Acta Math. Univ. Comenianae
Vol. LXII, 2(1993), pp. 191–195

ON THE MINIMAL REDUCTION AND MULTIPLICITY OF $(X^m, Y^n, X^kY^l, X^rY^s)$

E. BOĎA, D. ORSZÁGHOVÁ AND Š. SOLČAN

Abstract. An explicite formula to calculate the multiplicity of the ideal $(X^m, Y^n, X^kY^l, X^rY^s)\cdot A$ in $A = K[X,Y]_{(X,Y)}$ is given.

Let $K[X,Y]$ be a polynomial ring over a field K, $A = K[X,Y]_{(X,Y)}$ be a local ring with the maximal ideal $M = (X,Y)\cdot A$ and Q be an M-primary ideal in A. The multiplicity $e_0(Q,A)$ of Q in A is defined to be the leading coefficient of the Hilbert-Samuel polynomial $L_A(A/Q^t)$, $t \gg 0$ (see e.g. [**Z–S,** Vol. II, Chap. VIII, §10]).

Our main result is the following theorem.

Theorem. *Let* $Q = (X^m, Y^n, X^kY^l, X^rY^s)\cdot A$ *be an ideal in the local ring* $A = K[X,Y]_{(X,Y)}$. *Then*

$$e_0(Q,A) = \min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\}\,.$$

(assuming $m \geq n$, $m > k > r$ *and* $n > s > l$, *without loss of generality).*

The idea of counting the multiplicity $e_0(Q,A)$ is based on the notions "minimal reduction" and "analytic spread" introduced by Northcott and Rees, see [**N–R**]. Recall that an ideal $J \subseteq I$ of A is called a **reduction** of I, if $J\cdot I^{t-1} = I^t$ for an integer $t > 1$. If J is a reduction of I and $\dim(I) = 0$ then $\dim(J) = 0$ and $e_0(J,A) = e_0(I,A)$. If Q is an M-primary ideal in a local ring (A,M), then there exists an M-primary ideal $Q' \subseteq Q$ which is a reduction of Q such that Q' is generated by a system of parameters (see [**N–R**, §6, Theorem 2]).

In the following we show how to construct such a parametrical ideal Q'.

Let $Q = (X^m, Y^n, X^kY^l, X^rY^s)\cdot A$ in $A = K[X,Y]_{(X,Y)}$ and $A[X^mt, Y^nt, X^kY^lt, X^rY^st] = R_A(Q)$ be the Rees ring of A with respect to the

Received May 25, 1993.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 13H15, 13A30; Secondary 13B25.
Key words and phrases. Multiplicity, System of parameters, Reduction of an ideal, Rees ring.

ideal Q, $R_A(Q) = \underset{N\geq 0}{\oplus} Q^N t^N$. Let a, b, c, d be independent indeterminates over K and

$$\varphi\colon A[a,b,c,d] \twoheadrightarrow A[X^m t, Y^n t, X^k Y^l t, X^r Y^s t] = R_A(Q)$$

be the natural epimorphism which sends a, b, c, d onto $X^m t$, $Y^n t$, $X^k Y^l t$, $X^r Y^s t$ respectively.

Let J be an ideal contained in Ker φ, $J \subseteq$ Ker φ. Then there is an epimorphism

$$\varphi^*\colon A[a,b,c,d]/J \twoheadrightarrow A[X^m t, Y^n t, X^k Y^l t, X^r Y^s t]$$

and an epimorphism

$$\varphi'\colon A[a,b,c,d]/(J+M) \twoheadrightarrow R_A(Q)/M \cdot R_A(Q)$$

of factor rings $A[a,b,c,d]/(J+M) = K[a,b,c,d]/J'$ with $J' = (J+M)/M$ and

$$R_A(Q)/M \cdot R_A(Q) = \underset{N\geq 0}{\oplus} Q^N/M \cdot Q^N .$$

Let 's take J such that $\dim(J') = 2$. This is always possible for

$$\dim(R_A(Q)/M \cdot R_A(Q)) = l(Q) = 2$$

(by $l(Q)$ we denote the analytic spread of Q).

Let $\{\alpha, \beta\}$ be a system of parameters for $K[a,b,c,d]/J'$, i.e.

$$\dim(K[a,b,c,d]/(J',\alpha,\beta)) = 0 .$$

Then there exists an epimorphism Φ induced by φ'

$$\Phi\colon K[a,b,c,d]/(J',\alpha,\beta) \twoheadrightarrow R_A(Q)/(M,\overline{\alpha}t,\overline{\beta}t) \cdot R_A(Q) ,$$

with $\varphi(\alpha) = \overline{\alpha}t$, and $\varphi(\beta) = \overline{\beta}t$. Then the ring

$$R_A(Q)/(M,\overline{\alpha}t,\overline{\beta}t) \cdot R_A(Q) = \oplus\, Q^N/(M \cdot Q^N, (\overline{\alpha},\overline{\beta}) \cdot Q^{N-1})$$

is 0-dimensional. For (J',α,β) is (a,b,c,d)-primary and Φ is an epimorphism of graded rings, it follows that there is an integer $N_0 > 0$ such that for all $N > N_0$ it is $Q^N/(M \cdot Q^N, (\overline{\alpha},\overline{\beta}) \cdot Q^{N-1}) = 0$, i.e.

$$Q^N = M \cdot Q^N + (\overline{\alpha},\overline{\beta}) \cdot Q^{N-1} .$$

Then $Q^N/(\overline{\alpha},\overline{\beta}) \cdot Q^{N-1} = M \cdot Q^N/(\overline{\alpha},\overline{\beta}) \cdot Q^{N-1} = M \cdot (Q^N/(\overline{\alpha},\overline{\beta}) \cdot Q^{N-1})$. Using Nakayama's Lemma we get that $Q^N = (\overline{\alpha},\overline{\beta}) \cdot Q^{N-1}$.

Summarizing we can formulate a following result.

Proposition. *Let (A, M), Q, α, β be as above. Then the ideal $(\overline{\alpha}, \overline{\beta}) \cdot A$ is a reduction of Q and therefore $e_0(Q, A) = e_0((\overline{\alpha}, \overline{\beta}) \cdot A, A)$.*

Before proving the Theorem we prove an easy but useful lemma.

Lemma. *Let $m \geq n$, $m > k > r$ and $n > s > l$; all m, n, k, l, r, s are positive integers. Then*

(a) $mn = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $mn = \min\{mn, ml + nk, ms + nr\}$.

(b) $ml + nk = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $ml + nk = \min\{mn, ml + nk, ml + nr + ks - lr\}$.

(c) $ml + nr + ks - lr = \min\{mn, ml + nk, ms + nr, ml + nr + ks - lr\}$ *if and only if* $ml + nr + ks - lr = \min\{ml + nk, ms + nr, ml + nr + ks - lr\}$.

Proof.

(a) It is enough to prove the following:
If $mn \leq ml + nk$ and $mn \leq ms + nr$, then $mn \leq ml + nr + ks - lr$.

It is easy to see that the inequality $mn \leq ml + nk$ is equivalent to

$$\frac{m}{n}(n - l) \leq k \tag{1}$$

and $mn \leq ms + nr$ to

$$\frac{n}{m}(m - r) \leq s \tag{2}$$

From (1) and (2) we get $(n - l)(m - r) \leq ks$ and $mn \leq ml + nr + ks - lr$ as required.

(b) It is again enough to prove the implication:
If $ml + nk \leq mn$ and $ml + nk \leq ml + nr + ks - lr$, then $ml + nk \leq ms + nr$.

The first inequality is equivalent to $\frac{k}{n-1} \leq \frac{m}{n}$ and the second one to $\frac{k-r}{s-l} \leq \frac{k}{n-1}$. Then it follows $\frac{k-r}{s-l} \leq \frac{m}{n}$. But this is equivalent to $ml + nk \leq ms + nr$ as wanted.

(c) It is sufficient to prove that $ml+nr+ks-lr \leq ml+nk$ and $ml+nr+ks-lr \leq ms + nr$ imply $ml + nr + ks - lr \leq mn$.

The assumed inequalities are equivalent to

$$k \leq \frac{(k - r)(n - l)}{s - l} \qquad \text{and} \qquad s \leq \frac{(m - r)(s - l)}{k - r}.$$

Then $ks \leq (n - l)(m - r)$ and this is equivalent to $ml + nr + ks - lr \leq mn$. The proof of the lemma is complete. □

Now we are ready to prove the Theorem.

Proof of the Theorem. We will make it in 4 steps.

Step 1. Let $\min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\} = mn$.
By the previous lemma this is equivalent to $mn \le ml+nk$ and $mn \le ms+nr$. From the first inequality we get $nk \ge m(n-l)$ and

$$(X^kY^l)^n \in ((X^m)^{n-l} \cdot (Y^n)^l) \cdot A\,.$$

This implies that $c^n - a^{n-l}b^l \in J'$ (if $nk = m(n-k)$) or $c^n \in J'$. From the second inequality we have $nr \ge m(n-s)$ and

$$(X^rY^s)^n \in ((X^m)^{n-s} \cdot (Y^n)^s) \cdot A\,.$$

From this it follows that $d^n - a^{n-s}b^s \in J'$ (if $nr = m(n-s)$ or $d^n \in J'$.

Now J' is the ideal generated by u_1, v_1, $J' = (u_1, v_1)$, where u_1 is one of the elements c^n or $c^n - a^{n-l}b^l$ and v_1 equals to either d^n or $d^n - a^{n-s}b^s$. In all these cases the ideal (J', a, b) is (a,b,c,d)-primary and by the Proposition the ideal $(X^m, Y^n)\cdot A$ is a reduction of Q. Therefore $e_0(Q,A) = e_0((X^m,Y^n)\cdot A, A) = mn$.

Step 2. Now let $ml+nk = \min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\}$, i.e. $ml+nk \le mn$ and $ml+nk \le ml+nr+ks-lr$ by lemma. Then it follows $m(n-l) \ge nk$ and $(X^m)^{n-l}\cdot(Y^n)^l \in ((X^kY^l)^n)\cdot A$. Therefore either $a^{n-l}b^l - c^n \in J'$ (in case of equality $ml+nr+ks-lr = ml+nk$) or $a^{n-l}b^l \in J'$ from the first inequality and $r(n-l) \ge k(n-s)$, $(X^rY^s)^{n-l} \in ((Y^n)^{s-l} \cdot (X^kY^l)^{n-s}) \cdot A$ and therefore either $d^{n-l} - b^{s-l}c^{n-s} \in J'$ (if $ml+nr+ks-lr = ml+nk$) or $d^{n-l} \in J'$ from the second one. If $J' = (u_2, v_2)$, $u_2 \in \{a^{n-l}b^l - c^n, c^n\}$, $v_2 \in \{d^{n-l} - b^{s-l}c^{n-s}, d^{n-l}\}$, the ideal $J' = (J, a+b, c)$ is again (a,b,c,d)-primary and the ideal $(X^m+Y^n, X^kY^l)\cdot A$ is a reduction of Q by the Proposition. Therefore

$$\begin{aligned} e_0(Q,A) &= e_0((X^m+Y^n, X^kY^l)\cdot A, A) \\ &= e_0((X^m+Y^n, X^k)\cdot A, A) + e_0((X^m+Y^n, Y^l)\cdot A, A) \\ &= nk + ml\,. \end{aligned}$$

Step 3 is equivalent to the second one (changing the roles of $ml+nk$ and $ms+nr$).

Step 4. Let $ml+nr+ks-lr = \min\{mn, ml+nk, ms+nr, ml+nr+ks-lr\}$. This is again equivalent to

$$ml+nr+ks-lr = \min\{ml+nk, ms+nr, ml+nr+ks-lr\}\,.$$

From $ml+nr+ks-lr \le ml+nk$ one gets $k(n-s) \ge r(n-l)$. Then

$$(Y^n)^{s-l} \cdot (X^kY^l)^{n-s} \in ((X^rY^s)^{n-l}) \cdot A\,.$$

This implies either $b^{s-l}c^{n-s} - d^{n-l} \in J'$ (in case $ml + nr + ks - lr = ml + nk$) or $b^{s-l}c^{n-s} \in J'$.
From $ml + nr + ks - lr \leq ms + nr$ we have $s(m-k) \geq l(m-r)$ and

$$(X^m)^{k-r} \cdot (X^rY^s)^{m-k} \in ((X^kY^l)^{m-r}) \cdot A\,.$$

But this implies either $a^{k-r}d^{m-k} - c^{m-r} \in J'$ (if $ml + nr + ks - lr = ms + nr$) or $a^{k-r}d^{m-k} \in J'$.

Put $J' = (u_4, v_4)$, with $u_4 \in \{b^{s-l}c^{n-s}, b^{s-l}c^{n-s} - d^{n-l}\}$, $v_4 \in \{a^{k-r}d^{m-k}, a^{k-r}d^{m-k} - c^{m-r}\}$. If $ml + nr + ks - lr = ml + nk = ms + nr$ then $ml + nr + ks - lr = mn$ by the lemma and $e_0(Q, A) = mn = ml + nr + ks - lr$ by the step 1. In the rested cases the ideal $J' = (J, a + d, b + c)$ is (a, b, c, d)-primary, thus $(X^m + X^rY^s, Y^n + X^kY^l) \cdot A$ is a reduction of Q.

Therefore

$$\begin{aligned}
e_0(Q, A) &= e_0((X^m + X^rY^s, Y^n + X^kY^l) \cdot A, A) \\
&= e_0((X^r, Y^l) \cdot A, A) + e_0((X^{m-r} + Y^s, Y^l) \cdot A, A) \\
&\quad + e_0((X^r, Y^{n-l} + X^k) \cdot A, A) + e_0((X^{m-r} + Y^s, Y^{n-l} + X^k) \cdot A, A) \\
&= rl + (m-r)l + r(n-l) + sk = ml + nr + ks - lr
\end{aligned}$$

for $(X^{m-r} + Y^s, Y^{n-l} + X^k) \cdot A = (X^k, Y^s) \cdot A$.

The proof of the theorem is now complete. □

Acknowledgment. The authors would like to thank P. Schenzel, Halle, for stimulating discussions on the subject of this paper.

References

[**B–S**] Boďa E. and Solčan Š., *On the multiplicity of* $(X_1^m, X_2^n, X_1^kX_2^l)$, Acta Math. Univ. Comenianae **LII-LIII** (1987), 297–299.

[**N–R**] Northcott D. G. and Rees D., *Reductions of ideals in local rings*, Proc. of the Cambridge Phil. Soc. **50** (1954), 145–158.

[**Z–S**] Zariski O. and Samuel P., *Commutative algebra I, II.*, D. V. Nostrand Comp., Princeton, 1958 and 1960.

E. Boďa, Department of Geometry, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovakia

D. Országhová, Department of Mathematics, Agricultural University, 949 01 Nitra, Slovakia

Š. Solčan, Department of Geometry, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Slovakia