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A HAMILTONIAN PROPERTY OF CONNECTED SETS IN THE ALTERNATIVE SET THEORY

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Abstract. The representation of indiscernibility phenomena by π-equivalences and of accessibility phenomena by σ-equivalences enables a graph-theoretical formulation of topological notions in the alternative set theory. Generalizing the notion of Hamiltonian graph we will introduce the notion of Hamiltonian embedding and prove that for any finite graph without isolated vertices there is a Hamiltonian embedding into any infinite set connected with respect to some π- or σ-equivalence. Roughly speaking, in some sense this means that each such an infinite connected set, (in particular, each connected set in a complete metrizable topological space), contains each finite graph inside, and even is exhausted by the images of its edges. Moreover, the main Theorem 3, dealing with the so called deeply connected sets, is in fact a theorem of nonstandard arithmetic.

1. A Brief Outline of Alternative Set Theory

The alternative set theory (AST), similarly as nonstandard analysis, enables to treat the infinity phenomenon emerging on formally finite collections of objects. The basic collections of objects, considered itselves as selfstanding objects, AST deals with, are called classes. Sharp classes with definite boundaries are called sets. From the point of view of classical set theory, all the sets in AST are finite, as they are subject to the axiom scheme of induction for set-theoretical formulas. Enormously large "finite" sets, however, will be called infinite. In AST we define a class to be finite if each its subclass is a set. Every finite class obviously is a set, but not vice versa. Accepting the existence of infinite sets, we accept henceforth the existence of proper semisets, i.e., of proper classes which are subclasses of (formally finite!) sets. Thus we can see that, in contradistinction to proper classes in, e.g., Gödel-Bernays set theory, proper classes in AST need not be "larger" then sets. Proper semisets rather arise as mathematical counterparts of nonsharp collections without definite boundaries.

Small characters always denote sets, capital ones are used for classes. When a capital letter denotes a set, it will explicitly be pointed out.

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Natural numbers in AST are introduced, following von Neumann, as sets of smaller natural numbers. The class of all natural numbers is denoted by $\mathbf{N}$; the class of all finite natural numbers (i.e. of elements of $\mathbf{N}$ which are finite sets) is denoted by $\mathbf{FN}$. Both $\mathbf{N}$ and $\mathbf{FN}$ with canonically defined operations are models of Peano arithmetic. However $\mathbf{N}$ satisfies the induction scheme only for properties expressable by formulas of the language of PA (or, which turns out to be equivalent, by set-theoretical formulas), whereas $\mathbf{FN}$ satisfies the second order induction, i.e. the induction for arbitrary properties.

Every set A is equivalent through a set-function to a unique natural number denoted by $|A|$, and A is finite iff $|A| \in \mathbf{FN}$. On the other hand, as a consequence of the axiom of two cardinalities, any two infinite sets are equivalent through some class-function.

For the basic axiomatic system of AST, as well as for further notions, conventions and results we refer to Vopěnka's book [**V 1979**]. More detailed exposition can be found in [**V 1989**]. Nevertheless, for the reader not acquainted with AST, we will list some few facts which could help understanding the paper.

Let us begin reviewing some notation. For any class X, relations (i.e. classes of ordered pairs) R, S, and $n \in \mathbf{N}$ we put

$$[X]^n = \{s \subseteq X;\ |s| = n\},$$
$$\mathrm{dom}(R) = \{x;\ (\exists\, y)\ y\, R\, x\}, \qquad \mathrm{rng}(R) = \{y;\ (\exists\, x)\ y\, R\, x\},$$
$$R\text{"}X = \{y;\ (\exists\, x \in X)\ y\, R\, x\}, \qquad R|X = \{(y,x) \in R;\ x \in X\},$$
$$R \circ S = \{(x,y);\ (\exists\, z)(x\, R\, z\ \ \&\ \ z\, S\, y)\},$$
$$R^2 = R \circ R, \qquad R^3 = R \circ R \circ R, \qquad \text{etc.}$$

Perhaps the most important and typical axiom of AST is the following:

Axiom of Prolongation. *Every class-function G with domain $\mathbf{FN}$ is of the form $G = g|\mathbf{FN}$ for some set-function g.*

As a consequence we have:

Overspill Principle. *Let $\{\varphi_k(x);\ k \in \mathbf{FN}\}$ be a sequence of set-theoretical formulas and G be a function with domain $\mathbf{FN}$ such that $\varphi_k(G|n)$ holds for all $k, n \in \mathbf{FN}$. Then there is an infinite natural number ν and a set-function g with domain $\nu + 1 = \{0, 1, \dots, \nu\}$, such that $G = g|\mathbf{FN}$ and $\varphi_k(g|n)$ holds for all $n \le \nu$ and $k \in \mathbf{FN}$.*

A reflexive and symmetric relation on a class X will simply be called a symmetry on X.

A class X will be called a π-class (σ-class, respectively) provided there is a sequence $\{\varphi_k(x);\ k \in \mathbf{FN}\}$ of set-theoretical formulas such that

$$X = \{x;\ (\forall\, k \in \mathbf{FN})\ \varphi_k(x)\} \qquad \big(X = \{x;\ (\exists\, k \in \mathbf{FN})\ \varphi_k(x)\},\ \text{resp.}\big).$$

Terms like π-equivalence, σ-symmetry, set-symmetry etc., are selfexplanatory.

In what follows we will utilize the fact that every π-symmetry (σ-symmetry, respectively) R on a set X can be written in the form

$$R = \bigcap_{n\in\mathbf{FN}} S_n \qquad \Big(R = \bigcup_{n\in\mathbf{FN}} S_n, \text{ resp.}\Big)$$

where $\{S_n;\, n \in \mathbf{FN}\}$ is a sequence of set-symmetries on X, such that

$$S_{n+1} \subseteq S_n \qquad (S_n \subseteq S_{n+1}, \text{ resp.})$$

for each n. Moreover, R is a π-equivalence (σ-equivalence, respectively) iff the sequence $\{S_n;\, n \in \mathbf{FN}\}$ can be chosen in such a way that even

$$S_{n+1} \circ S_{n+1} \subseteq S_n \qquad (S_n \circ S_n \subseteq S_{n+1}, \text{ resp.})$$

holds for each n. In any of the above cases $\{S_n;\, n \in \mathbf{FN}\}$ will be called a generating sequence of R.

According to the basic ideas of the alternative set theory, one could expect that the discrete combinatorial notions, methods and results of finitary mathematics will play an important role in mathematics based on AST, even in areas traditonally dominated by infinitary "continuous" mathematics.

So far the most developed topic which can serve as an example is the topology (see e.g. [**V 1979**], [**V 1989**], [**G–Z 1985a**], [**G–Z 1985b**] and the survey article [**Z 1989**]). In most cases a topological space can be represented in AST as a pair (X, R) where X is a set and R is a π-equivalence on X, serving as mathematization of the phenomena of indiscernibility and continuity. Such pairs correspond, in some sense which can be made fully precise, to complete metrizable spaces in classical topology, and all the common topological notions can be introduced in terms of the relation R. E.g., a class $K \subseteq X$ is called open in (X, R) if for each $x \in K$ there is a set Y such that $R"\{x\} \subseteq Y \subseteq K$. However, by such an attempt one should be aware of the fact that the mathematical models of *geometrical* points are the monads $R"\{x\}$ and not the single elements $x \in X$.

As R in general is a proper semiset, set-functions $f\colon \nu + 1 \longrightarrow X$, such that $\nu \in \mathbf{N}$ and $f(n)\, R\, f(n{+}1)$ for each $n < \nu$, can be used to represent the phenomenon of continuous motion of point in the spirit of Poincaré (see [**P 1902**], [**P 1905**]). Indeed, given such a function f, the conclusion $f(0)\, R\, f(\nu)$ can be proved only for $\nu \in \mathbf{FN}$. For ν infinite, though R is an equivalence, $(f(0), f(\nu)) \notin R$ may well happen.

The notion of σ-equivalence was introduced in [**G–Z 1985a**] (see also [**G–Z 1985b**] and [**Z 1989**]) in order to enrich the domain of topology by simultaneous study of indiscernibility and accessibility phenomena, the latter of which can be faithfully represented by σ-equivalences.

On the other hand, every pair (X, R) where R is an arbitrary symmetry on the class X, can be viewed as a "graph" $(X, \hat{R})$ where

$$\hat{R} = \{\{x, y\};\ x \neq y \ \ \&\ \ x\, R\, y\}.$$

Hence all usual graph-theoretical notions and the corresponding results can directly be transfered into the AST version of topology. In particular, this applies to the notions of path and connectedness.

2. Deep Connectedness and Hamiltonian Embeddings

Under the term "graph" always a nonoriented graph without loops and multiple edges will be understood. However, as our underlying set theory is the AST now, we accept the following definition.

A graph is a pair of **classes** (X, E) such that $X \neq \emptyset$ and $E \subseteq [X]^2$. As usual, the elements of X will be called vertices and the elements of E edges. A vertex x is called isolated (in (X, E)) if it belongs to no edge $p \in E$. If (X, E), (Y, F) are graphs then (X, E) is called a subgraph of (Y, F), notation $(X, E) \subseteq (Y, F)$, if $X \subseteq Y$ and $E \subseteq F$. A graph (X, E) will be called a set-graph if both X and E are sets; it will be called finite if X (hence E, as well) is finite. Obviously, every finite graph is a set-graph.

Similarly as each symmetry R on X yields a graph $(X, \hat{R})$, every graph (X, E) induces a symmetry

$$\breve{E} = \{(x, y);\ x = y \ \vee\ \{x, y\} \in E\}$$

on X. Thus there is a natural one-to-one correspondence between symmetries and graph structures on any nonempty class X.

Let (X, E) be a graph. A path in (X, E), briefly an E-path, is an injective *set*-function $f\colon \nu + 1 \longrightarrow X$ such that $\nu \in \mathbf{N}$ and $\{f(n), f(n+1)\} \in E$ for each $n < \nu$. We put

$$\ddot{f} = \{f(0), f(\nu)\}$$

for any set-function f such that $\operatorname{dom}(f) = \nu + 1$, $\nu \in \mathbf{N}$. Two paths f, g are called internally disjoint if

$$\operatorname{rng}(f) \cap \operatorname{rng}(g) \subseteq \ddot{f} \cap \ddot{g}.$$

If R is a symmetry on X, then an R-path simply means a path in the graph $(X, \hat{R})$, i.e. an $\hat{R}$-path.

Let (X, E) be a graph. A set $A \subseteq X$ will be called connected (in (X, E)) if for every nonempty proper subset B of A there exist elements $x \in B$, $y \in A \setminus B$ such that $\{x, y\} \in E$. Similarly, if R is a symmetry on X, then a set $A \subseteq X$ is called R-connected iff it is connected in the graph $(X, \hat{R})$.

The following theorem already belongs to the folklore of AST.

Theorem 1. *Let R be a π- or a σ-symmetry on a class X and $A \subseteq X$ be a set. Then the following conditions are equivalent:*
(a) *A is R-connected.*
(b) *For all $x, y \in A$ there is an R-path f such that*

$$\operatorname{rng}(f) \subseteq A \qquad \textit{and} \qquad \ddot{f} = \{x, y\}.$$

(c) *There is a $\nu \in \mathbf{N}$ and a surjective set-function $f \colon \nu + 1 \longrightarrow A$ such that $f(n) \, R \, f(n+1)$ for each $n < \nu$.*

In the original version of the manuscript of [**V 1989**] it was asserted that for a π-equivalence R the conditions (b) and (c) can be put together, i.e., that for an R-connected set A the R-path f from (b) can be required to satisfy additionally $\operatorname{rng}(f) = A$ as in (c). But the presented proof was not correct. Within short A. Vencovská and the author of this article independently have shown that this common "Hamiltonian" strengthening of (b) and (c) is still true. However, as communicated to the author by P. Vopěnka, the proof by Vencovská made use of some higher axioms of AST, namely of the axiom of choice and the axiom of two cardinalities.

The proof we are going to present here is rather an elementary one, and relies on some well known results of graph theory. But this will be only a very particular case of an (at least for the author) unexpected result with both graph-theoretical and topological flavour announced in the Abstract, with a connection, though rather a loose one, to space filling curves.

Let R be a symmetry on a class X. A set $A \subseteq X$ will be called deeply R-connected if there is a set-symmetry $S \subseteq R$ such that A is S-connected. Obviously, if A is deeply R-connected, then it is R-connected. However, for some symmetries also the converse is true.

Proposition. *Let R be a π- or a σ-symmetry on a class X. Then every R-connected set $A \subseteq X$ is deeply R-connected.*

Proof. Let $A \subseteq X$ be R-connected and $\{S_n;\, n \in \mathbf{FN}\}$ be a generating sequence of R.

(a) If R is a π-symmetry, then, as $R \subseteq S_n$, A is S_n-connected for each $n \in \mathbf{FN}$. As this property can be expressed by a set-theoretical formula, using the overspill principle one can find a set-symmetry $S \subseteq R$ such that A is S-connected.

(b) If R is a σ-symmetry, then A already is S_n-connected for some $n \in \mathbf{FN}$. In the opposite case, using overspill again, one could find a set-symmetry $S \supseteq R$, such that A is not S-connected — a contradiction. □

Remark. Notice that for any infinite $\nu \in \mathbf{N}$ the equivalence

$$R = \mathbf{FN}^2 \cup (\nu \setminus \mathbf{FN})^2$$

on the set ν is a union of a σ-class and of a π-class, and ν obviously is R-connected (it even satisfies the mentioned common strengthening of (b) and (c) from Theorem 1). However, as it can easily be seen, ν is not deeply R-connected.

Let us repeat once more that as far as dealing with sets and set-theoretical formulas only, everything established in classical mathematics based on Cantor set theory for finite sets using "standard" methods, as, e.g., the induction over the number of elements, can directly be transferred to all sets in AST, no matter that finite or infinite. In particular, this applies to the following theorem, contributiting to the topic of Hamiltonian circles, which was proved by M. Sekanina [**Sk 1963**] and J. Karaganis [**Kr 1968**]. We will formulate it in terms of symmetries rather than graphs.

Theorem 2. *Let R be a set-symmetry on X and $A \subseteq X$ be an R-connected set. Then for each pair of distinct vertices $x, y \in A$, there is an R^3-path f such that*

$$\operatorname{rng}(f) = A \qquad \text{and} \qquad \ddot{f} = \{x, y\}.$$

Now, we have an immediate consequence.

Corollary. *Let R be an arbitrary symmetry on X and $A \subseteq X$ be a deeply R-connected set. Then for each pair of distinct vertices $x, y \in A$ there is an R^3-path f such that*

$$\operatorname{rng}(f) = A \qquad \text{and} \qquad \ddot{f} = \{x, y\}.$$

Obviously, the above f is an R-path if R is an equivalence. Also, choosing x, y in such a way that $\{x, y\} \in \hat{R}$, f can immediately be extended to a Hamiltonian circle.

In the following definition, for a function θ, as its values will be functions again, we put $\theta_p = \theta(p)$ for $p \in \operatorname{dom}(\theta)$.

Definition. Let (X, E) be a set-graph without isolated vertices and R be an arbitrary symmetry on a set A. A pair (h, θ) will be called a *Hamiltonian embedding* of (X, E) into (A, R) provided the following conditions are satisfied:

(a) $h\colon X \longrightarrow A$ is an injective set-function.

(b) θ is a set-function, such that $\operatorname{dom}(\theta) = E$ and for each edge $p = \{x, y\} \in E$, θ_p is an R-path satisfying

$$\ddot{\theta}_p = \{h(x), h(y)\}.$$

(c) For distinct edges $p, q \in E$ the paths θ_p, θ_q are internally disjoint.

(d) Each point $a \in A$ lies on the image of some edge, i.e.,

$$A = \bigcup_{p \in E} \operatorname{rng}(\theta_p).$$

Remark. Obviously, a finite graph (Y, F) is Hamiltonian iff there is a Hamiltonian embedding of the cycle of length three (or more) into $(Y, \breve{F})$.

Now, we are able to state and prove our main result.

Theorem 3. *Let A be an infinite set and R be an arbitrary equivalence on A. If A is deeply R-connected, then for each finite graph (X, E) without isolated vertices and every injective function $h\colon X \longrightarrow A$, there is a set-function θ such that the pair (h, θ) is a Hamiltonian embedding of (X, E) into (A, R).*

Proof. We will proceed by the induction over the number of edges $|E| \in \mathbf{FN}$. The case $|E| = 1$ follows from the Corollary to Theorem 2. Let $|E| > 1$ and $p = \{x, y\}$ be an arbitrary edge in (X, E). Let f be an R-path, such that

$$\operatorname{rng}(f) = A \qquad \text{and} \qquad \ddot{f} = \{h(x), h(y)\},$$

garanteed by the same Corollary. Let us denote $\operatorname{dom}(f) = \mu + 1$. We put

$$F = E \setminus \{p\}, \qquad Y = \{z \in X;\ (\exists\, q \in F)(z \in q)\}$$

and

$$a = \{0, \mu\} \cup \{m \le \mu;\ f(m) \notin h\text{"}Y \ \ \&\ \ m \text{ is even}\},$$
$$B = (A \setminus f\text{"}a) \cup h\text{"}Y\,.$$

Then $B \subseteq A$ obviously is a deeply R-connected set, $(Y, F) \subseteq (X, E)$ is a finite graph without isolated vertices and $|F| < |E|$. Thus we can assume that there is a θ such that $(h|Y, \theta)$ is a Hamiltonian embedding of (Y, F) into $(B, R \cap B^2)$. Now, one can enumerate the nonempty set a in its natural order $a = \{\alpha_0, \ldots, \alpha_\kappa\}$ and put $g(k) = f(\alpha_k)$ for $k \le \kappa$. As R is an equivalence and Y is finite, g is an R-path and $\ddot{g} = \{h(x), h(y)\}$. Then the set-function η, given by

$$\eta_q = \begin{cases} g & \text{for } q = p, \\ \theta_q & \text{for } q \in F, \end{cases}$$

raises to a Hamiltonian embedding (h, η) of (X, E) into (A, R). $\square$

The Theorem just proved and the Proposition have the following consequence:

Corollary 1. *Let R be a σ-equivalence on an infinite set A. If A is R-connected, then for each finite graph (X, E) without isolated vertices and every injecive function $h\colon X \longrightarrow A$, there is a set-function θ such that the pair (h, θ) is a Hamiltonian embedding of (X, E) into (A, R).*

This result can even be strengthened for π-equivalences.

Corollary 2. *Let R be a π-equivalence on an infinite set A. If A is R-connected, then there is an infinite natural number ν, such that for each set-graph (X, E) without isolated vertices, satisfying $|X| \leq \nu$, and every injective set-function $h\colon X \longrightarrow A$ there is a set-function θ such that the pair (h, θ) is a Hamiltonian embedding of (X, E) into (A, R).*

Proof. Let us denote M the class of all natural numbers n such that for each set-graph (X, E), satisfying $|X| \leq n$, and each injective set-function $h\colon X \longrightarrow A$, there is a set-function θ such that the pair (h, θ) is a Hamiltonian embedding of (X, E) into (A, R). Obviously, $\mathbf{FN} \subseteq M$. As R is a π-equivalence, M is a π-class, as well. Since $\mathbf{FN}$ is not a π-class (see, e.g., [**V 1979**]), there has to be an infinite element $\nu \in M$. □

Concluding remarks. (1) Under a slight and inessential modification of the Definition, the notion of Hamiltonian embedding with (X, E) replaced by an arbitrarry nonoriented set-graph (i.e. loops, multiple edges and isolated vertices are allowed) can be introduced, and Theorem 3 and its Corrolaries still can be proved.

(2) Neither Theorem 3 nor its Corollaries assert that each finite graph can be *drawn* in any connected metrizable topological space. In fact the notion of topological representation of a graph (X, E) in (A, R) where R is a symmetry on A would require the following modification of the Definition:

(a) h has to satisfy

$$x \neq y \;\Rightarrow\; (h(x), h(y)) \notin R,$$

for all $x, y \in X$.

(b) The paths $f = \theta_p$ have to satisfy additionally

$$f(m)\, R\, f(n) \;\Rightarrow\; (\forall\, k, l)\big(m \leq k \leq l \leq n \;\Rightarrow\; f(k)\, R\, f(l)\big),$$

for $m, n \in \mathrm{dom}(f)$, $m < n$.

(c) Paths θ_p, θ_q corresponding to distinct edges p, q have to satisfy even

$$\mathrm{rng}(\theta_p) \cap R\text{"}\mathrm{rng}(\theta_q) \subseteq R\text{"}(\ddot{\theta}_p \cap \ddot{\theta}_q).$$

(d) is omitted.

However, the corresponding restatements of the mentioned results are no way true. In particular, this would imply the triviality of a great deal of topological graph theory — a contradiction (cf. e.g. [**Wh 1984**]).

(3) The main Theorem 3 can also be regarded as a theorem of nonstandard arithmetic. The Prolongation Axiom is used only in the proof of the Proposition, verifying the applicability of the main Theorem to σ- and π-equivalences, needed to obtain the Corrolaries 1 and 2.

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