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# ENTROPY–MINIMALITY

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In this note, we introduce a dynamical property of continuous maps, which we call **entropy-minimality**, lying between minimality and topological transitivity. We pay special attention to maps of the interval, showing that topological transitivity implies entropy-minimality for piecewise monotone maps but not for maps of the interval in general.

Let $f\colon X \to X$ be a continuous self-map of a compact metric space. Recall that $f$ is **minimal** if the only nonempty, closed, $f$-invariant subset of $X$ is $X$ itself, and $f$ is **topologically transitive** if the only closed, $f$-invariant subset of $X$ with nonempty interior is $X$ itself. We say that $f$ is **entropy-minimal** if the only nonempty, closed, $f$-invariant subset $Y$ of $X$ such that $\mathrm{ent}\,(f|Y) = \mathrm{ent}\,(f)$ is $Y = X$. (Here $\mathrm{ent}\,(\cdot)$ denotes topological entropy [**AKM**].)

Clearly every minimal map is entropy-minimal. The converse is false. Any topologically transitive, piecewise monotone map of the interval provides a counterexample (see Theorem 2 below), as does any infinite, topologically transitive shift of finite type.

**Theorem 1.** *Every entropy-minimal map is topologically transitive.*

*Proof.* Let $f\colon X \to X$ be an entropy-minimal map. Let $\Omega = \Omega(f)$ denote the nonwandering set of $f$, defined by $x \in \Omega$ if and only if for every open set $U$ containing $x$, there exists $n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. $\Omega$ is nonempty, closed, $f$-invariant, and [**W**, Corollary 8.6.1(iii)] $\mathrm{ent}\,(f) = \mathrm{ent}\,(f|\Omega)$. Therefore $\Omega = X$. By [**GH**, Theorem 7.21], $\Omega(f^n) = X$ for every $n \geq 1$.

We use the following equivalent formulation of topological transitivity: $f$ is topologically transitive if and only if for every nonempty open set $U$, $\mathrm{cl} \cup_{n\geq 1} f^n(U) = X$. For ease of notation, if $E$ is a subset of $X$, we write $E^*$ in place of $\mathrm{cl} \cup_{n\geq 1} f^n(E)$. If $f$ is not topologically transitive, there exists a nonempty open set $U$ such that $U^* \neq X$. Let $V = X - U^*$. Since $X = U^* \cup V^*$, we have $\mathrm{ent}\,(f) = \max\{\mathrm{ent}\,(f|U^*), \mathrm{ent}\,(f|V^*)\}$ [**AKM**, Theorem 4]. Since $U^* \neq X$, we have $\mathrm{ent}\,(f|U^*) < \mathrm{ent}\,(f)$. Therefore $\mathrm{ent}\,(f|V^*) = \mathrm{ent}\,(f)$ and hence $V^* = X$.

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From the equivalent formulation of topological transitivity, there exists $n \geq 1$ such that $f^n(V) \cap U \neq \emptyset$. Let $W$ be a nonempty open subset of $V$ such that $f^n(W) \subseteq U$. Then $f^{kn}(W) \subseteq U^*$ for every $k \geq 1$. Since $U^* \cap V = \emptyset$, we have $f^{kn}(W) \cap W \subseteq f^{kn}(W) \cap V = \emptyset$ for every $k \geq 1$. But then no point of $W$ is in $\Omega(f^n)$. □

We now turn to the question: when does topological transitivity imply entropy-minimality? Recall that an $f$-invariant, Borel probability measure $\mu$ on $X$ is called a **measure of maximal entropy** if $\operatorname{ent}_\mu(f) = \operatorname{ent}(f)$. Here $\operatorname{ent}_\mu(f)$ denotes the measure-theoretic entropy [**W**] of the system $(X, f, \mu)$.

**Theorem 2.** *Every topologically transitive, piecewise monotone map of the interval is entropy-minimal.*

*Proof.* Let $f\colon [a,b] \to [a,b]$ be such a map. By [**P**, Corollary 3], $f$ is topologically conjugate to a piecewise linear map, each of whose linear pieces has slope $\pm\beta$, where $\operatorname{ent}(f) = \log \beta$. Without loss of generality, we may assume that $f$ itself has this property and hence satisfies the hypotheses of [**H**]. By [**H**, Theorem 8], $f$ has a unique measure $\mu$ of maximal entropy and $\mu$ is positive on nonempty open sets.

Let $a = a_0 < \cdots < a_n = b$, where the intervals $[a_{i-1}, a_i]$ are maximal with respect to "$f$ is monotone on $J$", and let $A = \{a_1, \ldots, a_{n-1}\}$. For $x \in [a,b] - \cup_{j\geq 0} f^{-j}(A)$, define $\varphi(x) \in \prod_0^\infty \{1, \ldots, n\}$ by $[\varphi(x)]_j = i$ if and only if $f^j(x) \in [a_{i-1}, a_i]$. The map $\varphi^{-1}$ is uniformly continuous on $\varphi([a,b] - \cup_{j\geq 0} f^{-j}(A))$ and so extends to a continuous map $\psi$ from $\Sigma = \operatorname{cl} \varphi(x \in [a,b] - \cup_{j\geq 0} f^{-j}(A))$ onto $[a,b]$. Then $\#\psi^{-1}(x) = 1$ or $2$ for every $x \in [a,b]$ and $\psi \circ \sigma = f \circ \psi$, where $\sigma$ is the shift on $\Sigma$.

Let $X$ be a closed, $f$-invariant subset of $[a,b]$ and let $\Sigma' = \psi^{-1}(X)$. Then [**W**, Theorems 8.2, 8.7(v)] $\sigma|\Sigma'$ has a (not necessarily unique) measure $\nu'$ of maximal entropy. Let $\nu$ be the measure defined on $X$ by $\nu(E) = \nu'(\psi^{-1}(E))$. Then $\operatorname{ent}(f|X) = \operatorname{ent}(\sigma|\Sigma') = \operatorname{ent}_{\nu'}(\sigma|\Sigma') = \operatorname{ent}_\nu(f|X)$, the first and last equalities because finite-to-one factor maps preserve topological entropy [**B**, Theorem 17], [**NP**, Corollary to Lemma 1]. Extend $\nu$ to all of $[a,b]$ by defining $\nu([a,b]-X) = 0$. If $X \neq [a,b]$, then $\nu \neq \mu$, and so $\operatorname{ent}_\nu(f|X) < \operatorname{ent}_\mu(f) = \operatorname{ent}(f)$. □

The proof above contains the easy proof of the following statement: **if a shift has a unique measure of maximal entropy, then the restriction of the shift to the support of this measure is entropy-minimal and has the same entropy as the original shift.** The converse is false: consider any minimal shift with entropy zero which has more than one invariant measure. See, for example, [**O**].

Below is an example which shows that Theorem 2 need not hold if the map is not piecewise monotone. Our example is a modified version of the map constructed by M. Barge and J. Martin [**BM**, Example 3]. It is defined on $[0,1]$ and has the property that for every $\varepsilon > 0$, there is a closed, $f$-invariant set $X_\varepsilon \subseteq [0, \varepsilon]$ such that $\operatorname{ent}(f|X_\varepsilon) = \operatorname{ent}(f)$. B. Gurevich and A. Zargaryan [**GZ**] used a similar construction to produce a map of the interval with no entropy-maximizing measure.

**Example.** Let $(a_n)$ be a doubly infinite increasing sequence such that $\lim_{n\to-\infty} a_n = 0$ and $\lim_{n\to\infty} a_n = 1$. Let $f\colon [0,1] \to [0,1]$ be a map such that $f(0) = 0, f(1) = 1$, and for all $n$, $f(a_n) = (a_n)$ and $f$ maps $[a_n, a_{n+1}]$ piecewise linearly onto $[a_{n-1}, a_{n+2}]$ with three linear pieces, as in Figure 1.

**Figure 1.**

As in [**BM**], it is easy to show that $f$ is topologically transitive. We show that $\operatorname{ent}(f) = \log 5$ and that $f$ is not entropy-minimal.

For $k = 2, 3, \ldots$, let

$$X_k = \{x \in [0,1] : f^i(x) \in [a_{-k}, a_k] \quad \text{for } i = 0, 1, \ldots\}.$$

Then $\operatorname{ent}(f) \geq \limsup_{k\to\infty} \operatorname{ent}(f|X_k)$, and $\operatorname{ent}(f|X_k) = \operatorname{ent}(f_k)$, where $f_k\colon [0,1] \to [0,1]$ is defined by

$$f_k(x) = \begin{cases} a_{-k}, & \text{if } f(x) \leq a_{-k}; \\ f(x), & \text{if } a_{-k} \leq f(x) \leq a_k; \\ a_k, & \text{if } f(x) \geq a_k. \end{cases}$$

Since $f_k \to f$ and entropy is $C^0$ lower semicontinuous [**M**, Theorem 2], $\operatorname{ent}(f) \leq \liminf_{k\to\infty} \operatorname{ent}(f_k)$. It follows that $\operatorname{ent}(f) = \lim_{k\to\infty} \operatorname{ent}(f_k)$.

Now $f_k = f$ on $[a_{-k+1}, a_{k-1}]$, and on $[a_{-k}, a_{-k+1}]$ and $[a_{k-1}, a_k]$, the graphs of $f_k$ are as in Figure 2.

**Figure 2.**

By [**ALM**, Theorem 4.4.5], $\operatorname{ent}(f_k)$ is the logarithm of the spectral radius, denoted $\rho(\cdot)$, of the $(2k+1)\times(2k+1)$ matrix $B_k = (b_{i,j})$, indexed by $\{-k, \dots, k\}$ and defined by

$$\begin{aligned} b_{i,i} &= 1\,, \\ b_{i,i-1} &= b_{i,i+1} = 2\,, \\ b_{i,j} &= 0 \quad \text{otherwise.} \end{aligned}$$

We show that $5 - \frac{4}{k+1} \le \rho(B_k) \le 5$, from which it follows that $\operatorname{ent}(f) = \log 5$. We use the fact from Perron-Frobenius theory (see, for example, [**S**]) that for any irreducible nonnegative matrix $B$ and any positive vector $\mathbf{v} = (v_i)$,

$$\min_i \frac{(B\mathbf{v})_i}{v_i} \le \rho(B) \le \max_i \frac{(B\mathbf{v})_i}{v_i}\,.$$

It is clear that $B_k$ is irreducible. Setting $v_i = 1$ gives $\rho(B_k) \le 5$. To prove the other inequality, set

$$v_i = \begin{cases} k+1+i, & i \le 0\,; \\ k+1-i, & i \ge 0\,. \end{cases}$$

Then

$$\frac{(B\mathbf{v})_i}{v_i} = \begin{cases} 5, & i \ne 0; \\ 5 - \frac{4}{k+1}\,, & i = 0\,. \end{cases}$$

To show that $f$ is not entropy-minimal, let

$$X = \{x \in [0,1] : f^i(x) \le a_0 \quad \text{for } i = 0, 1, \dots\}.$$

As above, $\operatorname{ent}(f|X) = \log 5$.

Replacing $a_0$ by $a_{-m}$ in the definition of $X$ yields the statement that the entropy of $f$ is concentrated on arbitrarily small closed intervals containing 0.

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