Acta Math. Univ. Comenianae
Vol. LXII, 2(1993), pp. 155–159

GENERATING HAMILTONIAN CYCLES IN COMPLETE GRAPHS

H. FLEISCHNER, P. HORÁK AND J. ŠIRÁŇ

Abstract. We prove that hamiltonian cycles of complete graphs can be generated in a Gray code manner by means of small local interchanges.

1. Introduction

Let C and C' be two hamiltonian cycles in a (simple) graph G. We say that C and C' are **switching-equivalent** (symbolically, $C \sim C'$) if the symmetric difference of their edge sets induces a quadrangle in G, i.e., if $E(C) \triangle E(C') = \{a, b, c, d\}$ where a, b, c, d are the consecutive edges of a cycle of length 4 in G. It is easy to see that $C \sim C'$ if and only if the cyclic sequences of vertices representing C and C' have the form $C = (u_1u_2v_1 \dots v_ku_3u_4w_1 \dots w_m)$, $C' = (u_1u_3v_k \dots v_1u_2u_4w_1 \dots w_m)$; in this case $E(C) \triangle E(C')$ is the edge set of the quadrangle $u_1u_2u_4u_3$ in G. Roughly speaking, C' is then obtained from C by "switching" the pairs of edges u_1u_2, u_3u_4 and u_1u_3, u_2u_4. If $k = 0$, i.e., if $C = (u_1u_2u_3u_4 \dots)$ and $C' = (u_1u_3u_2u_4 \dots)$, then we say that C and C' are **strongly switching-equivalent**.

We note that analogous concepts have been studied in operations research in connection with the travelling salesman problem. For example, the transformation used to define switching-equivalent hamiltonian cycles is the basic operation on travelling salesman tours called "2-opt" (see e.g. [**L**]). Also, the transformation for strong switching-equivalence (called "2-swap" in [**J**]) has been considered in local optimization of travelling salesman algorithms.

Let G be a hamiltonian graph. We associate with G two new graphs $H(G)$ and $H_s(G)$ as follows: The vertices of both $H(G)$ and $H_s(G)$ are the hamiltonian cycles of G; two vertices of $H(G)$ or $H_s(G)$ are adjacent if the corresponding hamiltonian cycles are switching equivalent or strongly switching-equivalent, respectively.

The idea of defining $H(G)$ and $H_s(G)$ is to express how "close" two structures (in our case, hamiltonian cycles) are, and how the switching operation can be used in generating all hamiltonian cycles of a graph. Similar situations are often encountered in the theory of generating combinatorial objects: the task is to

Received September 3, 1992.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05C45.

generate all structures of a certain type in a Gray Code manner, that is, where successively generated objects are "close" in some sense (they are obtained from each other by a "small local perturbation"). The combinatorial structures which have so far been studied from this point of view include, for example, permutations [**KL, RuS**], spanning trees of a given graph [**C, S**], eulerian trails and eulerian orientations [**MP, ZX1**], 0-1 matrices [**BL**], perfect matchings [**ZX2**], polyhedra [**NP**], and linear extensions of posets [**Ru**]. Most of those results say that the associated "local perturbation" graph is edge hamiltonian. We are interested in the same problem for the "hamiltonian cycle graph" $H_s(G)$.

Closely related to our problem, in fact we were inspired by it, is a problem of Gary Meisters and Janusz Olech concerning knight tours on chessboard, which were considered already by Euler. In our terminology, a knight tour (knight cycle) is a hamiltonian path (hamiltonian cycle) of a graph G on 64 vertices corresponding to the squares of a chessboard where two vertices are adjacent if a knight can get from one to the other in one move. The Meisters and Olech problem asks: Is the graph $H(G)$ connected? How many components are there?

2. The Result

Observe that if G is a hamiltonian graph of girth at least 5, then $H(G)$ consists of isolated vertices only. On the other hand, if G is a complete graph, then one would expect that even $H_s(G)$ is a graph of fairly rich structure.

Our aim is to prove that for every $n \geq 4$ the graph $H_s(K_n)$ is edge-hamiltonian. Clearly, $H_s(K_4) \cong K_3$. It is an easy exercise to show that $H_s(K_5)$ is isomorphic to $K_{6,6}$ minus a perfect matching.

For the sake of convenience put $H_n = H_s(K_n)$. Let x be an edge of K_n; consider the subgraph of H_n induced by those vertices which correspond to hamiltonian cycles containing the edge x. Denote this subgraph by $H_n(x)$. Obviously, H_n and $H_n(x)$ have $(n-1)!/2$ and $(n-2)!$ vertices, respectively.

Let P_{n-2} be the path $u_1u_2\dots u_{n-2}$ ($n \geq 3$) on $n-2$ vertices. A bijection $f\colon V(P_{n-2}) \to \{1, 2, \dots, n-2\}$ is called a **labelling**. The graph $L(P_{n-2})$ of all labellings of P_{n-2} is defined as follows. The vertices of $L(P_{n-2})$ are all the $(n-2)!$ labellings of P_{n-2}. Two labellings f and g are adjacent in $L(P_{n-2})$ if they differ "along" just one edge of P_{n-2}, i.e., if there is an edge u_iu_{i+1} of P_{n-2} such that $f(u_i) = g(u_{i+1})$, $f(u_{i+1}) = g(u_i)$, and $f(u) = g(u)$ for every $u \notin \{u_i, u_{i+1}\}$. Our first observation relates the labelling graph $L(P_{n-2})$ to $H_n(x)$.

Lemma 1. *$H_n(x) \cong L(P_{n-2})$ for $n \geq 3$.*

Proof. Let $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and let $x = v_{n-1}v_n$. The mapping $v_i \mapsto u_i$, $1 \leq i \leq n-2$ induces a bijection Φ of the vertex sets of $H_n(x)$ and $L(P_{n-2})$ which assigns to a hamiltonian cycle $C = (v_nv_{n-1}v_{i_1}v_{i_2}\dots v_{i_{n-2}})$ the labelling $f = \Phi(C)$ for which $f(u_j) = i_j$, $1 \leq j \leq n-2$. Moreover, it is easy to check that

C and C' are adjacent in $H_n(x)$ if and only if $\Phi(C)$ and $\Phi(C')$ are adjacent in $L(P_{n-2})$. □

By a result of [**RuS**] (see also [**RSZ**]) we know that there is a hamiltonian cycle in $L(P_{n-2})$ through any two specified edges. We thus have:

Corollary. *If $n \geq 4$ then for any two given edges of $H_n(x)$ there is a hamiltonian cycle in $H_n(x)$ containing the two edges.*

This corollary will be of central importance in the proof of the next theorem.

Theorem 2. *For $n \geq 4$ the graph $H_n = H_s(K_n)$ is edge-hamiltonian.*

Proof. As we have already seen, the statement is true for $n = 4$ or 5. We proceed as follows. Fix a vertex $u \in V(K_{n+1})$, $n \geq 5$, and consider a hamiltonian cycle, say, $(vuw\dots)$ in K_{n+1}. Suppressing the vertex u in the cycle yields the hamiltonian cycle $(vw\dots)$ in the graph $K_{n+1} - u \cong K_n$ which passes through the edge $x = vw$. Clearly, the subgraph H_n^x of H_{n+1} induced by those hamiltonian cycles of K_{n+1} that pass through the edges vu and uw is isomorphic to $H_n(x)$ (note again that $x = vw$). Moreover, if $y = v'w'$ is another edge not incident to u, then the subgraphs H_n^x and H_n^y are vertex-disjoint. We thus may put $V(H_{n+1}) = \cup_x V(H_n^x)$ where x runs through all edges of $K_n = K_{n+1} - u$; the union here is considered as a disjoint union.

Now, for convenience we assume that $V(K_{n+1}) = \{0, 1, \dots, n\}$ and that $u = 0$. Consider the following cyclic sequence S of all edges of $K_n = K_{n+1} - 0$:

$$(*) \qquad S = (1n, 1n-1, \dots, 12, 2n, 2n-1, \dots, 23, 3n, \dots, \dots, n-1n).$$

Note that any two consecutive edges in S are adjacent in K_{n+1}; this also holds for the last and first edge. In what follows, this sequence will play only an auxiliary role.

Let x and y be two consecutive edges in the cyclic ordering of S, say, $x = ij$ and $y = jk$. Consider the following four hamiltonian cycles of K_{n+1}: $C_1 = (j0ik\dots lm\dots)$, $C_2 = (j0ik\dots ml\dots)$, $D_1 = (j0ki\dots lm\dots)$, $D_2 = (j0ki\dots ml\dots)$. Obviously, (C_1, C_2, D_2, D_1) forms a 4-cycle in the graph H_{n+1}. Notice that by suppressing the vertex 0 we obtain the cycles $C_1(x) = (jik\dots lm\dots)$ and $C_2(x) = (jik\dots ml\dots)$ which are adjacent in H_n^x; the same holds true in H_n^y with respect to the cycles $D_1(y) = (jki\dots lm\dots)$ and $D_2(y) = (jki\dots ml\dots)$. For consecutive x and y in S we therefore have the following "local picture" of the graph H_{n+1} (we put $C_1(x)$ instead of C_1, etc., see Fig. 1).

Let us now do the same procedure with **each** of the $\binom{n}{2}$ consecutive pairs in our cyclic sequence S. Then, if x is an arbitrary edge of K_n, y is the successor of x, and z is the predecessor of x in S, the local picture of H_{n+1} extends to the one shown in Fig. 2.

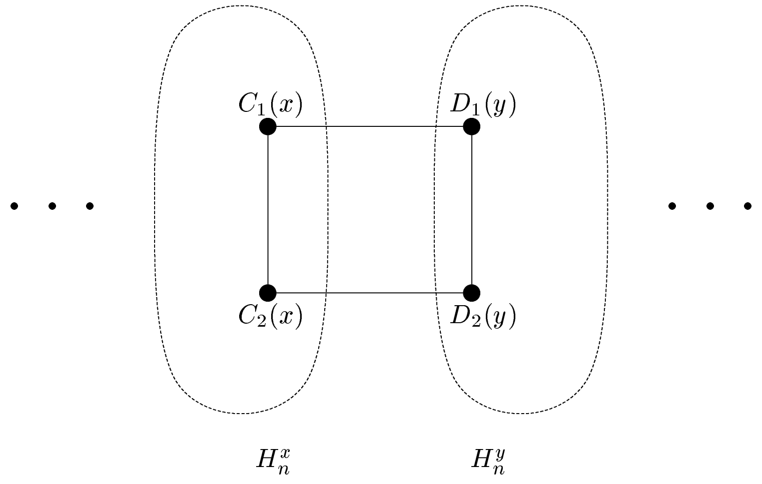

Figure 1. The local picture of H_{n+1}.

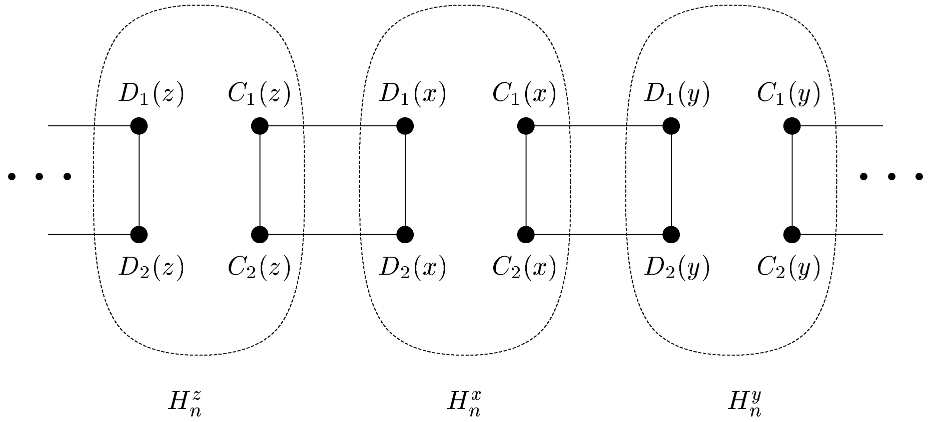

Figure 2.

By the Corollary, for each x in S there is a hamiltonian cycle $B(x)$ in H_n^x passing through the edges $C_1(x)C_2(x)$ and $D_1(x)D_2(x)$. For consecutive x, y in S let $Q(x,y)$ denote the quadrangle $(C_1(x)C_2(x)D_2(y)D_1(y))$. Consider the symmetric difference

$$F = (\bigcup_{x \in S} E(B(x))) \,\triangle\, (\bigcup_{xy} E(Q(x,y)))$$

where xy in the second union runs through all consecutive pairs in S except $1n$ and $n-1n$. Clearly, the edge set F induces a hamiltonian cycle in H_{n+1}.

It remains to show that for every given edge e of H_{n+1} there exists a hamiltonian cycle in H_{n+1} containing e. Consider first the case when e lies in H_n^z for some $z \in S$. Without loss of generality we may assume that $z = 1n$. Moreover, re-labelling the vertices of K_{n+1} (if necessary) we clearly may achieve that e is different from the edge $f = C_1(z)C_2(z)$. Then we proceed as above, with the only exception that for $B(z)$ we take a hamiltonian cycle in H_n^z passing through both e and f. The resulting hamiltonian cycle of H_n will contain e. Finally, let e be an edge traversing from H_n^x to H_n^y for some $x \neq y \in E(K_n)$, say, e joins the vertices $(j0i\dots)$ and $(l0k\dots)$. Then, without loss of generality, $j = l$ but $i \neq k$. Again, using a suitable re-labelling if necessary we may consider $x = ij$ and $y = kj$ to be consecutive in the ordering given by S. Therefore, we may identify e with, say, the edge $C_1(x)D_1(y)$. This completes the proof. $\square$

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H. Fleischner, Austrian Academy of Science, Vienna, Austria

P. Horák, Slovak Technical University, Bratislava, Slovakia

J. Širáň, Comenius University, Bratislava, Slovakia