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RECONSTRUCTION OF GRAPHS WITH CERTAIN DEGREE–SEQUENCES

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Introduction

The purpose of this note is to present some results concerning the Kelly-Ulam conjecture for graphs whose degree-sequences satisfy some extra conditions. Throughout, the graphs considered are finite, simple and undirected.

A subgraph of G obtained by deleting a vertex v together with all edges incident with v will be referred to as a **vertex-deleted subgraph** and denoted be $G - v$.

A graph G^* will be called a **reconstruction** of a graph G if there is bijection $f\colon V(G) \to V(G^*)$ such that $G - u$ is isomorphic to $G^* - f(u)$ for all $u \in V(G)$. A graph G will be called **reconstructible** if each of its reconstructions is isomorphic to G. The famous Kelly-Ulam reconstruction conjecture (see [**1**], [**2**], [**3**]) states that any graph with more than two vertices is reconstructible. The conjecture is apparently very hard, and only a few classes of reconstructible graphs are known (cf. [**3**], [**4**] for a survey).

It is easily seen that regular graphs are reconstructible. Going a step further, Bondy and Hemminger [**3**] define a vertex v of G to be bad if G has some vertex of degree $d(v) - 1$, and remark that a graph with at least 3 vertices is reconstructible whenever it contains a vertex with no bad neighbours. This result was extended by Širáň in [**5**] by proving that graph G with more than two vertices is reconstructible provided that G contains a vertex v such that for any its neighbours w all vertices of G of degree $d(w) - 1$ that are distinct from v are neighbours of v. It is also obvious that G is reconstructible if $\sum_{v \in B} d(v) < n$, where B is the set of bad vertices of G and n is number of vertices of G. In [**6**] this observation was extended by proving that G is reconstructible if $\sum_{v \in B} (d(v) - 1/2) < n$.

The first part of the paper is devoted to proving results of similar flavour as the above. In the second part we introduce the concept of a reconstruction matrix of a graph and establish a Kelly-Ulman type result for graphs with certain reconstruction matrices.

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1. Reconstruction with Help of Certain Degree-sequences

Let G be a graph and u, v be vertices of G. The vertices u and v will be referred to as G-**similar** if $G-u$ is isomorphic to $G-v$. By the symbol $N_G(v)$ we will denote the set of neighbours of v in the graph G.

Theorem 1. *Let G be a graph with n vertices $u_1, u_2, \dots, u_n$ of degrees $\deg(u_i) = d_i$ $(1 \le i \le n)$, such that $d_1 \le d_2 \le \dots \le d_n$. Assume that the following two conditions are fulfilled:*

(1) *For some $k \le n$ it holds that $d_{k-1} + 2 \le d_k < d_{k+1}$.*
(2) *In the vertex-deleted subgraph $G - u_k$ (where k is as in (1)), no vertex from $N_G(u_k)$ is $(G-u_k)$-similar to any other vertex in $G-u_k$.*

Then the graph G is reconstructible.

Proof. Let S be the collection of all the n vertex-deleted subgraphs of the graph G. It is easy to reconstruct the degree-sequence of G (see [**3**]). Let us identify all the subgraphs $G - u_k$ from S where the degree of the vertex u_k is d_k and (1) holds for k. Let A be the set of those values of k for which (1) holds. Among these subgraphs $G - u_k$ we have to find the one for which (2) holds and in this one we have to identify all neighbours of the vertex u_k. If we show (using (1), (2) and the information contained in S) that the set of neighbours of u_k is uniquely determined in $G - u_k$, then it will obvious that each reconstruction of G will be isomorphic to G.

Let us take each subgraph $G-u_k$, $k \in A$, one after another, and do the following: Define D as the set of those vertices from $G-u_k$ that are $(G-u_k)$-similar to some vertices from $G-u_k$. With help of the set D we verify whether some vertices of $N_G(u_k)$ are $(G-u_k)$-similar to some vertices from $G-u_k$; this is done by means of the following procedure. Let us find all the subgraphs $G - u_i$ from S where a vertex of degree $d_k - 1$ exists. Let us put these i into the set B. It is easily seen that $|B| = d_k$. Similarly, we successively take the subgraphs $G - u_i$, $i \in B$, step by step, and do the following: Let $G^h = (G - u_i) - u_k$ (the degree of the vertex u_k is $d_k - 1$ in the subgraph $G - u_i$). Now we successively remove one vertex from $G - u_k$ and test whether this subgraph is isomorphic to G^h. If these subgraphs are not isomorphic then we put the vertex back to $G - u_k$ and remove the next one. If now these subgraphs are isomorphic then we can assume that the removed vertex is the vertex is the vertex u_i (we can assume it because if the vertex u_i was $(G-u_k)$-similar to any other vertex in $G-u_k$ then it would be in the set D, and then we would take the next subgraph $G - u_k$, $k \in A$). Let us distinguish the vertex u_i in $G - u_k$. Obviously, if we go through all the subgraphs $G - u_i$, $i \in B$ then we have all neighbours of u_k distinguished in $G - u_k$. The fact that this process necessarily terminates for some $k \in A$ is guaranteed by the condition (2). The proof is complete. □

If $\overline{G}$ is the complement of G, we have the following obvious consequence.

Corollary. *Let G be a graph with n vertices $u_1, u_2, \ldots, u_n$ with degree $\deg(u_i) = d_i$ $(1 \le i \le n)$, such that $d_1 \le d_2 \le \cdots \le d_n$. Assume that the following two conditions are fulfilled:*

(1) *For some $k \le n$ it holds that $d_{k-1} < d_k \le d_{k+1} - 2$.*
(2) *In the vertex-deleted subgraph $\overline{G} - u_k$ (where k is an in (1)), no vertex from $N_{\overline{G}}(u_k)$ is $(\overline{G} - u_k)$-similar to any other vertex in $\overline{G} - u_k$.*

Then the graph G is reconstructible.

Remark 1. The reader can easily see that if the condition (1) from Theorem 1 is replaced by either $d_1 < d_2 \le \cdots \le d_n$ or $d_1 \le \cdots \le d_{n-1} < d_n$ and the condition (2) remains unchanged, then G is reconstructible as well.

The subgraph of G obtained by deleting a set of vertices M (where $|M| = k$), together with all the edges incident to at least one of the vertices in M will be referred to as a **k-vertex-deleted subgraph** and denoted by $G - M$.

Theorem 2. *Let G be a graph with n vertices $u_1, u_2, \ldots, u_n$ of degrees $\deg(u_i) = d_i$ $(1 \le i \le n)$, such that $d_1 \le d_2 \le \cdots \le d_n$. Assume that the following two conditions are fulfilled:*

(1) *For some $k \le n$ it holds that*

$$d_k + 2 \le d_{k+1} = d_{k+2} = \cdots = d_{k+q} \le d_{k+q+1} - 2\,.$$

(2) *Let $M = \{u_{k+1}, u_{k+2}, \ldots, u_{k+q}\}$ where k is as in (1). Then in the q-vertex-deleted subgraph $G - M$ at least for one u_i, $i = k+1, k+2, \ldots, k+q$ no vertex from $N_G(u_i)$ is $(G - M)$-similar to any other vertex in $G - M$.*

Then the graph G is reconstructible.

Proof. The proof is similar to the proof of Theorem 1, therefore we give here only a sketch. Instead of one subgraph $G - u_k$ we have subgraphs $G - u_{k^*}$, $k^* = k+1, k+2, \ldots, k+q$ from which we take one, for example $G - u_{k+j}$, in which we remove the vertices $u_{k+1}, u_{k+2}, \ldots, u_{k-j}, u_{k+j}, \ldots, u_{k+q}$ so that we obtain the subgraph $G - M$. Instead of the subgraphs $G - u_i$ we have subgraphs $G - u_{i^*}$ in which at least one vertex of degree $d_{k+1} - 1$ exists and $u_{i^*} \ne u_{k^*}$, $k^* = k+1, k+2, \ldots, k+q$. Let us put these i^* into the set B. In the subgraph $G - u_{i^*}$, $i^* \in B$ we remove all vertices of degree d_{k+1} and $d_{k+1} - 1$ (these will be the vertices $u_{k+1}, u_{k+2}, \ldots, u_{k+q}$), and we denote these subgraphs by G^h (with the same meaning as in Theorem 1). Now it holds that G^h and $G - u_{k+j}$ differ only in the fact that the subgraph $G - u_{k+j}$ contains the vertex u_{i^*} while subgraph G^h does not. In the way described in Theorem 1 we can find out whether the vertices u_{i^*} and u_{k+j} are adjacent. The only difference will be the following: even if the subgraph $G - u_{i^*}$ contains the vertex of degree $d_{k+1} - 1$, it still does not have to mean that the vertex u_{i^*} is adjacent with the vertex u_{k+j} (it can be adjacent with some other vertex of degree d_{k+1}). If the removed vertex from $G - u_{k+j}$ has the same degree as the vertex u_{i^*} (which is removed from $G - u_{i^*}$), then the vertices considered above are not adjacent. Further it is necessary to mention that we can uniquely determine each vertex of degree d_{k+1} which is adjacent to u_{k+j} in the subgraph $G - u_{k+j}$ (its degree is $d_{k+1} - 1$). The remaining parts of this proof are similar to the proof of Theorem 1. □

2. Reconstruction of Simple, Connected Graphs with Help of Reconstruction Matrix

Let G be a graph on n vertices $u_1, u_2, \dots, u_n$ and let S be a collection of vertex-deleted subgraphs of the graph G. We define the $n \times n$ reconstruction matrix of G as follows. The elements of its first row are the subgraphs $G - u_1, G - u_2, \dots, G - u_n$. Let $S_1, S_2, \dots, S_n$ be the collections of vertex-deleted subgraphs of the graphs $G - u_1, G - u_2, \dots, G - u_n$. In this way we obtain the subgraphs $(G - u_i) - u_j$. To the i-th column we add subgraphs $(G - u_i) - u_j$, for $j = 1, 2, \dots, i-1, i+1, \dots, n$. The above considerations can be illustrated in the following picture.

$G - u_1$	$G - u_2$	$G - u_3$		$G - u_n$
$(G - u_1) - u_2$	$(G - u_2) - u_1$	$(G - u_3) - u_1$		
$(G - u_1) - u_3$	$(G - u_2) - u_3$	$(G - u_3) - u_2$		
$(G - u_1) - u_4$				
$(G - u_1) - u_n$				

Theorem 3. *Let G be a graph on n vertices. Let at least one pair of vertices u_i, u_j exist in G so that the following two conditions are fulfilled:*

(1) *If $M = \{u_i, u_j\}$, a subgraph isomorphic to $G - M$ appears in the reconstruction matrix exactly twice.*

(2) *At least for one of the vertices u_i, u_j the following holds: either the vertices in its neighbourhood are not $(G-M)$-similar to any other vertices in $G-M$ or, if some of them are $(G-M)$-similar to $x_1, x_2, \dots, x_p$, then the vertices $x_1, x_2, \dots, x_p$ lie in its neighbourhood.*

Then the graph G is reconstructible.

Proof. Let us construct the reconstruction matrix of the graph G. There are all possible subgraphs $G - \{u_k, u_1\}$ in its columns. The reader will easy realise that the vertices for which (1) is fulfilled can be found with help of isomorphism of the subgraphs $G - \{u_k, u_1\}$. It is obvious that if two vertices suitable for (1) are found, then the subgraph $G - \{u_i, u_j\}$ will occur in the reconstruction matrix exactly twice (in two different columns). Let as assume that the vertices u_i, u_j are chosen so that (1) holds. Then, the corresponding subgraphs appear in the matrix as $(G - u_i) - u_j$ (in the i-th column) and $(G - u_j) - u_i$ (in the j-th column). We can unambiguously determine the neighbours of u_j because we know this vertex (it is the one removed from $G - u_i$) and similarly we can determine the neighbours of u_i in $(G - u_j) - u_i$. Now let us construct the collections of vertex-deleted subgraphs of $(G - u_i) - u_j$ and $(G - u_j) - u_i$ and denote them by S_{ij} and S_{ji}. With help of them we find out, whether or not for u_j from $(G - u_i) - u_j$ or for u_i from $(G - u_j) - u_i$ the condition (2) holds (we do not go into details because they are described in Theorem 1). Let us assume that (2) holds for u_i (the considerations in other case are similar). We can uniquely determine the neighbours of u_i in $(G - u_j) - u_i$. With help of the isomorphism between $(G - u_i) - u_j$ and $(G - u_j) - u_i$ we can determine the neighbours of u_i in $(G - u_i) - u_j$. Further we must find out whether the edge $u_i u_j$ exists in G. If the degree of the vertex u_j in $G - u_i$ is the same as the degree of u_j in G, then the edge $u_i u_j$ does not exist. By identifying all neighbours of u_i in $G - u_i$ it is obvious that each reconstruction of G is isomorphic to G. This completes the proof. $\square$

References

1. Ulam S. M., *A Collection of Mathematical Problems*, Wiley, New York, 1960.

2. Harary F., *Graph Theory*, Addison-Wesley, Reading, Mass., 1969.

3. Bondy J. A. and Hemminger R. L., *Graph Reconstruction – a survey.*, J. Graph Theory **1** (1977), 227–268.

4. Nash-Williams C. St. J. A., *The Reconstruction Problem*, Selected Topics in Graph Theory (Beineke and Wilson, eds.), Acad. Press, 1978, pp. 205–236.

5. Širáň J., *Reconstruction of graphs with special degree sequences*, Math. Slovaca **32** No. 4 (1982), 403–404.

6. Nash-Williams C. St. J. A., *Reconstruction using degree sequences*, Unpublished, but cited in [**3**].

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