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EDGE NEIGHBORHOODS IN LINE GRAPHS

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Abstract. By an edge-neighborhood of an edge f in a graph we mean the subgraph induced by nodes outside f which are adjacent to some node on f. Connected graphs whose line graphs have the same edge-neighborhood of any edge are characterized. There are P_4, stars, complete graphs and regular triangle-free graphs in which any two nodes with the distance two have the same number of common neighbors.

Objects which possess certain properties of symmetry have been widely studied in mathematics, partly because of esthetic reasons. Their characterizations often demonstrate the strength of mathematical theories. The current paper deals with one kind of them, e-locally homogeneous graphs, characterizes those of them which are line graphs, and reveals their connections to other intensively studied areas of combinatorics.

The terminology is based on [**2**]. Our graphs are finite, undirected, without loops and multiple edges. If v is a node, then $N(v)$ means the set of nodes adjacent to v. By the **neighborhood** $\langle N(v)\rangle$ **of a node** v we mean the subgraph induced by $N(v)$ and by $N^k(v)$ we mean the set of nodes with the distance k to v. A graph is said to be a **locally-**H **graph,** or a **locally homogeneous graph**, if for all its nodes v, $\langle N(v)\rangle$ is isomorphic to a given graph H. Analogously, $N^e(f)$ is the set of nodes outside the edge f which are adjacent to some endnode of f, $\langle N^e(f)\rangle$ is referred to as the **edge-neighborhood of the edge** f. A graph is said to be an **e-locally-**F **graph**, or an **e-locally homogeneous graph**, if for all its nodes v, $\langle N^e(v)\rangle$ is isomorphic to a given graph F.

Zelinka [**17**] proposed the e-neighborhood version of the well-known Zykov problem [**18**] (concerning neigbors of nodes):

Characterize the graphs F for which there exists an e-locally-F graph.

Such graphs will be called **e-realizable.** Fronček [**11**] showed that stars, complete graphs and $K_{1,2,\dots,2}$ are the only e-realizable graphs of the radius one. He also described e-realizable complete multipartite graphs ($= K_{n_1,n_2,\dots,n_k}$, where $n_1+1=n_2+1=n_3=\cdots=n_k$) [**6**] and double stars (= two stars which share a common edge; e-realizable if and only if each node has an odd degree) [**10**]. A class of non e-realizable trees homeomorphic to a star is described in [**8**]. Combining

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the results in [**6**] and [**14**], we obtain that a cycle C_n is e-realizable if and only if n is even or $n = 3$. $K_{2k+2} - (k+1)K_2$ is the only graph which is both locally homogeneous and e-locally homogeneous and in which the neighborhoods have the same number (k) of nodes as edge-neighborhoods have [**5**]. An upper bound for the number of edges in e-locally acyclic graphs was given in [**9**]. e-locally path graphs were studied in [**7**].

In this paper we characterize line graphs in which the edge-neighborhoods of any two edges have the same number of nodes. Before doing this, we need to introduce the atomic and king graphs.

A graph is (a,b)**-biregular**, if each edge joins nodes with the degrees a and b, $a \neq b$. A connected graph is (a,b)**-atomic** if it can be obtained from an (a,b)-biregular graph G after replacing each its node v with the degree a by the complete graph K_a and joining K_a to $N_G(v)$ by a pairwise independent edges, each of them joins a node from K_a to a node of $N_G(v)$. Note that we obtain a $(2,b)$-atomic graph if we insert two nodes into each edge in a b-regular multigraph, while $(1,b)$-atomic graphs are stars $K_{1,b}$. A node of a degree d is referred to as a d**-node**. A graph is a **king graph** if it can be obtained from a union of several crowns $K_2 + \overline{K_3}$ after joining each 2-node to just one 2-node from another crown by an edge.

Theorem 1. *Let G be a connected nontrivial graph and n be an integer. Then every edge in $L(G)$ has the edge-neighborhood with n nodes if and only if one of the following statements holds.*

(S1) *$G = P_4$, $n = 1$; $G = K_{1,3} + e$, $n = 2$; $G = K_4 - e$, $n = 3$.*

(S2) *G is the star on $n+3$ nodes.*

(S3) *G is the complete graph on $(n+8)/3$ nodes, $n \equiv 1 \pmod 3$.*

(S4) *G is an $((n+4)/3)$-regular triangle-free graph, $n \equiv 2 \pmod 3$.*

(S5) *G is $(2,n)$-atomic.*

(S6) *G is $((n+5)/3, (n+2)/3)$-atomic, $n \equiv 1 \pmod 3$*

(S7) *G is a king graph, $n = 6$.*

Proof. It is easy to verify that if G satisfies some of the conditions (S1), (S2), ..., (S7), then every edge in $L(G)$ has the edge-neighborhood with n nodes, but we reccomend the reader to read the next paragraph at first.

Now assume that each edge-neighborhood in $L(G)$ has the order n. If $S \subseteq G$, then $f(S) = \sum_{x \in V(S)} \deg_G(x)$ is called the **force of** S. We prove that the force of any induced P_3 is $n+4$ and the force of any K_3 is $n+5$. Note that there is a one-to-one correspondence between paths on three nodes, say $P = abc$, in G, and edges, $(ab)(bc)$, in $L(G)$. Futhermore, the order of the edge-neighborhood of the edge $(ab)(bc)$ in $L(G)$ equals to the number of edges in G adjacent to the edges ab or bc, but distinct from them. Hence $n = f(\langle a,b,c \rangle) - 5$, if $\langle a,b,c \rangle \cong K_3$, and $n = f(\langle a,b,c \rangle) - 4$ otherwise.

Since the force of any 3-element set of nodes in a k-regular graph G is $3k$, it follows from the above facts that either there is no induced P_3 in G, or there is no K_3 in G. In the former case G is complete and (S3) holds, while in the latter case (S4) holds.

Suppose G is not regular. Let v be a node in G with the maximal degree Δ. If $\Delta = 2$, then G is P_3 or P_4, hence assume $\Delta \geq 3$. Note that the following three observations hold:

(O1) If $xaby$ is an induced path, then $\deg(x) = \deg(y)$.

(O2) For any y, $z \in V(G)$ there is $x \in V(G)$ such that the distance from y to x is at most two and $\deg(x) = \deg(z)$.

(O3) If $x, y \in N(v)$; $x \neq y$, then

$$\begin{aligned} \deg(x) + \deg(y) &= n + 5 - \Delta \quad \text{if } x \text{ and } y \text{ are adjacent} \\ \deg(x) + \deg(y) &= n + 4 - \Delta \quad \text{if } x \text{ and } y \text{ are not adjacent.} \end{aligned}$$

The first observation holds, since $\langle x, a, b\rangle \cong \langle a, b, y\rangle \cong P_3$ gives

$$\deg(x) + \deg(a) + \deg(b) = f(\langle x, a, b\rangle) = f(P_3) = \deg(a) + \deg(b) + \deg(y).$$

To prove (O2), note that (O1) gives that the nodes with the distance three have the same degrees. Hence if P is a shortest y-z-path and its length is d, then the node on P with the distance $d - 3\lfloor d/3\rfloor$ from y has the same degree as z has and (O2) follows.

Finally we prove (O3). If x and y are adjacent, then $\langle v, x, y\rangle \cong K_3$, hence

$$\deg(x) + \deg(y) + \Delta = \deg(x) + \deg(y) + \deg(v) = f(\langle x, y, v\rangle) = f(K_3) = n + 5.$$

If x and y are not adjacent, then $\langle x, y, v\rangle \cong P_3$, and

$$\deg(x) + \deg(y) + \Delta = \deg(x) + \deg(y) + \deg(v) = f(\langle x, y, v\rangle) = f(P_3) = n + 4.$$

Denote $\rho(S) = \{\deg_G(x) \mid x \in S\}$ for $S \subseteq V(G)$. If $\rho(\langle N(v)\rangle)$ contains at least three elements, then the difference between the maximal and the minimal value of $\deg(x) + \deg(y)$ for x, $y \in N(v)$ is at least two, a contradiction to (O3). Thus $\mid \rho(\langle N(v)\rangle) \mid \leq 2$. Put $D_i = \{x \in V(G) \mid \deg_G(x) = i\}$. We distinguish two cases.

Case 1. Let $\mid \rho(N(x)) \mid = 1$ for any Δ-node x, and let all neighbors of v have the same degree r. Then, due to (O3), $\langle N(v)\rangle$ is either complete or trivial. If $\langle N(v)\rangle$ is complete, then $G \cong K_\Delta + 1$, since Δ is the maximal degree, a contradiction, since G is not regular. Otherwise $\langle N(v)\rangle$ is trivial, so $n + 4 = \Delta + 2r$. If v is adjacent to all other nodes of G, then G is a star and (S2) holds. Otherwise each node z from $N^2(v)$ is adjacent to an r-node $v_1 \in N(v)$; $f(\langle v, v_1, z\rangle) = n + 4 = \Delta + 2r$ gives

$\deg(z) = r$ and (O2) gives $\rho(G) = \{\Delta, r\}$, $\Delta > r$, since G is not regular. Now we distinguish two subcases.

Subcase 1.1. Let $r \geq 3$ and $u,w \in N(v_1) - \{v\}$, $u \neq w$. Then $\langle v_1, u, w\rangle \cong K_3$, since $f(\langle v_1, u, w\rangle) = 3r \neq n+4$. Hence $n+5 = 3r < \Delta+2r = n+4$, a contradiction.

Subcase 1.2. Let $r = 2$ (thus $n+4 = \Delta+4$). Since $N(v)$ and $N^2(v)$ are subsets of D_2, a node x is in D_Δ if the distance $d(u,v)$ is divisible by three and $x \in D_2$ otherwise, $\rho(V(G)) = \{\Delta, 2\}$, because of (O1). Bearing in mind $f(P_3) = \Delta + 4$ and $f(K_3) = \Delta + 5$, one can prove that

(1) $\langle D_\Delta\rangle$ is independent. (Since $\mid \rho(N(x)) \mid= 1$ for any Δ-node x).
(2) $\langle N(x)\rangle$ is independent for each Δ-node x.
(3) Each 2-node is adjacent to one 2-node and one Δ-node.

(1), (2) and (3) together give G is $(2, \Delta)$-atomic, hence (S6) holds.

Case 2. Let $\rho(\langle N(v)\rangle) = \{r, s\}$, $r > s$. Then all nodes from $N(v)$ but one have the same degree, since otherwise the sum of the degrees of two neighbors of v could equal to all the numbers $2r$, $r+s$, $2s$, a contradiction to (O3). Moreover, it follows from (O3) that $r = s+1$. We distinguish two subcases.

Subcase 2.1. Assume v_1 is the only neighbor of v with the degree s. At first we prove that $\rho(G) = \{\Delta, \Delta-1\}$ or (S6) holds. Note that, due to (O3), $\langle N(v)\rangle$ is the union of v_1 with the complete graph induced by $N(v) - v_1$. If $\Delta \geq 4$, then $K = \langle v \cup N(v) - v_1\rangle$ is a complete graph with at least four nodes. Hence each triple of its nodes has the same force. This gives that all nodes of K have the same degree, thus $\Delta = r = s+1$. Further, $\rho(G) = \{\Delta, \Delta-1\}$, because of (O2) and since any node in $N^2(v)$ lies on an induced P_4, in which the other nodes are in $v \cup N(v)$. If $\Delta = 3$, then $s \leq 2$. If $s = 1$, then $G = K_{1,3} + e$ and (S1) holds. If $s = 2$, then again $\rho(G) = \{\Delta, \Delta-1\}$.

Bearing in mind $f(K_3) = 3\Delta$ and $f(P_3) = 3\Delta - 1$, one can verify that:
Each Δ-node is adjacent to just one $(\Delta-1)$-node.
D_Δ induces a union of K_Δ, since $\langle D_\Delta\rangle$ is P_3-free.
$\langle N(y)\rangle$ is independent for any $(\Delta-1)$-node y.
This implies that G is $(\Delta, \Delta-1)$-atomic and (S6) holds.

Subcase 2.2. Assume v_1 is the only neighbor of v with the degree $s+1$. Then, due to (O3), $\langle N(v)\rangle$ is the star with the center v_1, hence $\deg(v_1) \geq \Delta$, this gives $\Delta = s+1$. Further $N^2(v) \subseteq D_{\Delta-1}$ and $\rho(G) = \{\Delta, \Delta-1\}$. If $\Delta = 3$, then $G = K_4 - e$. Let $\Delta \geq 4$. Then a node $u \in N(v) - v_1 \subseteq D_{\Delta-1}$ is adjacent to at most one $(\Delta-1)$-node, since $\langle D_{\Delta-1}\rangle$ is P_3-free and K_3-free. Further, v and v_1 are the only Δ-nodes adjacent to u, since $N^2(v) \subseteq D_{\Delta-1}$. Hence $\deg(u) = \Delta - 1 = 3$. Now we may assume G does not contain K_4, since this would imply that a 4-node is adjacent to three 4-nodes, and that case was studied in the Subcase 2.1. Bearing in mind $f(P_3) = 10$ and $f(K_3) = 11$, one can prove that D_3 induces a matching, D_4 induces a matching, since the two 4-nodes adjacent to a 3-node are adjacent, and $x \cup N(x)$ induces a crown for any 4-node x. This means G is a king graph and (S7) holds. □

Now we introduce a class of graphs which are neighborhoods in some line graphs. Let us color the edges of the union of three complete graphs K^1, K^2 and K^3 on d_1, d_2, and d_3 nodes by white color. A **gate graph** $\Gamma(d_1, d_2, d_3, \lambda_{123}, \lambda_{12}, \lambda_{13}, \lambda_{23}, T)$, $T \in \{0, 1\}$ is obtained from the just constructed graph after the addition of λ_{123} red triangles and λ_{ij} red edges joining K^i and K^j such that the components of the red graph are K_2 or K_3 and the red graph has precisely λ_{123} triangles. Moreover, if $T = 1$, then add a new node adjacent to all the nodes from K^1 and K^3.

It is easy to see that if $P = a_1a_2a_3$ is a path in a graph G, then the edge-neighborhood of the edge $(a_1, a_2)(a_2, a_3)$ in $L(G)$ is the gate graph $\Gamma(d_1, d_2, d_3, \lambda_{123}, \lambda_{12}, \lambda_{13}, \lambda_{23}, T)$, $T \in \{0, 1\}$, where $T = 1$ if and only if a_1, a_2, a_3 induces a triangle; $d_i = \deg(a_i) - 1 - T$, $i = 1, 3$; $d_2 = \deg(a_2) - 2$, $\lambda_{123} =\mid N(a_1) \cup N(a_2) \cup N(a_3) \mid$, λ_{ij} is the number of nodes adjacent to a_i and a_j , but not adjacent to the third node on P and distinct from it. Finally, after the addition of s independent edges to the union $K_r \cup K_r$ we obtain the graph $K_r \overset{s}{\cup} K_r$.

Note that $P_3 = L(P_4)$ is an e-locally-K_1 graph which is not locally homogeneous. The following theorem characterizes e-locally homogeneous line graphs and states that all such graphs but P_3 are also locally homogeneous.

Theorem 2. *Let $G \ncong P_4$ be a connected nontrivial graph. If $L(G)$ is locally edge-homogeneous with an edge-neighborhood F on n nodes, then F is a gate graph, $L(G)$ is localy homogeneous with a neighborhood H, and one of the following statements holds.*

(T1) $G = K_{1,n+2}$, $F = K_n$, $H = K_{n+1}$

(T2) $G = K_{(n+3)/8}$, $F = \Gamma((n-1)/3, (n-1)/3, (n-1)/3, (n-1)/3, 0, 0, 0, 1)$, $H = K_s \overset{s}{\cup} K_s$, *where* $s = (n+2)/3$ *and* $n \equiv 1 \pmod 3$.

(T3) *G is an $((n+4)/3)$-regular triangle-free graph and two nodes in G with the distance two have μ common neighbors,* $n \equiv 2 \pmod 3$, $1 \le \mu \le (n+4)/3$, $F = \Gamma((n+1)/3, (n-2)/3, (n+1)/3, 0, 0, \mu - 1, 0, 0)$, $H = K_s \cup K_s$, *where* $s = (n+1)/3$.

Proof. Straightforward from Theorem 1. □

Nedela [**15**] conjectured that every e-locally homogeneous graph is either locally homogeneous or bipartite. Theorem 2 verifies this conjecture for line graphs. Using similar approach one can prove that Nedela's conjecture holds also for complements of line graphs. It is proved in [**16**] that for a given graph H either all locally-H graphs are line graphs or none of them are. We predict a similar result in the case of a given edge-neighborhood.

Conjecture 1. *For a given graph F, either all e-locally-F graphs are line graphs or none are.*

One can verify that the only e-locally-K_n graphs are K_{n+2}, C_4, or P_3; hence Conjecture 1 holds for complete F.

Finally we will treat graphs which appear in the statement (T3), since the cases (T1) and (T2) are much more trivial. A number of results on these graphs can be found in Mulder [**13**] and Brouwer, Cohen, Neumaier [**1**]. Here we restate some of them. Let $R(k,\mu)$ be the set of all k-regular triangle-free connected graphs G in which any two nodes with the distance two have μ common neighbors ($1 \le \mu \le k$). Then the problem whether $R(k,\mu) = \emptyset$ is open. Clearly, $\mu = 1$ if and only if $girth(G) \ge 5$, hence $R(k,1)$ is infinite for any $k \ge 2$, but to find a smallest graph from $R(k,1)$ is another open problem. For $\mu \ge 2$ we have

$$\operatorname{diam}(G) \le \min\{3, k - 2\mu + 4\}, \tag{1}$$

[**1,** Cor.1.9.2, pp. 21], hence we have an algorithm which determines whether $R(k,\mu)$ is empty. Moreover, the equality in (1) holds if G is the k-dimensional cube or an Hadamard graph of degree 2μ.

Further, if $\mu < k \le 2\mu - 1$, then Mulder [**13**] gives that G is a bipartite graph with the diameter three and hence $R(k,\mu)$ is the set of the incidence graphs of symmetric block designs with certain parameters. The comlements of those block designs which correspond to finite projective planes are examples of them [**13**]. Further examples can be made out from the list of block designs given by Mathon and Rosa [**12**].

If $k = \mu$, then clearly G is complete multipartite. If $k = \mu - 1$, then G is $K_{k+1,k+1}$ with a one-factor removed or a cycle [**1,** p. 16].

Problems related to neighborhoods are difficult from an algorithmic point of view. Bulitko [**4**] and Bugata [**3**] gave different proofs of the result which states that there is no algorithm which can determine whether for a given finite graph H there exists a locally-H graph G (possibly infinite). It is not known whether the conclusion remains valid if we restrict ourselves to the case that G is also finite. One can expect that the analogous problem for e-neighborhoods is easier.

Conjecture 2. *There is an algorithm which for any finite graph F can determine whether there exists a finite e-locally-F graph.*

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