If $\alpha \mapsto [0]_\alpha$ is a bijection from $\mathrm{Con}\,(A)$ to the set of ideals of A, then A is said to be **ideal determined**. A famous theorem of Gumm and Ursini [**11**] asserts that a variety $\mathcal{K}$ is ideal determined iff $\mathcal{K}$ is permutable at 0 and 0-regular.

Motivated by this theorem, other compatible reflexive relations have also been studied from similar aspects, cf. e.g. [**1**] and [**5**]. Compatible reflexive symmetric binary relations are called **tolerances**; cf. [**4**] for basic facts about them. The tolerances of A form an algebraic lattice, which is denoted by $\mathrm{Tol}\,(A)$. If $\mathrm{Tol}\,(A) = \mathrm{Con}\,(A)$, then A is said to be **tolerance trivial**. A is called a 0-**tolerance regular algebra** if, for all $\alpha, \beta \in \mathrm{Tol}\,(A)$, the equality $[0]_\alpha = [0]_\beta$ implies $\alpha = \beta$. When $\alpha \mapsto [0]_\alpha$ is a bijection from $\mathrm{Tol}\,(A)$ resp. from the set of compatible reflexive relations of A to the set of ideals resp. clots of A, then A is called an **ideal tolerance-determined** resp. **clot determined algebra**. Agliano and Ursini have proved that every clot determined variety is congruence permutable, cf. [**1**, Thm. 2.7]. Notice that 0-tolerance regularity is a much weaker condition than being clot determined; first because "regular" is weaker than "determined", and secondly because it is a condition only on tolerances rather than all compatible reflexive relations. Hence the following theorem, the main achievement of the paper, seems to be an essential improvement of the above-mentioned result.

2. Results and Proofs

Theorem 1. *If a variety with* 0 *is* 0*-tolerance regular, then it is congruence permutable.*

Proof. Let $\mathcal{K}$ be a 0-tolerance regular variety. For $A \in \mathcal{K}$ and $R \subseteq A^2$ the tolerance relation generated by R will be denoted by $T(R)$. As usual, we will write $T(a,b)$ instead of $T(\{\langle a,b\rangle\})$. Consider the free algebra $A = F_{\mathcal{K}}(x,y)$ with two free generators x and y. Set $\alpha = T(x,y)$, $I = [0]_\alpha$ and $\beta = T(\{0\} \times I)$. Observe that $\{0\} \times I \subseteq \alpha$ implies $\beta \subseteq \alpha$, so we obtain $I \subseteq [0]_\beta \subseteq [0]_\alpha = I$. The 0-tolerance regularity of $\mathcal{K}$ gives $\alpha = \beta$, whence $\langle x,y\rangle \in \beta$. Now we need the following easy description of generated tolerances:

$$\begin{aligned}
&\langle a,b\rangle \in T(\{\langle a_1,b_1\rangle,\dots,\langle a_n,b_n\rangle\}) \text{ iff there are } m \geq 0,\\
&\text{elements } e_1 \dots, e_m, \text{ and a } (2n+m)\text{-ary term } r\\
&\text{such that } a = r(a_1,\dots,a_n,b_1,\dots,b_n,e_1,\dots,e_m)\\
&\text{and } b = r(b_1,\dots,b_n,a_1,\dots,a_n,e_1,\dots,e_m).
\end{aligned} \tag{1}$$

Note that (1) is just Lemma 1.7 in [**4**]; the reader can also prove it directly. Since $\mathrm{Tol}\,(A)$ is an algebraic lattice, there is a finite subset $\{c_1(x,y),\dots,c_n(x,y)\}$ of I such that

$$\langle x,y\rangle \in T(\{0\} \times \{c_1(x,y),\dots,c_n(x,y)\}).$$

By (1) there is a $(2n+m)$-ary term r and there are binary terms e_i such that

$$\begin{aligned}
x &= r(0,\dots,0,c_1(x,y),\dots,c_n(x,y),e_1(x,y),\dots,e_m(x,y)),\\
y &= r(c_1(x,y),\dots,c_n(x,y),0,\dots,0,e_1(x,y),\dots,e_m(x,y)).
\end{aligned}$$

For simplicity, let us consider the term $g(x_1, x_2,\dots,x_{2n+2}) = r(x_1, x_2,\dots,x_{2n}, e_1(x_{2n+1}, x_{2n+2}),\dots,e_m(x_{2n+1}, x_{2n+2}))$. Then we have

$$\begin{aligned}
x &= g(0,\dots,0,c_1(x,y),\dots,c_n(x,y),x,y),\\
y &= g(c_1(x,y),\dots,c_n(x,y),0,\dots,0,x,y).
\end{aligned} \tag{2}$$

We claim that the terms c_i satisfy

$$c_i(x,x) \approx 0 \qquad \text{for } i = 1,\dots,n. \tag{3}$$

Indeed, $\langle 0, c_i(x,y)\rangle \in \alpha = T(x,y)$. Hence, for each i, the description (1) provides us with $u_j(x,y) \in A$ and a term s such that $0 = s(x,y,u_1(x,y),\ \dots,\ u_k(x,y))$ and $c_i(x,y) = s(y,x,u_1(x,y),\ \dots,\ u_k(x,y))$. Therefore, using the fact that equations for the free generators are valid identities in $\mathcal{K}$, we obtain

$$c_i(x,x) \approx s(x,x,u_1(x,x),\ \dots,\ u_k(x,x)) \approx 0,$$

showing (3). Now define

$$p(x,y,z) = g(c_1(y,z),\dots,c_n(y,z),c_1(x,y),\dots,c_n(x,y),x,z).$$

From (2) and (3) we infer

$$p(x,x,y) \approx g(c_1(x,y),\dots,c_n(x,y),0,\dots,0,x,y) \approx y$$

and

$$p(x,y,y) \approx g(0,\dots,0,c_1(x,y),\dots,c_n(x,y),x,y) \approx x,$$

i.e., p is a Mal'cev term. Thus $\mathcal{K}$ is congruence permutable. $\square$

Corollary 2. *The following four conditions are equivalent for a variety $\mathcal{K}$ with* 0*.*

(a) $\mathcal{K}$ *is* 0*-tolerance regular;*
(b) $\mathcal{K}$ *is ideal determined and congruence permutable;*
(c) $\mathcal{K}$ *is ideal determined and tolerance trivial;*
(d) $\mathcal{K}$ *is congruence permutable and* 0*-regular.*

Proof. Since 0-regularity is an evident consequence of 0-tolerance regularity, the implication (a) $\Longrightarrow$ (d) follows from Theorem 1. By [**2**] or [**4**, Thm. 4.11], tolerance triviality and congruence permutability for varieties are equivalent conditions. This gives (d) $\Longrightarrow$ (a) and (b) $\iff$ (c). Since permutability at 0 trivially follows from congruence permutability, the mentioned result from Gumm and Ursini [**11**] yields (b) $\iff$ (d). $\square$

Note that there are known Mal'cev characterizations of the equivalent conditions of Corollary 2; indeed, [**11**] resp. Agliano and Ursini [**1**, Thm. 2.7] gives an appropriate Mal'cev condition equivalent to (d) resp. (b). Notice also that the five element non-modular lattice is 0-tolerance regular but not ideal determined. (Here and in the sequel, the description of lattice tolerances by their blocks, cf. [**7**] or [**4**, Corollary to Thm. 2.16] or [**8**], makes the verification of some examples easier.) Hence much less can be stated about tolerance kernels in case of single algebras than in case of varieties.

In the sequel, $\tau(a)$ will stand for $T(a,0)$, the tolerance generated by $\langle a,0\rangle$. Given an algebra A with 0, if for all $a,b\in A$ there exists a $c\in A$ with $\tau(c)=\tau(a)\circ\tau(b)=\tau(a)\vee\tau(b)$ (in $\mathrm{Tol}\,(A)$), then A is called **strongly** 0**-tolerance principal**. For example, using the results of [**2**] and [**3**], it is not too hard to show that distributive lattices with 0 are strongly 0-tolerance principal. To present an example of a different nature, let $C=\{0,a,1\}$ be a three element chain, and define $L=(C\times C)\cup\{b\}$ where $\langle 1,a\rangle\prec b\prec\langle 1,1\rangle$. Then L is not strongly 0-tolerance principal, for $\tau(\langle 1,a\rangle)\vee\tau(\langle 1,a\rangle)\neq\tau(\langle 1,a\rangle)\circ\tau(\langle 1,a\rangle)$. Finally, we formulate

Proposition 3. *Let A be a strongly* 0*-tolerance principal algebra. Then the following two conditions are equivalent:*

(i) *every ideal of A is a congruence kernel;*
(ii) *every ideal of A is a tolerance kernel.*

Proof. For $S\subseteq A$ let $I(S)$ denote the ideal generated by S. As usual, we will write $I(s_1,\dots,s_n)$ instead of $I(\{s_1,\dots,s_n\})$. Let us consider the condition

(iii) $I(s_1,\dots,s_n)=[0]_{\tau(s_1)\circ\dots\circ\tau(s_n)}$ holds for all $n>0$ and all $s_1,\dots,s_n\in A$.

Before showing that (i), (ii) and (iii) are equivalent, two easy properties of A are worth formulating. Firstly,

$$(*)\qquad\qquad \text{for all } a\in A,\ \tau(a)\in\mathrm{Con}\,(A);$$

indeed, for an appropriate $c \in A$, $\tau(c) = \tau(a) \vee \tau(a) = \tau(a) \circ \tau(a)$ gives the transitivity of $\tau(a) = \tau(c)$. Secondly, a straightforward induction shows that

$$(**) \qquad \text{for all } a_1, \dots, a_n \in A \text{ there exists a } c \in A \text{ such that}$$
$$\tau(c) = \tau(a_1) \circ \tau(a_2) \circ \dots \circ \tau(a_n) = \tau(a_1) \vee \dots \vee \tau(a_n) \quad (\text{in } \mathrm{Tol}\,(A)).$$

The implication (i) $\Longrightarrow$ (ii) is trivial.

Suppose (ii) and let $s_1, \dots, s_n \in A$. Then $I(s_1, \dots, s_n) = [0]_\alpha$ for some $\alpha \in \mathrm{Tol}\,(A)$. ¿From $s_i \in [0]_\alpha$ we conclude $\alpha \geq \tau(s_1) \vee \dots \vee \tau(s_n)$ in Tol (A), whence $I(s_1, \dots, s_n) \supseteq [0]_{\tau(s_1) \vee \dots \vee \tau(s_n)} = [0]_{\tau(s_1) \circ \dots \circ \tau(s_n)}$. The converse inclusion follows from $s_i \in [0]_{\tau(s_i)} \subseteq [0]_{\tau(s_1) \circ \dots \circ \tau(s_n)}$. This proves (ii) $\Longrightarrow$ (iii).

Now suppose (iii). By $(*)$ and $(**)$, every finitely generated ideal is a congruence kernel. Let J be an arbitrary ideal of A, and let H denote the set ot all finite subsets of J. For $X \in H$ the ideal $I(X)$ is a congruence kernel, hence it is the kernel of the congruence Θ_X generated by $\{0\} \times I(X)$. Set $\Theta = \bigvee_{X \in H} \Theta_X$. Since J is the union of its finitely generated subideals, $J \subseteq [0]_\Theta$. Conversely, let $a \in [0]_\Theta$. Since the Θ_X $(X \in H)$ form a directed system, $\langle 0, a \rangle \in \Theta_X$ holds for some $X \in H$, and we obtain $a \in I(X) \subseteq J$. Hence $J = [0]_\Theta$, proving (iii) $\Longrightarrow$ (i). □

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