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A NOTE ON THE HAMILTONIAN GENUS OF A COMPLETE BIPARTITE GRAPH

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Abstract. The Hamiltonian genus of a graph G (denoted by $\gamma_H(G)$) is the smallest number g such that the graph G is embeddable in the orientable surface with genus g and there is some face-boundary b which is a Hamiltonian cycle of G. In this paper we show that

$$\lceil (n-2)(n-1)/4 \rceil \leq \gamma_H(K_{n,n}) \leq \lceil n/2 \rceil^2 + \lceil n/2 \rceil .$$

Introduction

The **Hamiltonian genus** of a graph G (denoted $\gamma_H(G)$) is the smallest number such that the graph G is embeddable in the orientable surface with the genus g and there is some face-boundary b such that b is a Hamiltonian cycle of G. Such an embedding is called a **Hamiltonian embedding**. D. Bénard [**1**] proved the equality

$$\gamma_H(K_n) = \gamma(K_{n+1}) = \lceil (n-2)(n-3)/12 \rceil .$$

This is a corollary of the following result whose proof can be found in [**1**].

For every integer $n \geq 3$ there is an embedding of K_n into an orientable surface of genus $\gamma(K_n)$ such that at least one vertex of K_n is incident to triangular faces only.

In our paper we extend the known results on the Hamiltonian genus of graphs by investigating the Hamiltonian genus of complete bipartite graphs. For the sake of comparison it may be of interest to mention that $K_{n,n}$ admits an orientable embedding in which each face is bounded by a Hamiltonian cycle [4, 5]. By the Euler-Poincaré formula, its genus is $(n-1)(n-2)/2$.

Main Result

Theorem. *Let $K_{n,n}$ be a complete bipartite graph. Then*

$$\lceil (n-2)(n-1)/4 \rceil \leq \gamma_H(K_{n,n}) \leq \lceil n/2 \rceil^2 + \lceil n/2 \rceil .$$

The proof will be performed in the following two lemmas.

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Lemma 1. $\gamma_H(K_{n,n}) \geq \lceil (n-2)(n-1)/4 \rceil$.

Proof. Let $f_H : K_{n,n} \hookrightarrow S$ be a minimum Hamiltonian embedding of the graph $K_{n,n}$ into a surface S. Add one vertex and the corresponding edges into the face f whose boundary ∂f is the Hamiltonian cycle of G to obtain the graph $K_{n,n+1}$. From the construction of $K_{n,n+1}$ is clear, that it is still embedded into the surface S. Therefore

$$\gamma(K_{n,n+1}) = \lceil (n-2)(n-1)/4 \rceil \leq \gamma_H(K_{n,n}).$$

□

Lemma 2. $\gamma_H(K_{n,n}) \leq \lceil n/2 \rceil^2 - \lceil n/2 \rceil$, *i.e.*

$$\gamma_H(K_{n,n}) \leq n(n-2)/4 \quad \textit{if } n \textit{ is even and}$$
$$\gamma_H(K_{n,n}) \leq (n+1)(n-1)/4 \quad \textit{if } n \textit{ is odd.}$$

Proof. Let $m = \lceil n/2 \rceil$ and let

$$f_H : K_{m,m} \hookrightarrow S,$$
$$g_H : K_{m,m} \hookrightarrow T$$

be minimum Hamiltonian embeddings of $K_{m,m}$ such that g_H is a **mirror image** of f_H (i.e. if $v_1 v_2 \dots v_{t-1} v_t$ is the face-boundary in f_H, then $v'_t v'_{t-1} \dots v'_2 v'_1$ is the face-boundary in g_H). We express the number of faces of f_H (denoted $F(f_H)$) using the well-known Euler-Poincaré formula

$$F(G) = |E(G)| - |V(G)| + 2 - 2\gamma(G).$$

For the embedding f_H the number of faces is

$$F(f_H) = \lceil n/2 \rceil^2 - 2\lceil n/2 \rceil + 2 - 2\gamma_H(K_{m,m}). \tag{1}$$

Clearly, for g_H, $F(g_H) = F(f_H)$. Now, we will create the embedding

$$h \colon K_{2m,2m} \hookrightarrow R$$

from the embeddings f_H and g_H. First we cut the disks in all faces of f_H and g_H and glue the corresponding holes. By this operation we amalgamate two surfaces (S and T), each of genus $\gamma_H(K_{m,m})$ with $F(f_H)$ handles. Therefore by (1) the genus of the resulting surface R is

$$\gamma(R) = 2\gamma_H(K_{m,m}) + F(f_H) - 1 = m^2 - 2m + 1. \tag{2}$$

Let $v_1 v_2 v_3 \dots v_{t-1} v_t$ be some face-boundary of the embedding f_H (see Fig. 1) and let $v'_t v'_{t-1} \dots v'_3 v'_2 v'_1$ be the corresponding face-boundary of g_H. By amalgamating these two faces we obtain a non-cellular face (a handle) with two boundary

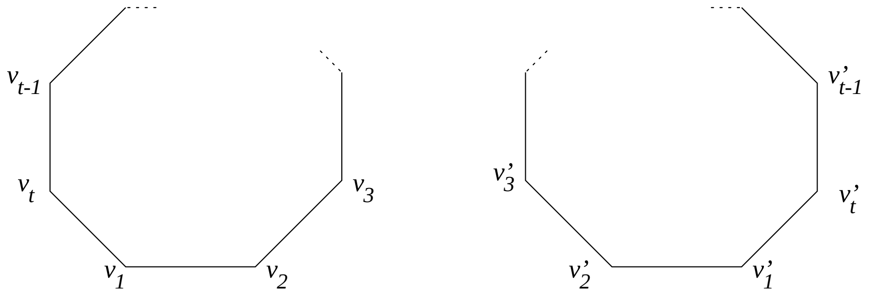

Figure 1. Corresponding face boundaries before glueing.

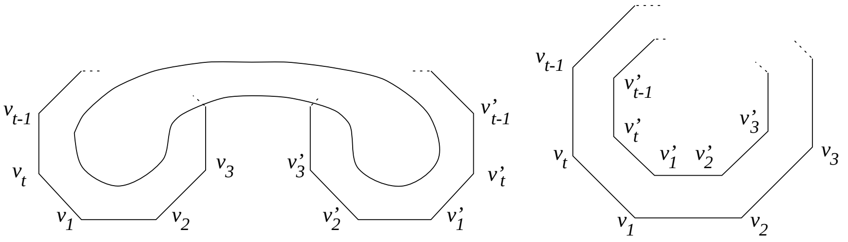

Figure 2. Amalgamation of corresponding faces and its planar projection.

cycles $v_1v_2v_3\ldots v_{t-1}v_t$ and $v'_tv'_{t-1}\ldots v'_3v'_2v'_1$ (as shown in Fig. 2). Add edges $v_1v'_2$, $v_2v'_3$, $\ldots$, $v_{t-1}v'_t$, $v_tv'_1$ into this face. It is easy to see that the orientability of the surfaces S and T guarantees that the addition of the relevant edges is possible. The necessary considerations can also be made using the rotation schemes. Fig. 3 illustrates the result of this operation.

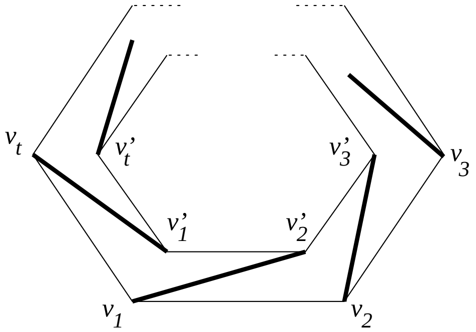

Figure 3. Adding the relevant edges.

Routine calculations show that by adding the appropriate edges into each face of R (precisely in the sense of the rotation) we obtain the requested embedding h of $K_{2m,2m}$. A similar construction is described in Pisanski [**3**] and Nedela and Škoviera [**2**].

Let $(u_1u_2\dots u_{m-1}u_m)$ be the permutation of the vertices $\{v_1,v_2,\dots,v_m\}$ such that $u_1u_2\dots u_{m-1}u_m$ is the Hamiltonian cycle in f_H and let $(u'_mu'_{m-1}\dots u'_2u'_1)$ be the permutation of the vertices $\{v'_1,v'_2,\dots,v'_m\}$ such that $u'_mu'_{m-1}\dots u'_2u'_1$ is the Hamiltonian cycle in g_H. From the method described above it follows that since the embeddings f_H and g_H have the Hamiltonian cycles $u_1u_2\dots u_{m-1}u_m$ and $u'_mu'_{m-1}\dots u'_2u'_1$ as the boundaries of some faces s and s', respectively, then in the embedding h of $K_{2m,2m}$ there are faces $r_1,r_2,\dots,r_m$ with boundaries $u_1u_2u'_3u'_2$, $u_2u_3u'_4u'_3$, $\dots$, $u_{m-1}u_mu'_1u'_m$, $u_mu_1u'_2u'_1$, respectively.

Without loss of generality, select one from these faces, say r_1. Join the other faces $r_2,\dots,r_m$ by deleting edges $u_3u'_4$, $\dots$, $u_{m-1}u'_m$, $u_mu'_1$ to one face t. It is easy to see that the boundary $u_2u_3\dots u_mu_1u'_2u'_1, u'_m\dots u'_4u'_3$ of t is a Hamiltonian cycle of a new graph. Moreover by adding at most $m-1$ handles we can add all deleted edges back to the graph to obtain the complete bipartite graph $K_{2m,2m}$ (as shown in Fig. 4).

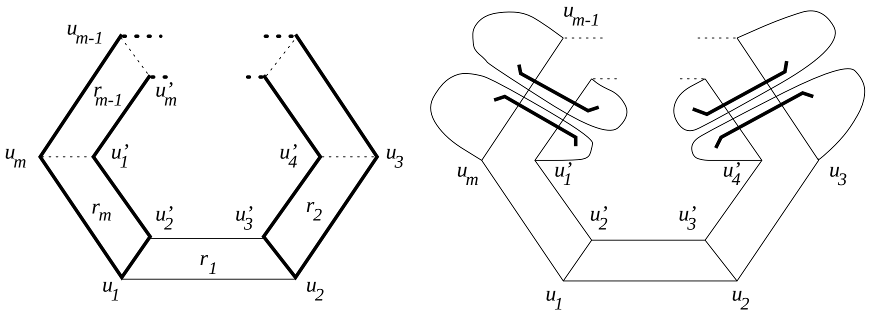

Figure 4. The Hamiltonian cycle and adding the deleted edges.

Thus we obtain a new surface R' from the surface R, and the genus of R' is

$$\gamma(R') = \gamma(R) + \lceil n/2 \rceil - 1. \tag{3}$$

The statement of the lemma now follows immediately from (2) and (3). ☐

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