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MAXIMAL PENTAGONAL PACKINGS

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Abstract. For $n \geq 5$, a pentagonal packing of size t is a set of t edge-disjoint pentagons (cycles of length five) in the complete graph K_n. A pentagonal packing $\mathcal{P}$ is maximal, denoted as $MPP(n)$, if the complement of the union of all pentagons from $\mathcal{P}$ is pentagon-free. The spectrum $S^{(5)}(n)$ for maximal pentagonal packings is the set of all possible sizes of $MPP(n)$. We formulate a conjecture on the structure of the spectrum $S^{(5)}(n)$, and prove the conjecture for all $n = 40k + 3$, $k \geq 2$.

1. Introduction

Let K_n be a complete graph on n vertices. By a pentagonal packing $\mathcal{P}$, shortly PP or $PP(n)$ we understand a set of edge-disjoint pentagons (cycles of length five) in K_n. The size of $\mathcal{P}$ is the number of pentagons in $\mathcal{P}$. The leave $L(\mathcal{P})$ of $\mathcal{P}$ is the graph which is the complement of the union of pentagons of $\mathcal{P}$. A PP is maximal, shortly MPP, if its leave is pentagon-free. The spectrum for MPP is defined to be the set

$$S^{(5)}(n) = \{t : \text{there exists an } MPP \text{ of } K_n \text{ of size } t\}.$$

The extremes of $S^{(5)}(n)$ are denoted by $m^{(5)}(n)$ and $M^{(5)}(n)$, respectively:

$$m^{(5)}(n) = \min S^{(5)}(n),\ M^{(5)}(n) = \max S^{(5)}(n).$$

The values of $m^{(5)}(n)$ and $M^{(5)}(n)$ have been determined in [**3**]. In this paper we concentrate on studying the structure of $S^5(n)$. Clearly, $S^{(5)}(n)$ is a subset of the interval $[m^{(5)}(n), M^{(5)}(n)]$. We believe that the following conjecture is true:

Conjecture 1. *For any $n \geq 6$, there is a number z_n (for $n \geq 45$, $z_n - m^{(5)}(n) \geq n/5 - 5$), so that*

i) if $t \in [m^{(5)}(n), z_n]$, then $t \in S^{(5)}(n)$ iff t has the same parity as $m^{(5)}(n)$.

ii) all integers from the interval $[z_n, M^{(5)}(n)]$ belong to $S^{(5)}(n)$.

To support our conjecture we first show that i) is true for all $n \geq 45$. In order to obtain this result we will need to determine the maximum number of edges in

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a pentagon-free, non-bipartite graph whose all vertices are of even (odd) degree. Since we believe this result is of some interest on its own we also determine all extremal graphs. For $n = 40k + 3$, $k \geq 2$, we prove the conjecture in full, i.e. for these values of n we prove also the part ii). It seems to us that a complete proof of the conjecture (if true) for all $n \in N$ would require an excessive number of ad hoc constructions.

2. Pentagon-Free Non-Bipartite Graphs

The maximum possible size of a pentagon-free graph has been determined in [**3**]. The graphs where this maximum size is achieved are bipartite. We determine here the maximum possible size of non-bipartite pentagon-free eulerian (the degrees of all vertices are even) and antieulerian (the degrees of all vertices are odd) graphs and describe all maximal graphs of these types. We make use of the following bounds from [**2**] and [**3**].

Theorem 1 ([**2**]). *For $n \geq 7$ the maximum size of a graph without pentagons is $\lfloor n^2/4 \rfloor$.*

Theorem 2 ([**3**]). *For $n \geq 11$ the maximum size of a non-bipartite graph without pentagons is $\lfloor n^2/4 \rfloor - n + 4$.*

Let G be a graph. We denote by $V(G), E(G)$ the set of all vertices and the set of all edges of G, respectively, by $e(G)$ the number of edges in G, and, for $V' \subset V(G)$, by $\langle V' \rangle$ the subgraph of G induced by V'.

Let the function $g^{\mathcal{E}}$ be defined for positive integers and the function $g^{\mathcal{A}}$ for positive even integers as follows.

$$\begin{aligned} g^{\mathcal{E}}(n) &= (n^2 - 4n + 12)/4 \text{ if } n \equiv 0 \pmod 4 \\ &= (n^2 - 6n + 17)/4 \text{ if } n \equiv 1 \pmod 4 \\ &= (n^2 - 4n + 16)/4 \text{ if } n \equiv 2 \pmod 4 \\ &= (n^2 - 6n + 21)/4 \text{ if } n \equiv 3 \pmod 4 \end{aligned}$$

$$\begin{aligned} g^{\mathcal{A}}(n) &= (n^2 - 8n + 40)/4 \text{ if } n \equiv 0 \pmod 4 \\ &= (n^2 - 8n + 44)/4 \text{ if } n \equiv 2 \pmod 4 \end{aligned}$$

We define, for $n \geq 22$, two classes $\mathcal{G}_n^{\mathcal{E}}$ and $\mathcal{G}_n^{\mathcal{A}}$ of C_5-free non-bipartite graphs on n vertices. All graphs in $\mathcal{G}_n^{\mathcal{E}}$ are eulerian, all graphs in $\mathcal{G}_n^{\mathcal{A}}$ are antieulerian (and therefore defined just for even n).

Assume first that n is odd. Set $n_1 = n_2 = 3$ for $n \equiv 3 \pmod 4$, and $n_1 = 1$, $n_2 = 5$ for $n \equiv 1 \pmod 4$. The class $\mathcal{G}_n^{\mathcal{E}}$ contains two different types of graphs. A graph of type **A** consists of $n - 7$ vertices inducing a complete bipartite subgraph

$K_{(n-n_1-4)/2,(n-n_2-4)/2}$ and 7 vertices $v_1, v_2, \ldots, v_7$. There are two subtypes of this type. In a graph of subtype **A1** the subgraph induced by the 7 vertices is C_7. All vertices from one part of the bipartite subgraph are adjacent to v_2 and v_4, and some even number of vertices from the other part to v_1 and v_3 and the remaining ones from the same part to v_3 and v_5. In a graph of subtype **A2** vertices v_1, v_2, v_3 induce a triangle, v_4 and v_5 are adjacent to v_2 and to all vertices of one part of the bipartite subgraph, v_6 and v_7 are adjacent to v_3 and to all vertices of the other part. A graph of type **B** consists of a bipartite graph of size $n-2$ and two vertices of degree 2 inducing a triangle with one vertex of the bipartite graph. The parts are of size $(n-n_1+2)/2$ and $(n-n_2)/2$, or $(n-n_1)/2$ and $(n-n_2+2)/2$, the degree of each vertex is even, and each vertex from the part being of even size is adjacent to all vertices but one from the other part. (A graph of type **B** may contain in the bipartite subgraph one isolated vertex x, while the remaining vertices in that subgraph form a complete bipartite graph — in a particular case the graph consists of an isolated triangle and a complete bipartite graph. From a graph of this shape other graphs in $\mathcal{G}_n^{\mathcal{E}}$ can be obtained by repeatedly replacing a pair of edges vw', vw'', with v always taken from the part of the bipartite graph containing x, by xw', xw''.)

Let now n be even; set $n_1 = n_2 = 2$ for $n \equiv 2 \pmod 4$, and $n_1 = 0, n_2 = 4$ for $n \equiv 0 \pmod 4$. A graph belongs to $\mathcal{G}_n^{\mathcal{E}}$ if it consists of $n-2$ vertices inducing a complete bipartite subgraph $K_{(n-n_1)/2,(n-n_2)/2}$ and two vertices of degree 2 inducing, with one vertex of the bipartite graph, a triangle. All graphs in $\mathcal{G}_n^{\mathcal{A}}$ contain 4 vertices inducing a subgraph H isomorphic to K_4, while the subgraph induced by the remaining vertices is bipartite with the partition $(A, B), |A| \geq |B|$. One vertex of H is adjacent to all vertices of A, no other vertex in H has a neighbour off H. Each vertex of B is adjacent to $|A|-1$ vertices of A. Each vertex in A is adjacent to an even number of vertices of B. For $n \equiv 0 \pmod 4$, there are two types of graphs in $\mathcal{G}_n^{\mathcal{A}}$, one with $|A| = |B| = n/2-2$, the other with $|A| = n/2$, $|B| = n/2-4$. For $n \equiv 2 \pmod 4$ there is just one type with $|A| = n/2-1$, $|B| = n/2-3$. (A graph in $\mathcal{G}_n^{\mathcal{A}}$ may contain one fixed vertex x in A not adjacent to any vertex in B, while $\langle A \cup (B - \{x\}) \rangle$ is a complete bipartite graph. From a graph of this shape, other graphs in $\mathcal{G}_n^{\mathcal{A}}$ can be obtained by repeatedly replacing a pair of edges vw', vw'', with $v \in B - \{x\}$, by xw', xw''.)

Theorem 3. *The maximum number of edges in a C_5-free non-bipartite graph on $n \geq 22$ vertices is $g^{\mathcal{E}}(n)$, if G is eulerian, and is $g^{\mathcal{A}}(n)$, if G is antieulerian. The extremal graphs are exactly the graphs from the classes $G_n^{\mathcal{E}}$, $G_n^{\mathcal{A}}$, respectively.*

Proof. Let $G = (V, E)$ be a non-bipartite C_5-free graph on n vertices, $n \geq 22$, either eulerian or antieulerian. Clearly, if G is antieulerian then n is even. We will deal with eulerian graphs on even number of vertices separately in the very last part of the proof. For the time being we assume that if G is eulerian then n is odd. Let $g, f^{\mathcal{E}}, f^{\mathcal{A}}$ be defined by $g(n) = g^{\mathcal{E}}(n)$ if n is odd, $g(n) = g^{\mathcal{A}}(n)$ if n is even, $f^{\mathcal{E}} = (n^2 - 6n + 17)/4$, and $f^{\mathcal{A}}(n) = (n^2 - 8n + 40)/4$; hence $g(n) \geq f^{\mathcal{A}}(n)$, $g^{\mathcal{E}}(n) \geq f^{\mathcal{E}}(n)$.

Throughout the proof, we will frequently specify a subset K of the vertex set V, $|K| = v_K$, and a subset E_K of E, $|E_K| = e_K$, such that (i) $\langle K \rangle$ is an empty graph in $G - E_K$, and (ii) each vertex from $V - K$ is adjacent to at most m_K vertices in K, m_K being an absolute constant. For $n \geq v_K + 7$, the total number of edges in G can be then estimated by

$$e(G) \leq e_K + m_K(n - v_K) + (n - v_K)^2/4. \tag{1}$$

The first term on the right hand side of (1) is the number of edges in E_K, the second term provides an upper bound on the number of edges having one endvertex in $V - K$ and the other in K, and the third term gives, according to Theorem 1, the maximum number of edges of the C_5-free subgraph $\langle V - K \rangle$. Hence

$$\begin{aligned} g(n) - e(G) &> (f^{\mathcal{A}}(n) - 1/4) - (e_K + m_K(n - v_K) + (n - v_K)^2/4) \\ &= (v_K/2 - m_K - 2)n - ((v_K^2 - 39)/4 - m_K v_K + e_K). \end{aligned} \tag{2}$$

In the case the graph $\langle V - K \rangle$ is non-bipartite, we can apply Theorem 2 in the same way as Theorem 1 in (1), getting for $n \geq v_K + 11$

$$\begin{aligned} g(n) - e(G) &> (f^{\mathcal{A}}(n) - 1/4) \\ &\quad - (e_K + m_K(n - v_K) + (n - v_K)^2/4 - (n - v_K) + 4) \\ &= (v_K/2 - m_K - 1)n - ((v_K^2 - 23)/4 - (m_K - 1)v_K + e_K). \end{aligned} \tag{3}$$

For eulerian graphs, since $g^{\mathcal{E}}(n) > g(n)$, we can get the following finer estimates:

$$\begin{aligned} g^{\mathcal{E}}(n) - e(G) &> (f^{\mathcal{E}}(n) - 1/4) - (e_K + m_K(n - v_K) + (n - v_K)^2/4) \\ &= ((v_K - 3)/2 - m_K)n - (v_K^2/4 - m_K v_K + e_K - 4). \end{aligned} \tag{4}$$

and, if $\langle V - K \rangle$ is not bipartite,

$$\begin{aligned} g^{\mathcal{E}}(n) - e(G) &> (f^{\mathcal{E}}(n) - 1/4) \\ &\quad - (e_K + m_K(n - v_K) + (n - v_K)^2/4 - (n - v_K) + 4) \\ &= ((v_K - 1)/2 - m_K)n - (v_K^2/4 - (m_K - 1)v_K + e_K). \end{aligned} \tag{5}$$

It is easy to observe that if $G \in (G_n^{\mathcal{E}} \cup G_n^{\mathcal{A}})$ then $e(G) = g(n)$. We will prove now that if $G \notin (G_n^{\mathcal{E}} \cup G_n^{\mathcal{A}})$ then $e(G) < g(n)$.

Being non-bipartite, G contains an odd cycle. Denote the length of the shortest odd cycle in G by l. Because of minimality of l, any vertex off such a cycle can be adjacent to at most 2 vertices on the cycle. If $l \geq 9$, for K consisting of any 9 vertices of the cycle, $v_K = 9$, $e_K \leq 9$, $m_K = 2$ we get from (2) $g(n) - e(G) >$

$(2n-3)/2 > 0$ (since $n \geq 22$). Let $l = 7$ and C be a cycle of length 7 in G. If $\langle V - V(C)\rangle$ is non-bipartite, from (3) for $K = V(C)$, $v_K = 7$, $e_K = 7$, $m_K = 2$ we get $g(n) - e(G) > (n-13)/2 > 0$. Let $\langle V - V(C)\rangle$ be bipartite with the bipartition (A, B), where $|A| = a = (n-7)/2 + c$, $|B| = b = (n-7)/2 - c$. Let a be odd. If G is antieulerian, then b is even and the degree of each vertex of A is odd, i.e. less than $b+2$. If G is eulerian, then b is odd and the degree of each vertex of A is even, i.e. again less than $b+2$. We get $e(G) \leq 7 + 2b + a(b+1)$. A routine calculation shows $g(n) - e(G) = g(n) - ((7+a+b) + ab + b) = g(n) - (n + ((n-7)^2/4 - c^2) + ((n-7)/2 - c)) > (c + 1/2)^2 \geq 0$. Let a be even. We may assume that G is eulerian, otherwise we interchange A and B obtaining the previous case. Hence b is even; we get $e(G) \leq 7 + 2(n-7) + ab = (n^2 - 6n + 17)/4 + 1 - c^2$. Since c is odd for $n \equiv 1 \pmod 4$ and is even otherwise, we get $e(G) \leq g^{\mathcal{E}}(n)$. The equality takes place iff $c = 0$ or $c = 1$, $\langle V - V(C)\rangle$ is a complete bipartite graph, and each its vertex is adjacent to exactly 2 vertices of C, i.e. iff G is a graph of type **A1** from $G_n^{\mathcal{E}}$. In the rest of the proof we therefore assume $l = 3$, i.e. G contains a triangle.

Let x, y be two adjacent vertices of G such that the set $V_{x,y}$ of all vertices of G adjacent to both x and y, is of a maximum possible size $t \geq 1$. Let $M = V_{x,y} \cup \{x, y\}$. Since M is C_5-free, no vertex of $V - M$ is adjacent to more than one vertex in M, and for $t \geq 3$ no two vertices of $V_{x,y}$ can be adjacent. If $n - 8 \leq t$ $(\leq n-2)$, then $\langle V - M\rangle$ contains at most 6 vertices and 15 edges, hence $e(G) \leq (2t+1) + 6 + 15 \leq 2n + 18 < g(n)$ (for $n \geq 22$). If $5 \leq t \leq n - 9$, then (2) can be applied for $K = M$, $v_K = t + 2, e_K = 2t + 1, m_K = 1$; we get (since $n \geq t + 9$) $g(n) - e(G) > (t/2 - 2)(t + 9) - (t^2 + 8t - 39)/4 = ((t+1)^2 - 34)/4 > 0$. If $t = 4$, let z be one vertex from $V_{x,y}$ if G is antieulerian, otherwise let z be the vertex x. Then, because of degree parity, there is a vertex $n_z \in V - M$ adjacent to z. Let $K = M \cup \{n_z\}$. Any vertex from $V - K$ adjacent to two vertices in K must be adjacent to z and n_z; there are at most $t = 4$ such vertices. For $v_K = 7$, $e_K \leq 9 + 4$, $m_K = 1$, (2) implies $g(n) - e(G) > (n - 17)/2 > 0$.

For $t \leq 3$ we will prove the assertion of the theorem separately for antieulerian and eulerian graphs.

Let G be antieulerian. If $t = 3$, then, because of degree parity, there are two distinct vertices n_x, n_y in $V - M$ adjacent to x, y, respectively. Let $K = M \cup \{n_x, n_y\}$. If a vertex from $V - K$ is adjacent to two vertices of K, then these are either x and n_x or y and n_y. There are at most $t = 3$ vertices of each kind, therefore, after removing 6 edges not in $\langle K\rangle$, we have $m_K = 1$. For $v_K = 7$, $e_K \leq 9 + 6$, (2) implies $g(n) - e(G) > (n - 21)/2 \geq 0$. Postponing the case $t = 2$, let us assume $t = 1$. Let $V_{x,y} = \{z\}$. As G is antieulerian, each of x, y, z has at least one neighbour in $V - M$; let these distinct neighbours be n_x, n_y, n_z, respectively, and let $V_6 = \{x, y, z, n_x, n_y, n_z\}$. There are at most three vertices in $V - V_6$ adjacent to two vertices of V_6 (at most one to each of the pairs x, n_x;

y, n_y; z, n_z). If there is such a vertex u, choose $K = V_6 \cup \{u\}$. After removing at most 2 edges not in $\langle K \rangle$ we have $m_K = 1$. For $v_K = 7$, $e_K \leq 8 + 2$, (2) implies $g(n) - e(G) > (n - 11)/2 > 0$. If there is no such vertex u, choose $K = V_6$. For $v_K = 6$, $e_K = 6$, $m_K = 1$, (2) implies $g(n) - e(G) > 3/4$. As the last possibility, assume $t = 2$, i.e. $V_{x,y} = \{u, z\}$, where u, z are two distinct, possibly adjacent, vertices.

Assume first that a vertex $v \in M$ forms a triangle with two vertices $v_1, v_2 \in V - M$. Let w be the vertex v_1 if v is adjacent to all the remaining three vertices of M, otherwise let w be the vertex v. Then, because of degree parity, there is one additional vertex $w' \in V - M$ adjacent to w. Denote $K = M \cup \{v_1, v_2, w'\}$. Either w' is adjacent to two or three of the vertices v, v_1, v_2, or there may be at most $t = 2$ vertices in $V - M$ adjacent to both w and w', and at most one adjacent to two or three of the vertices v, v_1, v_2. Every other vertex in $V - K$ is adjacent to at most one vertex in K. For $v_K = 7, e_K \leq (6 + 3) + 4, m_K = 1$, (2) implies $g(n) - e(G) > (n - 7)/2 > 0$. Let there now be no triangle of G having just one vertex in M.

If $\langle V - M \rangle$ is non-bipartite, then it contains a shortest odd cycle C. If the length of C is at least 7, take for K the set M together with any 7 vertices of C. Each of the 7 vertices is a neighbour of at most one vertex in M. Each vertex in $V - K$ is a neighbour of at most one vertex in M and, because of minimality of C, of at most two vertices in C. For $v_K = 11, e_K \leq 6 + 7 + 7, m_K = 3$, (2) implies $g(n) - e(G) > (n - 15)/2 > 0$. If C is a triangle, then, since G is C_5-free and no triangle has just one vertex in M, there is at most one edge incident with both M and C. Since G is antieulerian, either there are at least two edges incident with both M and $V - M$, or no such edge exists. In the former case there is a vertex $v \in V - (M \cup V(C))$ adjacent to M. Choose $K = M \cup \{v\}$. $\langle V - K \rangle$ is non-bipartite since it contains C. (3) can be applied for $v_K = 5$, $e_K \leq 7$, $m_K = 1$ yielding $g(n) - e(G) > (n - 15)/2 > 0$. In the latter case $\langle M \rangle$ is K_4. Choose $K = M \cup V(C)$. At most one vertex of $V - K$ can be adjacent to two vertices of C (in that case it may be adjacent to all three of them). For $v_K = 7$, $e_K \leq 6 + 3 + 2$, $m_K = 1$, (2) implies $g(n) - e(G) > (n - 13)/2 > 0$.

If $\langle V - M \rangle$ is bipartite, let the bipartition be (A, B), $|A| = a = (n - 4)/2 + c$, $|B| = b = (n - 4)/2 - c$, hence $ab = f(n) - 6 - c^2$. Let $d = e(\langle M \rangle)$ (i.e. $d = 6$ if $\langle M \rangle$ is K_4, otherwise $d = 5$) and denote by A_1 (by B_1) the set of vertices in A (in B) adjacent to M. Let $a_1 = |A_1|$, $b_1 = |B_1|$, and assume $a_1 \geq b_1$. No vertex of $A_1 \cup B_1$ can be adjacent to two vertices of M. No two vertices of $A_1 \cup B_1$ are adjacent, otherwise G contains either C_5 or a triangle with just one vertex in M. We obtain

$$\begin{aligned} e(G) &\leq d + (a_1 + b_1) + (ab - a_1 b_1) = (d + 1) + ab - (a_1 - 1)(b_1 - 1) \\ &= f(n) + (d - 5) - c^2 - (a_1 - 1)(b_1 - 1). \end{aligned} \tag{6}$$

For $a_1 \geq b_1 \geq 3$, (6) implies $e(G) < g(n)$. Suppose $2 \geq b_1$.

Let us first assume that a and b are odd. We will show that $e(G) \leq f(n) - c^2$. Since c is odd for $n \equiv 0 \pmod 4$ and is even otherwise, the assertion implies $e(G) < g(n)$. If b_1 is even then no vertex in A_1 can be adjacent to all vertices of $B - B_1$; we get (since $a_1 \geq b_1$) $e(G) \leq d + (a_1 + b_1) + (ab - a_1 b_1) - a_1 = d + ab + b_1(1 - a_1) \leq d + ab \leq f(n) - c^2$. If b_1 is odd, i.e. $a \geq b_1 \geq 1$, let $v_b \in B_1$. If $d = 5$ or $c \geq 1$, then the assertion follows from (6). If $c = 0$ and $d = 6$ (i.e. $\langle M \rangle = K_4$), then, because of degree parity in M, there is a vertex $v_a \in A_1$ adjacent to the same vertex of M as v_b. Since $c = 0$, there exist vertices $v'_a \in A - A_1, v'_b \in B - B_1$. One of the edges $v_a v'_b$, $v'_b v'_a, v'_a v_b$ cannot be present in G, otherwise there is a C_5 in G. Therefore there is one less edge in G than given by (6) and $e(G) \leq f(n) - c^2$.

Let now a, b be even. Let $2 \geq b_1 \geq 1$. For degree-parity reason, no vertex of $B - B_1$ can be adjacent to all vertices of A, i.e. G contains by at least $b - b_1$ less edges than given by (6). Then $e(G) \leq f(n) + (d-5+b_1) - c^2 - b \leq (d-3) + f(n) - (c-1/2)^2 - (2n-9)/2 < g(n)$. Let $b_1 = 0$. No vertex of B can be adjacent to all vertices of A. In this case $e(G) \leq d + a_1 + (a-1)b \leq 6 + ab + a - b = (d-5) + f(n) - (c-1)^2$. If $n \equiv 0 \pmod 4$, then c is even. For $c \neq 0, 2$ we get $e(G) < g(n)$. If $c = 0$, then $a = b = n/2 - 2$. If $c = 2$, then $a = n/2$, $b = n/2 - 4$. In both cases the equality $e(G) = g(n)$ implies $d = 6$ (i.e. $\langle H \rangle = K_4$), $A_1 = A$ and all vertices of A are connected to the same vertex of H (otherwise C_5 would be present). If $n \equiv 2 \pmod 4$, then c is odd. For $c \neq 1$ we get $e(G) < g(n)$. If $c = 1$, then $a = n/2 - 1$, $b = n/2 - 3$. The equality $e(G) = g(n)$ implies $d = 6$, $A_1 = A$ and all vertices of A are connected to the same vertex of H. Therefore the equality takes place exactly for the graphs from $\mathcal{G}_n^{\mathcal{A}}$. The assertion of the theorem on antieulerian graphs is proved.

Assume that G is eulerian. If $t = 3$, let $V_{x,y} = \{u, v, z\}$. If $\langle M \rangle$ is isolated, then for $K = M$, $v_K = 5$, $e_K = 7$, $m_K = 0$ from (2) we get $g(n) - e(G) > (n-7)/2 > 0$. If there is a vertex $w' \in V - M$ adjacent to some vertex $w \in M$ (w' cannot be adjacent to more vertices of M), let $K = M \cup \{w'\}$. A vertex of $V - K$ adjacent to two vertices of K must be adjacent to w and w'. There are at most $t = 3$ such vertices. For $v_K = 6, e_K = 8 + 2, m_K = 1$ from (4) we get $g^{\mathcal{E}}(n) - e(G) > (n - 18)/2 > 0$.

If $t = 2$, let $V_{x,y} = \{u, z\}$ where u, z are two distinct, possibly adjacent, vertices. Then, because of the degree parity, there are two distinct vertices n_x, n_y of $V - M$ adjacent to x, y, respectively. Let $K = M \cup \{n_x, n_y\}$. If a vertex of $V - K$ is adjacent to two vertices of K, then these are either x and n_x or y and n_y. There are at most $t = 2$ vertices of each kind. For $v_K = 6$, $e_K = 8 + 4$, $m_K = 1$ from (4) we get $g^{\mathcal{E}}(n) - e(G) > (n - 22)/2 > 0$. Finally, let $t = 1$, i.e. $V_{x,y} = \{z\}$. If there are two vertices in M, say x and y, not adjacent to any vertex in $V - M$, choose $K = \{x, y\}$,. If $\langle V - K \rangle$ is nonbipartite, then, for $v_K = 2, e_K = 3, m_K = 0$, (5) gives $g'(n) - e(G) > (n - 12)/2 > 0$. If $\langle V - K \rangle$ is bipartite, with bipartition

(A, B), $|A| = a = (n-1)/2 + c$, $|B| = (n-3)/2 - c$, then $a + b = n - 2$ is odd. Assume a is odd. Then no vertex of B can be adjacent to all vertices of A. Therefore $e(G) \leq 3 + (a-1)b = (n^2 + 6n + 21)/4 - c^2$. If $n \equiv 1 \pmod 4$, then c is odd, otherwise c is even. We have $e(G) \leq g^{\mathcal{E}}(n)$, and the equality takes place for $c \in \{-1, 1\}$ in the former case, and for $c = 0$ in th latter case, if the bipartite graph is complete. This is exactly for graphs of type **B** from $\mathcal{G}_n^{\mathcal{E}}$. Suppose now that two vertices in M, say again x and y, are adjacent to $V - M$, i.e. there are two distinct (since $t = 1$) vertices $n_x, n_y \in V - M$ adjacent to x, y, respectively. Let $V_5 = M \cup \{n_x, n_y\}$. A vertex of $V - V_5$ adjacent to two vertices of V_5 is adjacent either to x and n_x, or to y and n_y. Let there be such a vertex v, adjacent, say to x and n_x. Let $K = V_5 \cup \{v\}$. There is at most one vertex of $V - K$ adjacent to two vertices of K (to y and n_y). For $v_K = 6, e_K \leq 7 + 1, m_K = 1$ from (4) we get $g^{\mathcal{E}}(n) - e(G) > (n-22)/2 \geq 0$. Let there be no such vertex v. Then there are two more vertices $m_x, m_y \in V - M$, adjacent to x, y, respectively, such that no two vertices from $\{m_x, n_x, m_y, n_y\}$ are adjacent. Let $K = M \cup \{m_x, n_x, m_y, n_y\}$. No vertex of $V - K$ can be adjacent to more than two vertices of K. If $\langle V - K \rangle$ is nonbipartite, for $v_K = 7$, $e_K = 7$, $m_K = 2$, (5) gives $g^{\mathcal{E}}(n) - e(G) > (4n - 49)/4 > 0$. Let $\langle V - K \rangle$ be bipartite with the bipartition (A, B) with $|A| = a = (n-7)/2 + c$, $|B| = (n-7)/2 - c$, $c \geq 0$. If a, b are odd, then no vertex of A can be adjacent to all vertices of B and we get $e(G) \leq 7 + 2(n-7) + a(b-1) = (n^2 - 6n + 16)/4 - (c + 1/2)^2 - (n-10)/2 < g^{\mathcal{E}}(n)$. If a, b are even, we get $e(G) \leq 7 + 2(n-7) + ab = (n^2 - 6n + 21)/4 - c^2$.

If $n \equiv 1 \pmod 4$ then c is odd, otherwise c is even. In both cases $e(G) \leq g^{\mathcal{E}}(n)$. The equality takes place for $c = 1$ in the former case, and for $c = 0$ in the latter case, if the bipartite subgraph is complete, and each its vertex is adjacent to 2 vertices of K, i.e. either to m_x and n_x, or to m_y and n_y. This is true exactly for graphs of type **A2** from $\mathcal{G}_n^{\mathcal{E}}$.

Now let us return to the possibility that G is eulerian and n is even. Then by adding one isolated vertex we get a C_5-free non-bipartite eulerian graph G' with an odd number of vertices. Therefore $e(G) = e(G') \leq g^{\mathcal{E}}(n+1) = g^{\mathcal{E}}(n)$. The equality takes place iff G' is extremal. The only extremal graphs in $\mathcal{G}_{n+1}^{\mathcal{E}}$ having one isolated vertex are of type **B**. By removing the vertex we get a graph from $\mathcal{G}_n^{\mathcal{E}}$. The theorem is proved. □

3. Main Results

First we show that the part i) of the conjecture is satisfied by all $n \geq 45$.

Theorem 4. *For all $n \geq 45$ there is a number z_n, $z_n - m^{(5)}(n) \geq n/5 - 5$, so that if $t \in [m^{(5)}(n), z_n]$, then $t \in S^{(5)}(n)$ iff t has the same parity as $m^{(5)}(n)$.*

Proof. Consider a $PP(n)$ $\mathcal{P}$. Then the degrees of all vertices of $L(\mathcal{P})$ have the same parity as $n - 1$. As the number of edges in a bipartite graph equals the sum of degrees in either of its two parts we get:

(*) If the leaves of some two $PP(n)$ $\mathcal{P}'$ and $\mathcal{P}''$ are bipartite then their sizes have the same parity.

Suppose now that a $MPP(n)$ $\mathcal{P}$ has a nonbipartite leave. From Theorem 4 the size of $\mathcal{P}$ is at least $z_n = \lceil (n(n-1)/2 - g^{\mathcal{E}}(n))/5 \rceil$. Thus, there is no $MPP(n)$ of size strictly less than z_n having the same parity as $m^{(5)}(n)$. A routine calculation shows that $z_n - m^{(5)}(n) \geq n/5 - 5$. To finish the proof we provide a construction of $MPP(n)$ of size $m^{(5)}(n) + 2i$, $i = 1, \ldots, \lfloor n/10 \rfloor$.

From [**3**] follows that for arbitrary $n \geq 11$ there exists an $MPP(n)$ such that $L(MPP(n))$ is a bipartite graph $G(X,Y)$, where $|X \cap Y| \leq 1$, $|X| \geq |Y|$, $|X| - |Y| \leq 6$ and we can get $G(X,Y)$ from the complete bipartite graph $K_{|X|,|Y|}$ by removing edges which are incident with at most 4 vertices in Y.

Take a set P of $\lfloor n/10 \rfloor$ pentagons of MPP with all vertices in X, a set V of $3\lfloor n/10 \rfloor$ distinct vertices of Y other than the 4 vertices mentioned above. Suppose $C = x_1x_2x_3x_4x_5x_1$ is a pentagon from P and a, b, c are three distinct vertices of V. Then it is possible to replace C by three new pentagons $x_1x_2x_3x_4ax_1$, $x_4bx_2cx_5x_4$, $x_5bx_3cx_5$. After such a replacement the PP remains maximal. As we can carry out the above construction for arbitrary number of pentagons in P, it is possible to increase the initial total number of pentagons in the MPP, initially equal to $m^{(5)}(n)$, by any number $2i, i = 1, \ldots, \lfloor n/10 \rfloor$. □

Finally, we prove that the conjecture is valid for all $n = 40k + 3$, $k \geq 2$.

Theorem 5. *For any $n = 40k+3, k \geq 2$, the structure of $S^{(5)}(n)$ is as in the conjecture with $z_n = \lceil (n(n-1)/2 - g^{\mathcal{E}}(n))/5 \rceil = m^{(5)}(n) + 8k - 1 < m^{(5)}(n) + n/5$.*

Proof. One can easily observe that the assertion of the Theorem can be obtained by combining the assertions of the following Lemmas 1–5. □

Throughout the paragraph we consider $n = 40k + 3$, $k \geq 1$, and employ the following notation. $T = \{t_1, t_2, t_3\}, X' = \{x_1, \ldots, x_{10k}\}$, $X'' = \{x_{10k+1}, \ldots, x_{20k}\}$, $X = X' \cup X''$, $Y = \{y_1, \ldots, y_{20k}\}$ will be sets of vertices. If we consider, for an even t, $K_t^* = K_t - F$ on a set of vertices with indices $i = 1, \ldots, t$ then the 1-factor F comprises edges with endvertices of indices $2j-1$ and $2j$, $j = 1, \ldots, t/2$.

Lemma 1. *There is a resolvable decomposition $\mathcal{F}_V$ of K_{10m}^* on $V = \{v_1, \ldots, v_{10m}\}$ into $10m^2 - 2m$ pentagons which contains a 2-factor $\mathcal{Z}_V$ made up of cycles*

$$v_{10i+1}v_{10i+3}v_{10i+5}v_{10i+7}v_{10i+9}v_{10i+1} \text{ and } v_{10i+2}v_{10i+4}v_{10i+6}v_{10i+8}v_{10i+10}v_{10i+2},$$

$i = 0, \ldots, m-1$.

In addition, the pentagon $v_1v_8v_9v_3v_7$ belongs to $\mathcal{F}_V$.

Proof. In view of [**1**] there exists a resolvable decomposition of K_{10m}^*, $m \neq 2$, into pentagons such that four 2-factors comprise a resolvable decomposition of

$m \cdot K_{10}^*$, the other 2-factors comprise a resolvable decomposition of the m-partite graph $K_{10,10,\dots,10}$. So to prove our lemma it suffices to find a resolvable decomposition of K_{10}^* with the given properties. One such a decomposition (described for the vertex set $\{1,\dots,10\}$) consists of $\mathcal{Z}_{\{1,\dots,10\}}$: (1 3 5 7 9)&(2 4 6 8 10), and of the 2-factors (1 8 9 3 7)&(2 7 10 4 8), (1 6 7 4 5)&(2 5 8 3 6), (1 4 9 5 10)&(2 3 10 6 9).□

Now we introduce two $PP(20k+3)$, P_S and P_S^*. Let $V = \{v_1,\dots,v_{20k}\}$, $k > 1$, $S = V \cup T$. Let $\mathcal{F}_V$ and $\mathcal{Z}_V$ be as in Lemma 1. Then $P_V^* = \mathcal{F}_V \cup \{v_{2i-1}v_{2i}t_2v_{2i+10k}t_1v_{2i-1};\ i = 1,\dots,5k\} \cup \{v_{2i-1}v_{2i}t_3v_{2i-10k}t_1v_{2i-1};\ i = 5k+1,\dots,10k\}$, $P_V = (P_V^* - \{v_{10i+1}v_{10i+3}v_{10i+5}v_{10i+7}v_{10i+9}v_{10i+1};\ i = 0,1,\dots,k-1\}) \cup (\{v_{10i+j}t_2v_{10i+j+10k}t_3v_{10i+((j+2) \bmod 5)}v_{10i+j};\ i = 0,1,\dots,k-1;\ j = 1,3,5,7,9\}$.

Note that $L(P_V^*) = \{t_jv_{2i-1}; j = 2,3; i = 1,\dots 10k\} \cup \{t_1t_2, t_1t_3, t_2t_3\}$, i.e. $L(P_S^*)$ is C_5-free, and $|P_V^*| = 40k^2 + 6k$, while $L(P_V) = \{t_1t_2, t_1t_3, t_2t_3\}$ and $|P_V| = 40k^2 + 10k$.

Lemma 2. *Odd numbers in the interval* $\langle 80k^2 + 12k + 1, 80k^2 + 20k + 1\rangle$ *belong to* $S^{(5)}(n)$, $n = 40k+3$, $k > 2$.

Proof. Follows directly from Theorem 4 and the fact (see [**3**] that $m_5(n) = 80k^2 + 12k + 1$ for $n = 40k+3$. □

Lemma 3. *For any even number* b *in the interval* $\langle 52k^2, 100k^2\rangle$, *there is a pentagonal packing* R^* *of* $H_k = (X' \vee X'') \vee Y$ *with* b *pentagons, such that* $L(R^*)$ *is a subgraph of* $X \vee Y$ *(i.e.* $L(R^*)$ *is* C_5*-free and all edges of* $X' \vee X''$ *are covered by pentagons of* R^**) and*

a) for $52k^2 + 100k \le b \le 100k^2$ *the vertices* $y_1,\dots,y_{20}$ *are isolated vertices in* $L(R^*)$

b) for $52k^2 \le b \le 52k^2 + 100k$ *there are vertices* $x \in X$, $y', y'' \in Y$ *such that the path* $y'xy'' \in L(R^*)$ *and* y', y'' *have odd indices in* Y.

Proof. We prove the statement by induction with respect to k.

Case $k = 1$:

For the sake of convenience we use, only in this part of the proof, a different notation for vertices of X and Y. Namely, we partition X, Y into subsets $X_i = \{x_j^i, j = 0,\dots,4\}$, $Y_i = \{y_j^i, j = 0,\dots,4\}, i = 1,2,3,4$, and $X_1 \cup X_2 = X'$, $X_3 \cup X_4 = X''$. Our graph H_1 has 500 edges. At the beginning we decompose H_1 into 100 pentagons. For each of the edges $e = x'x''$ of $X' \vee X''$ we form a x'–x'' path in $X \vee Y$ of length four so that all 100 paths will be mutually edge disjoint. For $1 \le i,j \le 5$,

to the edge	$x_i^1x_j^3$	we assign the path	$x_i^1y_j^1x_i^2y_{i+j}^3x_j^3$,
to the edge	$x_i^2x_j^3$	we assign the path	$x_i^2y_j^2x_i^4y_{i+j}^4x_j^3$,
to the edge	$x_i^1x_j^4$	we assign the path	$x_i^1y_j^2x_i^3y_{i+j}^1x_j^4$,
to the edge	$x_i^2x_j^4$	we assign the path	$x_i^2y_j^4x_i^1y_{i+j}^3x_j^4$,

the subscripts are taken $\pmod 5$. Thus we get a decomposition of H_1 into pentagons.

Now, starting from the above decomposition R^*, we will gradually decrease the number of pentagons in R^* by two, by the following process. Take 3 edges of $X' \vee X''$ which form a path, say $x'x''\overline{x}'\overline{x}''$, and omit the pentagons $C_{x'x''}, C_{x''\overline{x}'}, C_{\overline{x}'\overline{x}''}$ of R^* covering the edges. The choice of edges is made so that there is $y \in Y$ with edges $x'y \in C_{x'x''}, y\overline{x}'' \in C_{\overline{x}'\overline{x}''}$. By adding y to the path, we form the pentagon $x'x''\overline{x}'\overline{x}''yx'$ and the other edges of $C_{x'x''} \cup C_{x''\overline{x}'} \cup C_{\overline{x}'\overline{x}''}$ will belong to $L(R^*)$.

$$\begin{array}{llll}
\text{To the path} & x_i^1 x_{i+1}^3 x_{i+2}^1 x_4^4 & \text{we add the vertex} & y_{i+1}^1, \\
\text{to the path} & x_i^1 x_{i+2}^3 x_{i+4}^1 x_3^4 & \text{we add the vertex} & y_{i+2}^1, \\
\text{to the path} & x_i^2 x_{i+1}^4 x_{i+2}^2 x_4^3 & \text{we add the vertex} & y_{i+1}^4, \\
\text{to the path} & x_i^2 x_{i+2}^4 x_{i+4}^2 x_3^3 & \text{we add the vertex} & y_{i+2}^4,
\end{array}$$

where $i = 1, 2, \ldots, 5$, the subscripts taken $\pmod 5$. We construct four more pentagons in a similar way, however, this time the edge $x'y$, or yx'' may originate from some of the 60 previously omitted pentagons, e.g. the edge $y_0^1 x_1^1$ will be taken from the pentagon originally covering the edge $x_{i+1}^3 x_{i+2}^1$, $i = 5$.

$$\begin{array}{llll}
\text{To the path} & x_0^4 x_0^1 x_1^4 x_1^1 & \text{we add the vertex} & y_0^1, \\
\text{to the path} & x_0^4 x_2^1 x_2^4 x_3^1 & \text{we add the vertex} & y_2^1, \\
\text{to the path} & x_0^3 x_0^2 x_1^3 x_1^2 & \text{we add the vertex} & y_0^4, \\
\text{to the path} & x_0^3 x_2^2 x_2^3 x_3^2 & \text{we add the vertex} & y_2^4.
\end{array}$$

Note that all new 24 pentagons are mutually edge disjoint, so we are able to replace gradually $3t$ pentagons, $t = 1, \ldots, 24$ by t pentagons.

Case $k > 1$:
Let $X_i = \{x_{10i+j}, j = 1, \ldots, 10\}, i = 0, \ldots, 2k-1$, $Y_i = \{y_{20i+j}, j = 1, \ldots, 20\}, i = 0, \ldots, k-1$. Thus $\bigcup_{i=0}^{k-1} X_i = X'$, $\bigcup_{i=k}^{2k-1} X_i = X''$, $\bigcup_{i=0}^{k-1} Y_i = Y$. Partition the edges of H_k into k^2 induced subgraphs isomorphic to H_1. For example, such a partition is given by the sets $X_i \cup X_j \cup Y_{i+j-1 \pmod k}$, $i = 1, \ldots, k$, $j = k+1, \ldots, 2k$. In each particular subgraph we can construct from 52 up to 100 pentagons, therefore in H_k we are able to form from $52k^2$ to $100k^2$ pentagons, and the leave is a subgraph of $X \vee Y$. If for $b \geq 52k^2 + 100k$ we take 100 pentagons in each subgraph generated by a set of vertices containing Y_0, then clearly R^* has property a). Property b) is straightforward. $\square$

Lemma 4. *Odd numbers in the interval* $\langle 80k^2 + 16k + 1, 112k^2 + 16k + 1\rangle$ *and even numbers in the interval* $\langle 80k^2 + 20k, 112k^2 + 20k\rangle$ *belong to* $S^{(5)}(n), n = 40k + 3, k > 1$.

Proof. First we form (for $k > 1$) two $MPP(40k + 3)$ A and B, of cardinalities $80k^2 + 16k + 1$ and $80k^2 + 20k$, respectively. Put $A = P_Y^* \cup \{t_2t_3y_1x_1y_3t_2\} \cup$

$P_X, B = P_X \cup P_Y$. Clearly, A and B have the required cardinalities, $L(B) = \{t_1t_2, t_2t_3, t_1t_3\} \cup X \vee Y$. $L(A)$ is a subgraph of a bipartite graph $(X \cup \{t_2, t_3\}) \vee (Y \cup \{t_1\})$, i.e. $L(A)$ is C_5-free.

We proceed in both cases the same way. We choose $8k^2+1$ pentagons of A (B) and show how to replace independently any $8k^2$ of them by 5 pentagons and the remaining one by 3 pentagons, which will finish the proof.

Take arbitrary $2k$ of the 2-factors of $\mathcal{F}_X$ (we recall that $\mathcal{F}_X$ is a part of P_X and each factor in $\mathcal{F}_X$ consists of $4k$ pentagons) which differ from the 2-factor Z_X and such that the edge x_2x_8 does not belong to U', the union of the chosen 2-factors. Consider one more pentagon $C = x_2x_4x_6x_8x_{10}x_2$. C is the pentagon which is to be replaced by 3 pentagons. One of the three pentagons is the pentagon $C' = x_2x_4x_6x_8y_8x_2$. Set $U = U' \cup \{x_2x_{10}, x_8x_{10}\}$. To each edge $e = x_ix_j$ of U we form an x_i–x_j path of length 4 in $X \vee Y$ such that all $40k^2+2$ paths will be mutually edge disjoint with the path $x_8y_8x_2$. This way we obtain the other new pentagons.

Let the x_i-x_j path be $x_iy_jx_ey_ix_j$. Call edges of the path incident with the vertex x_e inner edges, the other edges will be called outer edges of the path. Clearly, all $80k^2+4$ outer edges are distinct. In order to guarantee that all $80k^2+4$ inner edges are distinct, and that the sets of inner and outer edges are disjoint, we have to choose the vertex x_e such that a) if e and e' are adjacent edges of U then $x_e \neq x_{e'}$, b) $x_e \notin N_U(x_i) \cup N_U(x_j)$, where $N_U(x)$ is the neighbourhood of x in U. Associate with each $e = x_ix_j \in U$ a set $L_e = X - (N_U(x_i) \cup N_U(x_j))$. As $\Delta(U) = 4k+2$, we get $|L_e| \geq 20k - (8k+4) = 12k-4$. To assign to each vertex $e \in U$ a vertex x_e satisfying a) and b), we have to find a regular edge coloring of Uassigning to e a color $x_e \in L_e$. Since, for $k > 1$, $|L_e| \geq 2\Delta(U) - 1$, such a coloring can be found by applying a straightforward greedy algorithm. ☐

Lemma 5. *Odd numbers in the interval* $\langle 112k^2+16k+1, 160k^2+20k-1\rangle$ *and even numbers in the interval* $\langle 112k^2+20k, 160k^2+20k\rangle$ *belong to* $S^{(5)}(n)$, $n = 40k+3$, $k > 2$.

Proof. Define a $PP(20k+3)$ S on $T \cup X$ by

$$\begin{aligned} S = (&\mathcal{F}_{X'} \cup \mathcal{F}_{X''} - \{x_{10i+1}x_{10i+3}x_{10i+5}x_{10i+7}x_{10i+9}x_{10i+1}; i = 0, 1, \dots, k-1\}) \\ &\cup \{x_{2i-1}x_{2i}t_2x_{2i+10k}t_1x_{2i-1}; i = 1, \dots, 5k\} \\ &\cup \{x_{2i-1}x_{2i}t_3x_{2i-10k}t_1x_{2i-1}; i = 5k+1, \dots, 10k\} \\ &\cup (\{x_{10i+j}t_2x_{10i+j+10k}t_3x_{10i+((j+2) \bmod 10)}x_{10i+j}; j = 1, 3, 5, 7, 9\}. \end{aligned}$$

Then $L(S) = \{t_1t_2, t_1t_3, t_2t_3\} \cup (X' \vee X'')$, and $|S| = 20k^2 + 10k$.

To get the part of the statement for even numbers, it suffices to take $MPP(n)$ Q of the form $Q = S \cup P_Y \cup R^*$ where R^* is as in Lemma 3, because $|Q| = 20k^2 + 10k + 40k^2 + 10k + |R^*|$ and $|R^*|$ ranges over all even numbers of the interval $\langle 52k^2, 100k^2\rangle$.

To show that the odd numbers $b \in \langle 112k^2 + 16k + 1, 160k^2 + 16k + 1\rangle$ belong to $S^{(5)}(40k+3)$ we take $MPP(40k+3)$ $Q = S \cup P_Y^* \cup R^* \cup \{t_2t_3y'xy''t_2\}$, where $52k^2 \le |R^*| \le 52k^2 + 100k$ and y', x, y'' are as in Lemma 3 (note that $L(Q)$ is a subgraph of a bipartite graph $(X \cup \{t_2, t_3\}) \vee (Y \cup \{t_1\})$.

To get a $MPP(40k+3)$ Q' with $112k^2 + 116k + 3$ pentagons we omit from Q the cycles $y_1y_3y_5y_7y_9y_1$; $y_1y_8y_9y_3y_7y_1$ and $t_2t_3y'xy''t_2$ and add five cycles: $y_jt_2y_{10k+j}t_3y_{j+2}y_j$, $j = 1, 3, 5, 7$, and $y_9t_2t_3y_1y_8y_9$. The leave $L(Q')$ contains the quadrangle $y_1y_7y_3y_9y_1$ but is again a bipartite graph in view of a) of Lemma 3. In order to form an $MPP(40k+3)$ with b pentagons, b is odd, $b \in \langle 112k^2 + 16k + 3, 160k^2 + 20k - 1\rangle$, we replace a suitable number of $k-1$ cycles

$$y_{10i+1}y_{10i+3}y_{10i+5}y_{10i+7}y_{10i+9}y_{10i+1}, \qquad i = 1, \dots, k-1$$

by 5 new cycles (every edge y_jy_{j+2} will be contained in the pentagon $y_jt_2y_{10k+j}t_3$ $y_{j+2}y_j$), and take R^* satisfying a) of Lemma 3, of appropriate cardinality. The leave $L(Q')$ of the $MPP(40k+3)$ Q' with $160k^2 + 20k - 1$ pentagons contains two quadrangles, $y_1y_7y_3y_9y_1$ and $t_2t_1t_3y_{10k+9}t_2$. □

References

1. Alspach B., Schellenberg P. J., Stinson D. R. and Wagner D., *The Oberwolfach problem and factors of uniform odd length cycles*, J. Combinat. Theory (A) **52** (1989), 20–43.

2. Bondy J. A., *Large cycles in graphs*, Discrete Math. **1** (1971), 121–132.

3. Rosa A. and Znám Š., *Packing pentagons into complete graphs: how clumsy can you get?*, Discrete Math. **128** (1994), 305–316.

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