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DIVISION FOR STAR MAPS WITH THE BRANCHING POINT FIXED

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ABSTRACT. We extend the notion of division given for interval maps (see [**10**]) to the n-star and study the set of periods of star maps such that all their periodic orbits with period larger than one have a division. As a consequence of this result we get some conditions characterizing the star maps with zero topological entropy.

1. Introduction

The **n-star** is the subspace of the plane which is most easily described as the set of all complex numbers z such that z^n is in the unit interval $[0,1]$. We shall denote the n-star by $\mathbf{X}_n$. We shall also use the notation $\mathcal{X}_n$ to denote the class of all continuous maps from $\mathbf{X}_n$ to itself such that $f(0) = 0$.

We note that the 1-star and the 2-star are homeomorphic to a closed interval of the real line. Thus, in what follows, when talking about $\mathbf{X}_n$ or $\mathcal{X}_n$ we shall always assume that $n \geq 2$.

As usual, if $f \in \mathcal{X}_n$ we shall write f^k to denote $f \circ f \circ \cdots \circ f$ (k times). A point $x \in \mathbf{X}_n$ such that $f^k(x) = x$ but $f^j(x) \neq x$ for $j = 1, 2, \dots, k-1$ will be called a **periodic point of f of period k**. If x is a periodic point of f of period m then the set $\{f^k(x) : k > 0\}$ will be called a **periodic orbit** of f of period m (of course it has cardinality m).

The set of periods of all periodic points of a map $f \in \mathcal{X}_n$ will be denoted by $\operatorname{Per}(f)$.

In this paper we extend the notion of division given for interval maps (see [**10**]) and for maps from $\mathcal{X}_3$ (see [**4**]) to the n-star and we study the set of periods of maps from $\mathcal{X}_n$ such that all their periodic orbits have a division. As a consequence of this result we get some conditions characterizing the maps from $\mathcal{X}_n$ with zero topological entropy.

We start by fixing the notion of division. The components of $\mathbf{X}_n \setminus \{0\}$ will be called branches.

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Definition 1.1. Let $f \in \mathcal{X}_n$ and P be a periodic orbit of f with period larger than one. We say that P has a **division for** f if

(a) The orbit P lies in one branch and there is a partition of this branch into two connected sets W_1, W_2 such that $f(P \cap W_1) = P \cap W_2$ and $f(P \cap W_2) = P \cap W_1$.

(b) The orbit P lies in more than one branch and there exists a partition of $\mathbf{X}_n \setminus \{0\}$ into $p > 1$ sets $W_1, W_2, \dots, W_p$, which are union of branches, such that

$$f(P \cap W_i) = P \cap W_{i+1(\text{mod. } p)}, \; 1 \le i \le p.$$

The main result of this paper is the following.

Theorem 1.2. *Let $f \in \mathcal{X}_n$. Then* $\operatorname{Per}(f) \subset (\bigcup_{i=2}^n i \cdot \mathbf{N}) \cup \{1\}$ *if and only if each periodic orbit of f with period larger than one has a division.*

From the above theorem and its proof we easily obtain the following corollary which studies the set of periods of a map from $\mathcal{X}_n$ having a periodic orbit with no division (compare with Theorem 2.11 of [**5**]).

Corollary 1.3. *Let $f \in \mathcal{X}_n$ having a periodic orbit of period larger than one with no division. Then there exists $m \in \mathbf{N}$ such that* $\operatorname{Per}(f) \supset \{k \in \mathbf{N} : k \ge m\}$.

Similar problems for arbitrary tree maps will be studied in a forthcoming paper by the same authors.

As a consequence of the previous results we obtain a characterization of the maps from $\mathcal{X}_n$ with zero topological entropy. To state this result we need some more notation.

Topological entropy is a topological invariant to measure how a map mixes the points of the space by iteration. For a definition and basic properties of topological entropy see for instance [**1**] or [**8**].

The notion of simple periodic orbit with period a power of two was first introduced by Block in [**7**]. Here we will extend this notion to the case of $\mathbf{X}_n$ (see [**6**] for a generalization of this notion to trees). Let P be a subset of $\mathbf{X}_n$. We shall denote by $\operatorname{Span}(P)$ the smallest connected subset of $\mathbf{X}_n$ containing P.

Definition 1.4. Let $f \in \mathcal{X}_n$ have a periodic orbit P of period m. We say that P is **simple** if either it has period one or any set $Q \subset P$ satisfying that $\operatorname{Card}(Q) > 1$, $P \cap \operatorname{Span}(Q) = Q$ and $f^k(Q) = Q$ for some $1 \le k < m$ has a division for f^k.

Theorem 1.5. *Let $f \in \mathcal{X}_n$. Then the following statements are equivalent*

(a) *The topological entropy of f is zero.*

(b) *Every periodic orbit of f is simple.*

(c) $\operatorname{Per}(f) \subset \cup_{i=2}^n i \cdot \{1, 2, 2^2, \dots, 2^l, \dots\} \cup \{1\}$.

We note that the above theorem extends to maps from $\mathcal{X}_n$ a well known fact about interval maps (see for instance [**3**]). It is also a particular case of Theorem 2 of [**6**] for maps from $\mathcal{X}_n$. However, our proof of Theorem 1.5 is more direct than the one of Theorem 2 from [**6**] because we do not need to use the spectral decomposition of tree maps.

The following result, which is an easy corollary of Theorem 1.5, studies the set of periods of maps from $\mathcal{X}_n$ with positive topological entropy. It is a particular case of Corollary 3 of [**6**] for maps from $\mathcal{X}_n$ and of Theorem E from [**11**] for graph maps.

Corollary 1.6. *Let $f \in \mathcal{X}_n$. Then, f has positive topological entropy if and only if there exist $m, r \in \mathbf{N}$ such that* $\operatorname{Per}(f) \supset \{rk : k \geq m,\ k \in \mathbf{N}\}$.

The paper is organized as follows. In Section 2 we give some definitions and preliminary results. In Section 3 we prove Theorem 1.2 and Corollary 1.3. Lastly, in Section 4 we use the results proven in the previous section to show Theorem 1.5 and Corollary 1.6.

2. Preliminary Definitions And Results

To unify the notation we have to consider in a special way the case where a periodic orbit lies in one branch of $\mathbf{X}_n$. Assume that P is such a periodic orbit of $f \in \mathcal{X}_n$. Then f has a fixed point z in $\operatorname{Span}(P)$. Then we will force the point z to play the role of 0. According to this we shall call **branch** to any of the two connected components of $\mathbf{X}_n \setminus \{z\}$ (of course one of these branches is homeomorphic to $\mathbf{X}_n$). We note that, in this framework, **each periodic orbit of a map from $\mathcal{X}_n$ lies at least in two branches**.

For the space $\mathbf{X}_n$ the closed interval $[x, y]$ is defined to be $\operatorname{Span}(\{x, y\})$. Let P be a periodic orbit of $f \in \mathcal{X}_n$. Then the closures of components of $\operatorname{Span}(P) \setminus (P \cup \{0\})$ will be called **P-basic intervals**.

Assume now that $f \in \mathcal{X}_n$ and that P is a finite f−invariant set. We will say that f is P-linear if $f|_I$ is linear for each basic interval I and f is constant on each component of $\mathbf{X}_n \setminus \operatorname{Span}(P)$.

The following lemma relates the set of periods of a map with the set of periods of a P-linear version of it (for a proof see Corollary 2.5 of [**5**]).

Lemma 2.1. *Let $f \in \mathcal{X}_n$ have a periodic orbit P and let $g \in \mathcal{X}_n$ be a P-linear map such that $f|_P = g|_P$. Then* $\operatorname{Per}(f) \supset \operatorname{Per}(g)$.

If I and J are intervals, we say that I f−covers J if $f(I) \supset J$. In this case, we shall simply write $I \longrightarrow J$. If $f \in \mathcal{X}_n$ and P is a finite f-invariant set, then the **P-graph** of f is the oriented graph with all basic intervals as vertices and having an arrow from I to J if and only if I f−covers J.

A **path** of length k in a P-graph of f is a sequence of vertices $I_1, I_2, \dots, I_{k+1}$ such that I_i f-covers I_{i+1} for $i = 1, 2, \dots, k$. If $I_{k+1} = I_1$ we say this is a **loop** of length k. Such a loop will be written by $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_k \longrightarrow I_1$ and identified with all its **shifts**. That is, with each of the loops $I_i \longrightarrow I_{i+1} \longrightarrow \dots \longrightarrow I_1 \longrightarrow \dots I_{i+1} \longrightarrow I_i$.

Let $\alpha \colon L_0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow \dots \longrightarrow L_l$ and $\beta \colon K_0 \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \dots \longrightarrow K_k$ be two paths in a P-graph such that $L_l = K_0$. Then we shall write $\alpha\beta$ to denote the concatenation of α and β; that is, the path:

$$L_0 \longrightarrow L_1 \longrightarrow L_2 \longrightarrow \dots \longrightarrow L_l \longrightarrow K_1 \longrightarrow K_2 \longrightarrow \dots \longrightarrow K_k.$$

If α is a loop we shall also write α^n to denote $\alpha\alpha\dots\alpha$ (n times). The length of a path α will be denoted by $|\alpha|$. Of course we always have $|\alpha\beta| = |\alpha| + |\beta|$.

A loop which is not a repetition of any shorter loop will be called **nonrepetitive**. An **elementary loop** is a loop which cannot be formed by the concatenation of shorter loops. Of course each elementary loop is nonrepetitive.

The following lemma shows the relation between nonrepetitive loops and periodic orbits (for a proof see Lemmas 2.2, 2.4 and 2.6 of [**5**]).

Lemma 2.2. *Assume that $f \in \mathcal{X}_n$ and that P is a periodic orbit of f of period m.*

(a) *If α is a nonrepetitive loop of length k in the P-graph of f such that at least one of the intervals in the loop does not contain* 0 *then f has a periodic point of period k.*
(b) *If f is P-linear then for each basic interval there is a loop in the P-graph of f of length m passing through this interval.*

Let $f \in \mathcal{X}_n$, P a periodic orbit of f with period larger than one and B a branch in $\mathbf{X}_n$ such that $P \cap B \neq \emptyset$. We shall denote by sm_B the unique point from P such that $[0, sm_B]$ is a basic interval. Let $\mathcal{B}$ be the set of all branches from $\mathbf{X}_n$ which contain points of P. We define the map $\varphi \colon \mathcal{B} \longrightarrow \mathcal{B}$ such that, for each $B \in \mathcal{B}$, $\varphi(B)$ is the unique branch containing $f(sm_B)$. Since $\mathcal{B}$ is a finite set, the map φ has at least one periodic orbit. This notion plays an important role in [**5**]. It is used to define the **type** of a periodic orbit.

Remark 2.3. If a map $f \in \mathcal{X}_n$ has a periodic orbit P such that the map $\varphi \colon \mathcal{B} \longrightarrow \mathcal{B}$ has a periodic orbit of period t, then there exists an elementary loop of length t in the P-graph of f such that all basic intervals in this loop are adjacent to 0.

We shall denote by $(s_1, s_2, \dots, s_r)$ the greatest common divisor of the natural numbers $s_1, s_2, \dots, s_r$. The following result will be useful in the next section.

Proposition 2.4. *Let $f \in \mathcal{X}_n$. Then* $\operatorname{Per}(f) \subset (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$ *if and only if for each periodic orbit P of f with period $m \geq 2$ we have that $(s_1, s_2, \dots, s_r) > 1$,* *where $s_1, s_2, \dots, s_r$ are the lengths of all elementary loops in the P-graph of a P-linear map $g \in \mathcal{X}_n$ such that $f|_P = g|_P$.*

To prove this proposition we will need some technical lemmas. The following one follows easily from the proof of Lemma 6.1 of [**2**] (see also Proposition 2.2 of [**10**] or Lemma 2.1.6 of [**3**] for a version of these techniques for interval maps) and from the fact that each periodic orbit of a map from $\mathcal{X}_n$ lies at least in two branches (recall the unification of the notation made at the very beginning of this section).

In the sequel, the sets of natural numbers (without zero) and integer numbers will be denoted by $\mathbf{N}$ and $\mathbf{Z}$, respectively.

Lemma 2.5. *Let $f \in \mathcal{X}_n$ be a P-linear map, where P is a periodic orbit of f of period $m \geq 2$ such that one of the basic intervals f-covers itself. Then,* $\operatorname{Per}(f) \supset \{k \in \mathbf{N} : k \geq m\}$.

Lemma 2.6. *Let $f \in \mathcal{X}_n$ have a periodic orbit P and assume that f is P-linear. Let $S = (s_1, s_2, \dots, s_r)$, where $s_1, s_2, \dots, s_r$ are the lengths of all elementary loops in the P-graph of f. Then for each loop α in the P-graph of f we have that S divides $|\alpha|$.*

Proof. Since each loop is identified with all its shifts, the proof follows easily from the fact that each loop is a concatenation of elementary loops. ☐

Lemma 2.7. *Let $s_1, s_2, \dots, s_r$ with $r \geq 2$ be natural numbers such that $(s_1, s_2, \dots, s_r) = 1$. Then for each $q \in \mathbf{Z}$ and $N \in \mathbf{N}$ there exist natural numbers $n_1, n_2, \dots, n_r$ and k such that*

$$\sum_{i=1}^{r} n_i s_i = kN + q.$$

Proof. We will prove the lemma by induction on r. We start with $r = 2$. Take $\widetilde{n}_1$ and n_2' such that $n_2' s_2 = \widetilde{n}_1 s_1 + 1$. Then, choose $l, t \in \mathbf{N}$ such that $n_1' = lN - \widetilde{n}_1 \geq 0$, and $tN + q \geq 0$ and set $n_1 = (tN + q)n_1'$, $n_2 = (tN + q)n_2'$ and $k = (tN + q)ls_1 + t$. We have $n_1 s_1 + n_2 s_2 = (tN + q)[(n_1' + \widetilde{n}_1)s_1 + 1] = (tN + q)[lNs_1 + 1] = kN + q$.

Now suppose that the lemma holds for $r \geq 2$ and prove it for $r + 1$. Set $S = (s_1, s_2, \dots, s_r)$. Then $(S, s_{r+1}) = 1$ and $(s_1/S, s_2/S, \dots, s_r/S) = 1$. Take $\widetilde{n}_r$ and n_{r+1}' such that $n_{r+1}' s_{r+1} = \widetilde{n}_r S + 1$. By induction we get that

$$\frac{1}{S} \sum_{i=1}^{r} n_i' s_i = k'N - \widetilde{n}_r.$$

Therefore,

$$\sum_{i=1}^{r+1} n_i' s_i = S(k'N - \widetilde{n}_r) + \widetilde{n}_r S + 1 = Sk'N + 1.$$

Then, we choose again $t \in \mathbf{N}$ such that $tN + q \geq 0$ and we set $n_i = (tN + q)n_i'$ for $i = 1, 2, \dots, r + 1$ and $k = (tN + q)Sk' + t$. We have

$$\sum_{i=1}^{r+1} n_i s_i = (tN + q)(Sk'N + 1) = kN + q.$$

□

Proof of Proposition 2.4. Assume that $\mathrm{Per}\,(f) \subset (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$ and that P is a periodic orbit of f with period $m \geq 2$. By Lemma 2.5 we get that $s_i \geq 2$ for all $i = 1, 2, \dots, r$. Thus, if $r = 1$ we are done. Assume that $r \geq 2$ and fix $i \in \{1, 2, \dots, r\}$. Let α_i be the elementary loop

$$I_1^i \longrightarrow I_2^i \longrightarrow \cdots \longrightarrow I_{s_i}^i \longrightarrow I_1^i$$

of length s_i in the P-graph of g. Then, for $k \geq 0$, we define $V_k^i = g^k(I_1^i \cup I_2^i \cup \ldots \cup I_{m_i}^i)$. We note that $V_0^i \subset V_1^i \subset \ldots$ and, hence, by using the same arguments as in the proof of Lemma 6.1 of [**2**] we get that there exists a path from I_1^i to $I_1^{i+1(\mathrm{mod.}\ r)}$ in the P-graph of g. Let β_i be the shortest such path. In view of the fact that g is P-linear, β_i has an interval which does not contain 0.

Assume now that $(s_1, s_2, \dots, s_r) = 1$. By Lemma 2.7 there exist $n_1, n_2, \dots, n_r$ and k such that

$$\sum_{i=1}^{r} n_i s_i = k(n!) + 1 - \sum_{i=1}^{r} |\beta_i|.$$

Then look at the loop $\alpha = \alpha_1^{n_1}\beta_1\alpha_2^{n_2}\beta_2 \ldots \alpha_r^{n_r}\beta_r$. It has length $\sum_{i=1}^{r} n_i s_i + \sum_{i=1}^{r} |\beta_i| = k(n!) + 1$. By construction the loop α is nonrepetitive and at least one of its intervals does not contain 0. Hence, by Lemmas 2.2(a) and 2.1 we get that f has a periodic orbit of period $k(n!) + 1 \notin (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$; a contradiction.

Now we prove the converse. Fix a periodic orbit P of f with period $m \geq 3$ and let $s_1, s_2, \dots, s_r$ be the lengths of all elementary loops in the P-graph of g. By assumption we have $S = (s_1, s_2, \dots, s_r) > 1$. In view of Remark 2.3 we see that there exists j such that $s_j \leq n$. Thus, $1 < S \leq n$. Then, by Lemma 2.2(b) we get that there exists a loop of length m in the P-graph of g and, by Lemma 2.6, S divides m. Hence, $m \in (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$. Therefore, $\mathrm{Per}\,(f) \subset (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$.□

3. The Set of Periods

This section will be devoted to prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2. Assume $f \in \mathcal{X}_n$. If each periodic orbit of f with period larger than one has a division, then it follows trivially that $\mathrm{Per}\,(f) \subset (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$.

To prove the converse, suppose that $\mathrm{Per}\,(f) \subset (\bigcup_{i=2}^{n} i \cdot \mathbf{N}) \cup \{1\}$. Let P be a periodic orbit of f. Since the fact that P has or has not a division depends only on $f|_P$, it does not matter whether we work with the map f itself or with the P-linear map $\widetilde{f} \in \mathcal{X}_n$ such that $\widetilde{f}|_P = f|_P$. Moreover, in view of Lemma 2.1, we have that $\mathrm{Per}(\widetilde{f}) \subset \mathrm{Per}(f) \subset (\bigcup_{i=2}^n i \cdot \mathbf{N}) \cup \{1\}$. So, in the rest of the proof, we may assume without loss of generality that f is P-linear. Let $s_1, s_2, \dots, s_r$ be the lengths of all elementary loops in the P-graph of f. By Proposition 2.4 we get that $S = (s_1, s_2, \dots, s_r) > 1$.

Let $m_1, m_2, \dots, m_l$ be the periods of all cycles of the map $\varphi \colon \mathcal{B} \longrightarrow \mathcal{B}$ defined in Section 2. By Remark 2.3 we have that $\{m_1, m_2, \dots, m_l\} \subset \{s_1, s_2, \dots, s_r\}$. Hence, S divides m_i for each $i = 1, 2, \dots, l$.

Suppose that the cycle of φ of period m_i is $\{B_1^i, B_2^i, \dots, B_{m_i}^i\}$, for each $i = 1, 2, \dots, l$. We may assume that $\varphi(B_j^i) = B_{j+1 \text{ (mod. } m_i)}^i$ for each $i = 1, 2, \dots, l$. Then, by Remark 2.3, there exists an elementary loop $\alpha_i : I_1^i \longrightarrow I_2^i \longrightarrow \cdots \longrightarrow I_{m_i}^i \longrightarrow I_1^i$ such that $I_j^i \subset B_j^i$ for $j = 1, 2, \dots, m_i$ (we recall that all intervals I_j^i are adjacent to 0).

By using the same arguments as in the proof of Proposition 2.4 we get that there exists a shortest path β_1 from I_1^1 to I_1^2 in the P-graph of f. By adding some basic intervals from α_2 to β_1, if necessary, and relabeling the intervals of α_2 (and the corresponding branches) we can arrange that β_1 is still a path from I_1^1 to I_1^2, $|\beta_1|$ is a multiple of S and $I_j^i \subset B_j^i$ for $j = 1, 2, \dots, m_i$. In a similar way we can obtain paths β_i from I_1^i to I_1^{i+1} such that $|\beta_i|$ is a multiple of S and $I_j^{i+1} \subset B_j^{i+1}$ for $j = 1, 2, \dots, m_i$ and $i = 2, 3, \dots, l-1$. Now, let β_l be the shortest path in the P-graph of f from I_1^l to I_1^1. Consider the loop $\beta_1 \beta_2 \dots \beta_l$. By Lemma 2.6 we get that S divides $|\beta_1 \beta_2 \dots \beta_l|$. Hence, since S divides $|\beta_i|$ for $i = 1, 2, \dots, l-1$, we get that S also divides $|\beta_l|$.

Now, for $1 \le i \le l$ and $1 \le j \le S$ we set (see Figure 3.1)

$$\mathcal{B}_j^i = \{B \in \mathcal{B} : \varphi^k(B) = B_{j+k \text{ (mod. } s)}^i \quad \text{for} \quad k \in \mathbf{N}\}$$

(note that, in particular, $\mathcal{B}_j^i \supset \{B_k^i : 1 \le k \le m_i \text{ and } k \equiv j \text{ (mod. } S)\}$). Then, for $1 \le j \le S$, we define W_j as the union of all branches in $\mathcal{B}_j^i$ for $i = 1, 2, \dots, l$. We would like that the sets W_j form a partition of $\mathbf{X}_n \setminus \{0\}$. To achieve this we simply add to any of these sets the union of all branches which do not contain any point of P. In such a way we finally get a partition of $\mathbf{X}_n \setminus \{0\}$ into S nonempty subsets such that each one consists on a union of branches. We note that in view of the definition of the sets W_j we have that, for each $j \in \{1, 2, \dots, S\}$, all intervals of the form I_k^i with $k \equiv j$ (mod. S) are contained in W_j.

We claim that this is the partition we are looking for. That is, we have to show that $f(P \cap W_j) = P \cap W_{j+1 \text{ (mod. } S)}$, $1 \le j \le S$. Assume the contrary. Then there exist $k \in \{1, 2, \dots, S\}$ and $x \in P \cap W_k$ such that $f(x) \notin W_{k+1 \text{ (mod. } S)}$. Let B be the branch of $\mathbf{X}_n$ where x lies. By construction we have that $f(sm_B) \in W_{k+1 \text{ (mod. } S)}$. Therefore, there exists a basic interval $K = [y, z]$ contained in B such that $f(z) \in W_q$ with $q \not\equiv k+1$ (mod. S) but $f(y) \in W_{k+1 \text{ (mod. } S)}$ (see

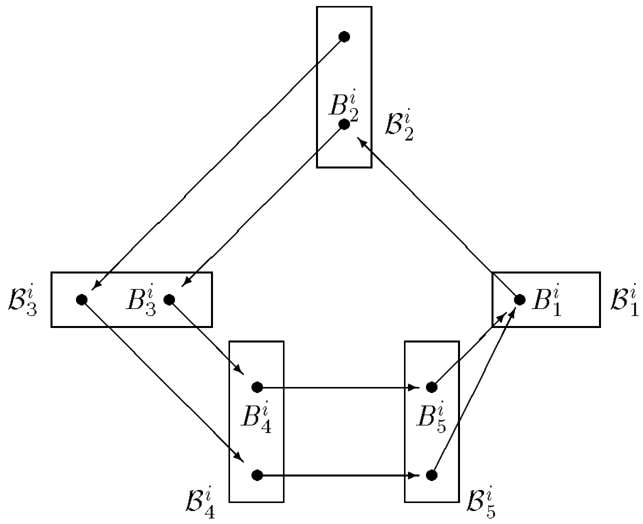

Figure 3.1. The construction of the partition (here $S = m_i = 5$).

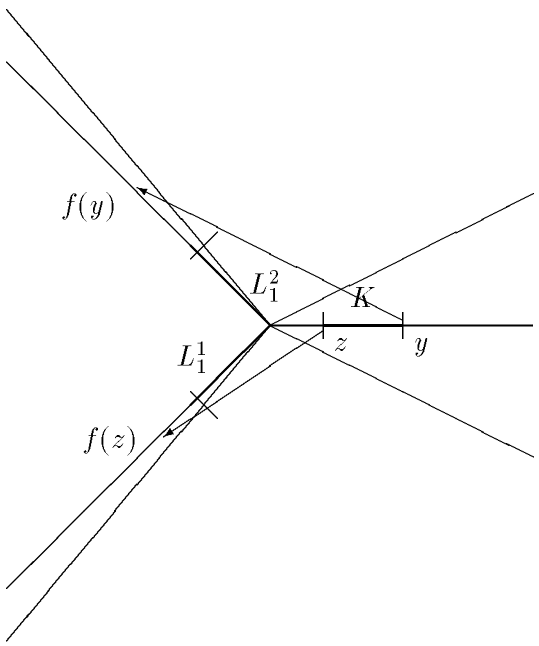

Figure 3.2. A no division case.

Figure 3.2). Let L_1^1 (resp. L_1^2) be the basic interval which is adjacent to 0 in the branch where $f(z)$ (resp. $f(y)$) lies. Then, by the definitions of φ and the sets W_j, we have that there exist a path γ_1 (resp. γ_2) from L_1^1 (resp. L_1^2) to an interval $I_{j_1}^{i_1}$ (resp. $I_{j_2}^{i_2}$) such that $j_1 \equiv q \pmod{S}$ (resp. $j_2 \equiv k+1 \pmod{S}$). We note that, again by the definition of the sets W_j, we have that S divides $|\gamma_1|$ and $|\gamma_2|$ (see Figure 3.1).

Now, let γ_3 be the a path from $I_{j_1}^{i_1}$ to $I_{j_2}^{i_2}$ (it can be constructed by connecting some pieces of the elementary loops α_{i_1} and α_{i_2} with β_i for $i = i_1, i_1 + 1 \pmod{S}, i_1 + 2 \pmod{S}, \ldots, i_2$). Finally, by using again the construction in the proof of Proposition 2.4, there exists a path γ_4 from $I_{j_2}^{i_2}$ to K (see Figure 3.3).

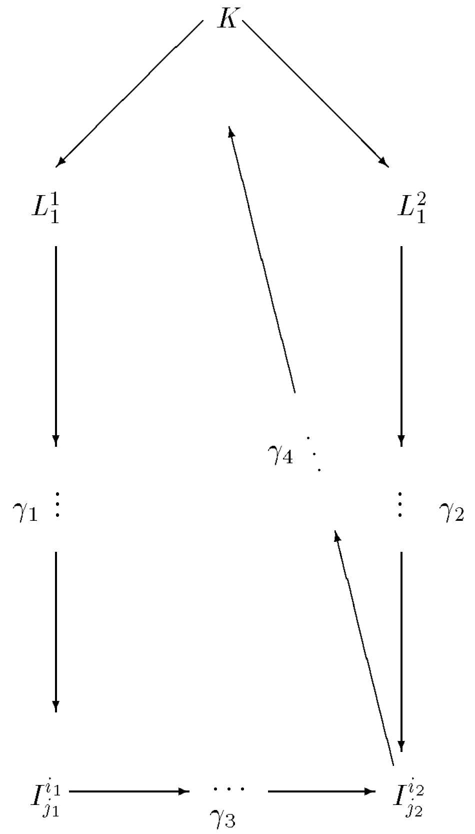

Figure 3.3. Some typical paths in the P-graph of f.

Now we look at the loops $(K \longrightarrow L_1^2)\gamma_2\gamma_4$ and $(K \longrightarrow L_1^1)\gamma_1\gamma_3\gamma_4$. Since they have length a multiple of S we get that $|\gamma_4| \equiv S-1 \pmod{S}$ and $1 + |\gamma_3| + |\gamma_4| \equiv$

0 (mod. S). This implies that $|\gamma_3| \equiv 0$ (mod. S) but this is impossible because $j_1 \not\equiv j_2$ (mod. S) (recall the construction of the loop γ_3 and that all paths β_i have length a multiple of S). $\square$

Proof of Corollary 1.3. Let P be a periodic orbit of f of period larger than one having no division. By Lemma 2.1 we can assume that f is P-linear. Let now $s_1, s_2, \dots, s_r$ be the lengths of all elementary loops in the P-graph of f. If $s_i = 1$ for some $i \in \{1, 2, \dots, r\}$ then, by Lemma 2.5, we are done. Thus, we assume that $s_i \geq 2$ for each $i = 1, 2, \dots, r$. By Theorem 1.2 and Proposition 2.4 we get that $(s_1, s_2, \dots, s_r) = 1$ and $r \geq 2$. Hence, for $i = 1, 2, \dots, r$ and $l = 0, 1, \dots, s_r - 1$ there exist $k_i^l \in \mathbf{Z}$ such that $\sum_{i=1}^r k_i^l s_i = l$ (where we fix $k_i^0 = 0$ for $i = 1, 2, \dots, r$). Now, take $k_i > \max\{|k_i^l| : l = 0, 1, \dots, s_r - 1\}$.

We also will use the notation from the proof of Proposition 2.4. That is, let $\alpha_i = I_1^i \longrightarrow I_2^i \longrightarrow \cdots \longrightarrow I_{s_i}^i \longrightarrow I_1^i$ be the elementary loop of length s_i in the P-graph of f and let β_i be the shortest path from I_1^i to $I_1^{i+1(\text{mod. } r)}$ (recall that such a path has an interval which does not contain 0).

We set

$$m = \sum_{i=1}^r k_i s_i + |\beta_i|.$$

Then, for each $k \geq m$ we write $k - m = t s_r + l$ with $t \geq 0$ and $l \in \{0, 1, \dots, s_r - 1\}$ and set $n_i = k_i + k_i^l$ for $i = 1, 2, \dots, r-1$ and $n_r = k_r + k_r^l + t$. Clearly, $k_i > 0$ for each $i = 1, 2, \dots, r$ and

$$k = \sum_{i=1}^r n_i s_i + |\beta_i|.$$

Then, the loop $\alpha_1^{n_1} \beta_1 \alpha_2^{n_2} \beta_2 \dots \alpha_r^{n_r} \beta_r$ has length k, is nonrepetitive and at least one of its intervals does not contain 0. Thus, by Lemma 2.2(a), we get that f has a periodic orbit of period k. This ends the proof of the corollary. $\square$

4. Topological Entropy

In this section we shall prove Theorem 1.5 and Corollary 1.6. The following result will be useful in this task. It follows from the proof of Theorem 2.11 of [**5**] (see also [**6**] and [**11**]).

Lemma 4.1. *Let $f \in \mathcal{X}_n$ and let P be a periodic orbit of f with period larger than one. If P has no division then the topological entropy of f, $h(f)$, is positive.*

Let $f \in \mathcal{X}_n$. We will denote by $P(f)$ the set of all periodic points of f. To get the main result of this section we need the following lemma from [**13**] (for a definition and main properties of the center of a map see for instance [**9**] or [**12**]).

Lemma 4.2. *The center of* $f \in \mathcal{X}_n$ *is* $\overline{P(f)}$.

Now we are ready to prove Theorem 1.5.

Proof of Theorem 1.5. Since $h(f^k) = kh(f)$ for each $k \in \mathbf{N}$, from Lemma 4.1 and the definition of a simple orbit, it follows that (a) implies (b).

To prove that (b) implies (c) we have to show that if P is a simple periodic orbit of f, then its period is of the form $i \cdot 2^l$ with $1 \leq i \leq n$ and $l \in \mathbf{N} \cup \{0\}$. Then, without loss of generality we may assume that f is P-linear. If P has period equal to one we are done. Then we assume that P has period m larger than one. Assume first that P lies in one branch. By the definition of a simple orbit and of a division in one branch, it follows easily that the period of P is of the form $2^j = 2 \cdot 2^{j-1} \in \cup_{i=2}^n i \cdot \{1, 2, 2^2, \dots, 2^l, \dots\} \cup \{1\}$.

Assume now that P lies in more than one branch. In view of the definition of a simple orbit P has a division. Then there exists $P_1 \subset P$ such that $P \cap \operatorname{Span}(P_1) = P_1$, P_1 lies on the smallest possible number of branches and $f^{k_1}(P_1) = P_1$ for some $k_1 \in \mathbf{N}$. Let m_1 be the period of P_1 by f^{k_1}. Then, clearly, $m = m_1 k_1$ and P_1 lies in at most n/k_1 branches. Again by the definition of a simple orbit, P_1 has a division for f^{k_1}. Thus, we can continue this process until we get a periodic orbit P_j of $f^{k_1 k_2 \cdots k_j}$ of period m_j, for some $j \geq 1$, lying in one branch. By construction we have $m = k_1 k_2 \cdots k_j m_j$ and $k_1 k_2 \cdots k_j \leq n$. Since P_j is a periodic orbit of $f^{k_1 k_2 \cdots k_j}$ lying in one branch, we know that m_j is a power of two. Thus $m \in \cup_{i=2}^n i \cdot \{1, 2, 2^2, \dots, 2^l, \dots\} \cup \{1\}$. On the other hand, by Corollary 1.6, it is not difficult to show that (c) implies (b).

To end the proof of this theorem we need to show that (b) implies (a). Let $g = f^N$ with $N = n!(n-1)! \dots 2!$. Then, by using the same techniques as above, it is not difficult to show the following facts

(i) The period of any periodic orbit of g is a power of 2 (perhaps one).
(ii) Every periodic orbit of g lies only on one branch.

Now let ρ_i be the natural retraction from $\mathbf{X}_n$ to the branch B_i, $1 \leq i \leq n$. Then, using Lemma 4.2 and the well known fact that Theorem 1.5 holds for interval maps (see for instance [**3**]), we get

$$\begin{aligned} h(f) &= 1/N \cdot h(g) = 1/N \cdot h(g|_{\overline{P(f)}}) \\ &= \max_{1 \leq i \leq n} 1/N \cdot h(\rho_i \circ g|_{\overline{P(f)} \cap B_i}) \\ &= \max_{1 \leq i \leq n} 1/N \cdot h(\rho_i \circ g|_{B_i}) = 0. \end{aligned}$$

□

Proof of Corollary 1.6. If $h(f) > 0$, then in view of Theorem 1.5 there exists a periodic orbit P of f which is not simple. Thus, there exist $r > 0$ and $\widetilde{P} \subset P$ such that $P \cap \operatorname{Span}(\widetilde{P}) = \widetilde{P}$ and $\widetilde{P}$ is a periodic orbit of f^r with no division. Thus, by

Corollary 1.6, $\mathrm{Per}(f^r) \supset \{k \in \mathbf{N} : k \geq m\}$ for some $m \in \mathbf{N}$. Hence, $\mathrm{Per}(f) \supset \{rk : k \geq m,\ k \in \mathbf{N}\}$. On the other hand, if $\mathrm{Per}(f) \supset \{rk : k \geq m,\ k \in \mathbf{N}\}$, then $h(f) > 0$ by Theorem 1.5. □

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