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VISUALIZATION OF VALUE FUNCTION IN TIME-OPTIMAL DIFFERENTIAL GAMES *

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Abstract. An effective method for numerical solution of time-optimal differential games is the computation of fronts of level sets of the value function in reverse time. The paper describes a specialized computer algorithm developed by the authors for the visualization of graphs of the value function. The computer program based on this algorithm is compatible with the numerical procedures for the computation of fronts. Graphs of the value function of the "homicidal chauffeur" differential game are given.

Key words. differential games, value function, numerical methods, scientific visualization

AMS subject classifications. 49N55, 90D25, 90D26, 68U05

1. Introduction. Typical problems in the theory of differential games [7]–[9] are those with the payoff be the time of attaining a given terminal set M. The first player minimizes the attaining time but the second player maximizes it. If the game has a stationary dynamics, the value function V is a function of the state vector x and does not depend on the time t. The value function $x \to V(x)$ is "usual" scalar function of two variables, if x is a two-dimensional vector. The plots of such a function can be useful in many cases.

Nowadays, numerical methods for solving differential games are developed intensively [1], [2], [5], [6], [13], [15]. Many works are devoted to time-optimal problems. The authors of this paper have an experience in the development of algorithms for solving time-optimal problems in the plane [11], [12], [14].

The basis of our method for solving time-optimal problems in the plane is the computation of level sets of the value function or, more precisely, the computation of fronts of the value function. The level set $W(T)$ is a collection of all points x such that $V(x) \leq T$. The front $F(T)$ contains all points on the boundary of the set $W(T)$ such that $V(x) = T$.

In this paper, a method for the visualization of graphs of the value function developed by the authors is described. Some examples of such graphs for a well-known in the theory of differential games "homicidal chauffeur" game are given.

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2. Differential game dynamics. We consider differential games in the plane with the following dynamics

$$\dot{x} = f(x,\ u,\ v), \quad x \in R^2,\ u \in P,\ v \in Q. \tag{2.1}$$

Here $x = (x_1,\ x_2)'$ is the two-dimensional state vector and u and v are control variables of the first and second player chosen from compact sets P and Q, respectively. The vector function f satisfies the standard conditions [8], [9] for the existence, uniqueness and continuation to the infinite time interval of solutions to the equation (1) where $u(t)$ and $v(t)$ with the values in P and Q are substituted instead of u and v. The first player strives to bring the state vector to a closed terminal set M for the minimal time, the objective of the second player is opposite.

The best guaranteed result of the first player for a given initial state x_0 is defined as the shortest time that the first player can guarantee using feedback controls. Similarly, the best guaranteed result of the second player is defined as the longest time that the second player can guarantee using feedback controls. The best guaranteed results of the players coincide [7]–[9] for a wide class of differential games. The common value is called the meaning of the value function for the state x_0. On the whole, the function $x \to V(x)$ is considered. The computation of the value function is one of the most important steps in solving differential games.

As an example of differential game dynamics, we consider the dynamics of the homicidal chauffeur [7] game in reduced coordinates

$$\begin{aligned} \dot{x}_1 &= -w^{(1)}x_2\, u/R + v_1 \\ \dot{x}_2 &= w^{(1)}x_1\, u/R + v_2 - w^{(1)}, \qquad |u| \le 1, \quad |v| \le \nu. \end{aligned} \tag{2.2}$$

Here $(x_1,\ x_2)'$ is the state vector which gives the relative position of the evader with respect to the pursuer, and $w^{(1)}$ and R are constants which define the pursuer's velocity and the minimal radius of turn, respectively. The first player' control is the scalar parameter u. The second player steers with the vector control v that is chosen from the circle of radius ν.

Usually, a circle with the center at the origin is used as a terminal set M. Sometimes, problem statements with other terminal sets can be of interest.

The homicidal chauffeur game is one of the most popular and also one of the most difficult problem in the theory of differential games. It was proposed by Isaacs and studied by many authors [4], [5], [6], [10], [12]. Using this problem, we will demonstrate our visualization tools.

3. Backward procedure for the computation of fronts. Let $T \ge 0$. The level set (the Lebesgue set) of the value function is denoted by $W(T)$. This is the set of all points in the plane such that the minimizing player using feedback strategies can guarantee the transition of trajectories of the system (2) to the terminal set M within time T.

The set $W(T)$ is formed via a step-by-step backward procedure giving a sequence of embedded sets

$$W(\Delta) \subset W(2\Delta) \subset W(3\Delta) \subset \ldots \subset W(i\Delta) \subset \ldots \subset W(T).$$

Here Δ is the step of the backward procedure. Each set $W(i\Delta)$ consists of all initial points from which the minimizing player guarantees the attainment of $W((i-1)\Delta)$ within time Δ. We put $W(0) = M$.

Let us explain the idea behind the algorithm for the construction of sets $W(T)$.

This is a dynamic programming method. In the theory of differential games, the fundamental ideas of the backward construction of level sets were considered in works of Isaacs, Fleming, Pontryagin, Krasovskii and Pschenichnyi.

The crucial point of our algorithm is the computation of "fronts". The front F_i (Fig. 1) is the set of all points of $\partial W(i\Delta)$ with the property that the minimal guaranteed time of attaining the previous set $W((i-1)\Delta)$ is equal to Δ. For other points of $\partial W(i\Delta)$, the optimal time is less than Δ. The line $\partial W(i\Delta) \setminus F_i$ possesses the properties of the barrier (see [7] for the definition). The front F_i is computed using the previous front F_{i-1}. For the first step of the backward procedure, F_0 coincides with the usable part (see [7] for the definition) Γ_0 of the boundary of M. It may be one or several usable parts. The computations are carried out separately from each usable part. One should take into account that the obtaining parts of the level set can collide with each other.

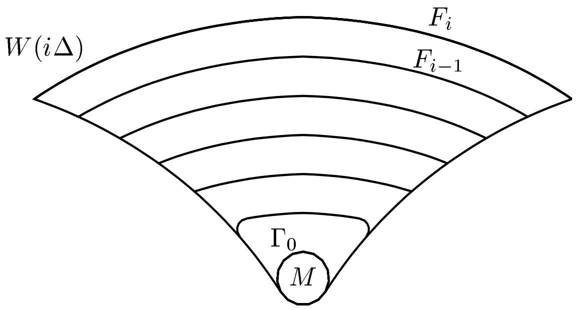

FIG. 3.1. *Construction of the sets $W(i\Delta)$.*

In the computation, each front is stored as an ordered collection of points, so fronts are polygonal lines. More details about the numerical procedure for the computation of fronts are given in [11], [12].

The collections of fronts ordered in the ascending reverse time are utilized for plotting the graphs of the value function. Any other information related to the game considered is not used. Therefore the visualization procedures developed for plotting the graphs of the value function can be applied to other interesting problems related to the propagation of fronts (for instance, the propagation of fire).

4. Visualization of the graphs of the value function. Since we use the representation of the value function as a collection of fronts F_i, a procedure for the reconstruction of the surface between two neighbor fronts in the three dimensional space is required for plotting the graph of the value function. To implement that, an algorithm for the reconstruction of the surface based on the triangulation of the gap between two neighbor fronts was developed.

The idea of this algorithm is as follows. The points of the fronts where the sharp bending occurs are found. These points divide each front into parts such that each part is smooth enough. Then the correspondence between the parts of the neighbor fronts is obtained. The parts of two neighbor fronts are considered to be correspondent, if they are close to each other in a geometric sense. This is the most difficult step of the algorithm because one should choose some threshold values. Also, heuristic arguments should be used in some cases. The final step of the algorithm is the triangulation between the corresponding parts.

The algorithm allows to localize regions where the smoothness and continuity of the function are violated. The surface constructed with this procedure is then plotted using photo-realistic computer graphics algorithms. To obtain plane pictures from three-dimensional plots, the perspective projection is utilized. The object is illumined with a point source of light that can be located either in a fixed point or be infinitely distant. The method of Gouraud is applied for the triangle coloring. The system allows to change the points of view, the properties of the surface "material" and the location of a light source. A special coloring mode is available, when the color for each point of the surface constructed is defined by the function value at this point.

The triangulation algorithm that we use presently is not based on a specific character of the location of fronts in the problem considered. This allows to apply the algorithm for other problems and, therefore, extend its application range. On the other hand, attempts to cover a general case makes the algorithm more complicated and does not provide an adequate result in some cases because of the presence of a heuristic component. For this reason, more close interface between triangulation and computation algorithms will be realized in the future. To this end, an additional information about the structure and mutual disposition of fronts that is available due to the program for the computation of fronts can be used effectively. This will help to develop more reliable triangulation algorithm.

Additionally, a program that allows to study the propagation of fronts of the value function in the plane was developed. An animation method is used to realize the movement of fronts. A special procedure that allows to examine some fronts more precisely and to detect some peculiarities is available. The program has convenient interface and allows to control the speed of the animation.

5. Examples. We begin the demonstration of examples for the homicidal chauffeur game with the examples corresponding to the classical statement of Isaacs.

In these examples, the set Q is a 25-polygon inscribed into the circle of radius ν with the center at $(0,0)$.

In Figure 2, the computation results for the following values of parameters are presented: $w^{(1)} = 3$, $\nu = 1$ and $R = 3$. The terminal set M is a 15-polygon approximating the unit circle with the center at the origin. The step Δ is 0.01. Every 10th front is plotted. The fronts are symmetric with respect to the x_2-axis. The left and right barrier lines terminate on the lower boundary of the sets A and B, respectively. After that, the left and right ends of the front begin to bend round the left and right barrier lines, and two symmetric corner points arise on the front. These corner points become more and more close, and at $\tau = 8.42$, a self-intersection of the front occurs. As a result, the front is divided into two parts: the internal part and the external one. The computations are carried out from each part separately. The internal part of the front propagates upwards sliding with its ends along the corresponding barriers. At $\tau = 10.6$, it collides with the terminal set, and two symmetric gaps which are filled out at $\tau = 11.3$ arise. The external part of the front propagates outwards and can fill out the whole plane with the time (the last external front in the picture corresponds to $\tau = 9$). Therefore, for each point of the plane, the minimal guaranteed time of approaching the set M is finite.

In Figure 3, a three-dimensional graph of the value function of Fig. 2 example is presented. The axes in the horizontal plane are x_1 and x_2, and the vertical axis measures the value function. The picture shows the value function for the region of (x_1, x_2) where the fronts are computed.

Both Fig. 2 and Fig. 3 show the graph of the mathematical function (Fig. 2 via

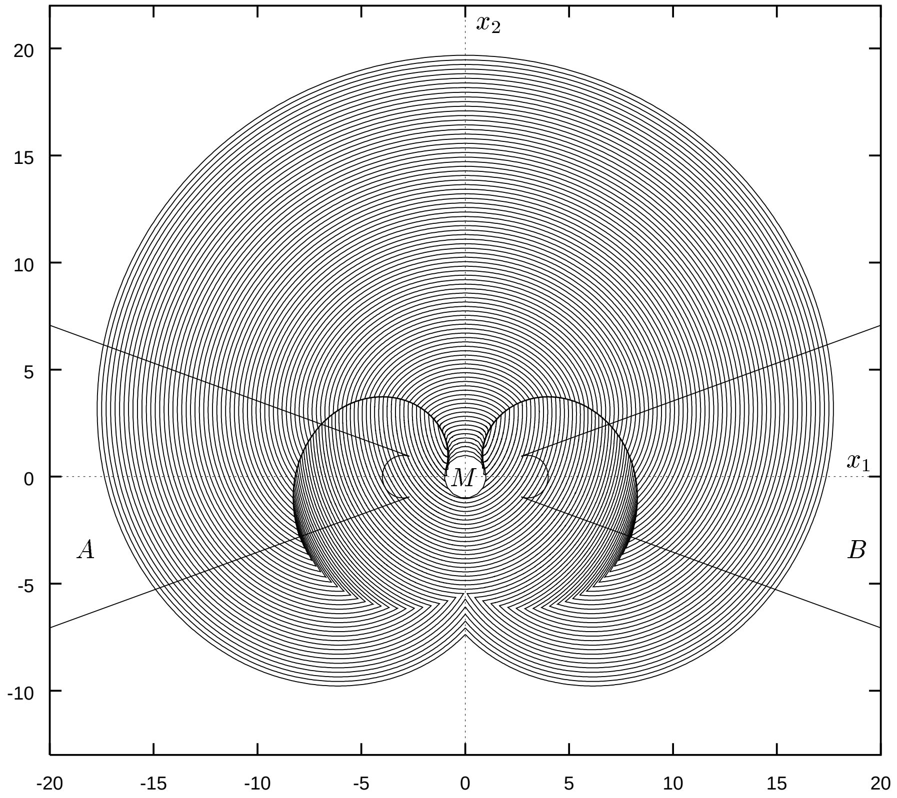

FIG. 5.1. *Level sets of the homicidal chauffeur game for* $w^{(1)} = 3$, $\nu = 1$ *and* $R = 3$.

level sets). However, the advantage of the Fig. 3 graph is that one can easily see the discontinuities, the form of nonsmooth parts and different degrees of steepness of various parts of the graph. Furthermore, the representation of functions as two-dimensional surfaces in three-dimensional space is habitual for human perception.

For the next example shown in Figs. 4, 5 and 6, the following values of parameters are used: $w^{(1)} = 2$, $\nu = 0.6$, $R = 0.2$. The set M is a regular polygon inscribed into the circle of radius 0.015. The center of the circle is $(0.2, 0.3)$. The step Δ is 0.001. The sets $W(8k\Delta, M)$, $k = 1, 2, ...$, are depicted.

Let us explain the constructions presented in Fig. 4. The right barrier line terminates on the lower boundary of the auxiliary set B. The front begins to bend round this barrier line. After some time, the left barrier line ends on the lower boundary of the set A, and the left part of the front bends round the left barrier. The left and the right parts of the front go towards one to other till the first self-intersection of the front occurs at $\tau = 0.725$. The front is divided into two parts (internal and external). For $\tau > 0.725$, only internal fronts that propagate into the "region of turn" are drawn. Here, very complicated structure of fronts arises.

At $\tau = 0.904$, the second self-intersection of the front which is drawn with the thick dash line produces two gaps that are filled out afterwards separately. The next front consists of three parts: an exterior part (which is not shown), and two interior parts (two loops inside the dash contour). The greatest value of τ in the region of

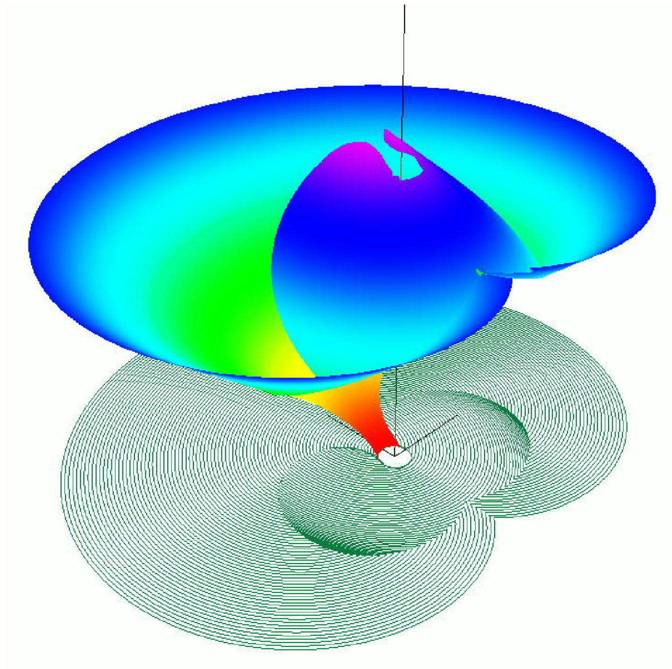

FIG. 5.2. *The graph of the value function for $w^{(1)} = 3$, $\nu = 1$ and $R = 3$.*

turn is 0.95. This corresponds to the time when the fronts complete filling the gap on the left hand side of the axis x_2. As a result, the sets $W(\tau, M)$ for $0.904 < \tau < 0.95$ are triply connected.

Figures 5 and 6 show three-dimensional graphs of the value function corresponding to the level sets of Fig. 4. Two different points of view were used. In Figure 6, level lines of the value function are additionally plotted onto the graph.

Next pictures are related to the case when the value function assumes infinite values. We consider a modified homicidal chauffeur dynamics comparing to (2). Namely, we assume additionally that the radius ν of the restriction Q on the control of the second player depends on the state x : the radius ν is constant and equal to w_e outside the circle of radius s with center at the origin, but the radius is proportional to $|x|$ inside this circle. Such a modification of the homicidal chauffeur game was proposed in [3], [6]. Following to [6], we consider the rectangle $\{(x_1, x_2) \in R^2 : -3.5 \le x_1 \le 3.5, -0.2 \le x_2 \le 0\}$ as the terminal set M.

Figure 7 corresponds to $w^{(1)} = 1$, $R = 0.8$, $s = 0.75$ and $w_e = 0.4$. Here, the value function is finite in the region where the solution was computed. However, it increases very rapidly in the central part for small negative values of x_2.

Figure 8 corresponds to an increased resource of the second player: $w_e = 1.1$, but other parameters are the same as before. The value function is bounded for positive values of x_2, the graph has the form of a hill. For negative values of x_2, two petals go to the infinity.

If we increase w_e up to $w_e = 1.5$, we obtain a hole for positive values of x_2, the value function being infinite within this hole. The graph (Fig. 9) takes the form of a tube.

Figure 10 is done for $w^{(1)} = 1$, $s = 0.75$, $w_e = 1.4$ and $R = 0.3$. For small negative

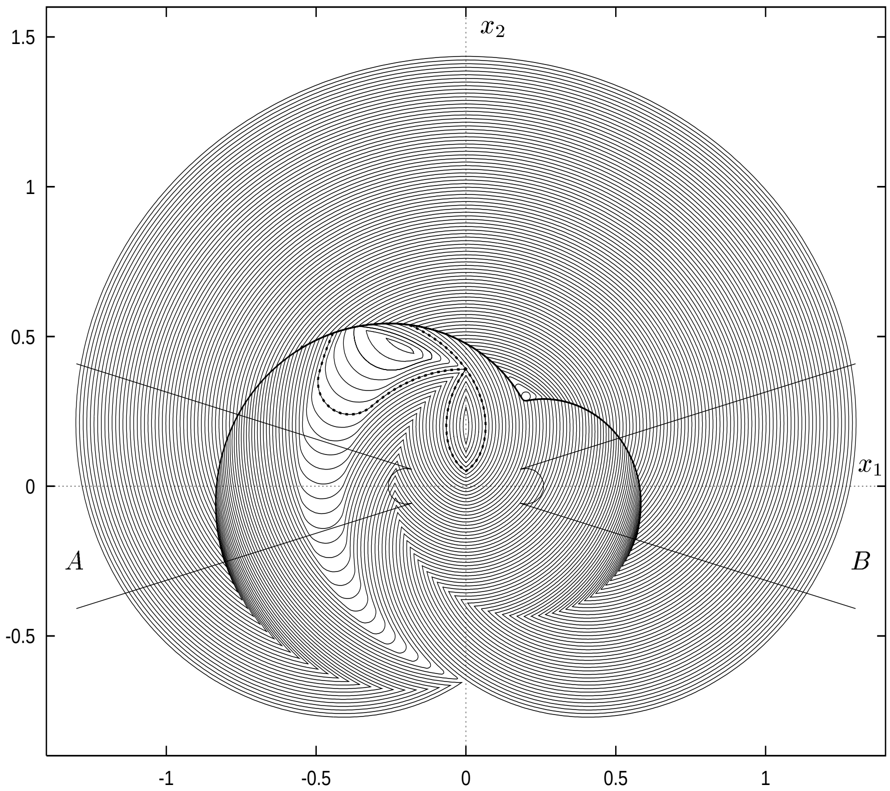

FIG. 5.3. *Complicated structure of level sets in the "region of turn." The terminal set is a small circle in the first quadrant,* $w^{(1)} = 2$, $\nu = 0.6$ *and* $R = 0.2$.

values of x_2, the graph of the value function has the form of a half-tube (a "gutter") extending to the infinity.

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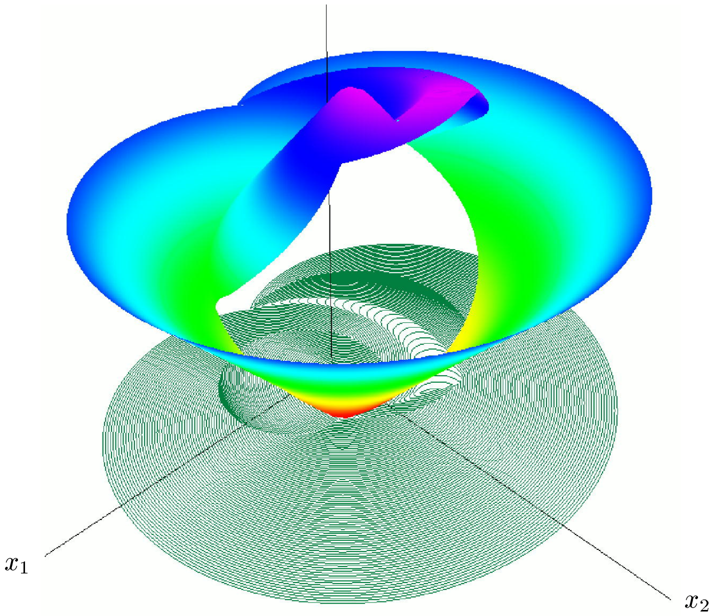

FIG. 5.4. *The graph of the value function for level sets in Fig. 4.*

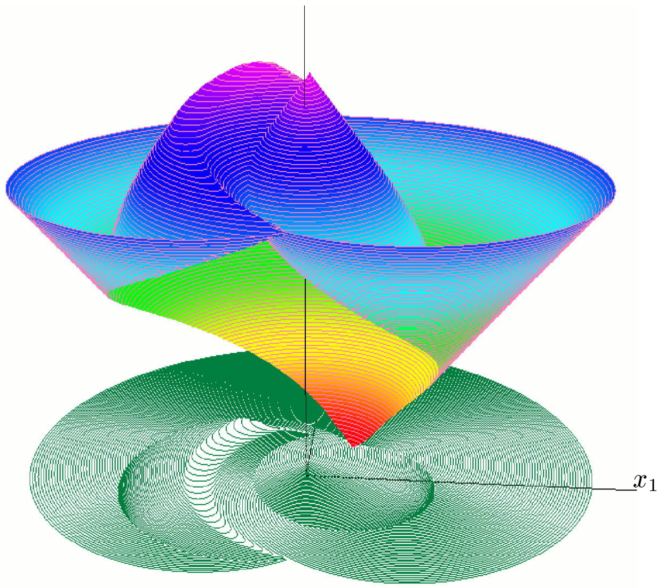

FIG. 5.5. *The graph of the value function from another point of view.*

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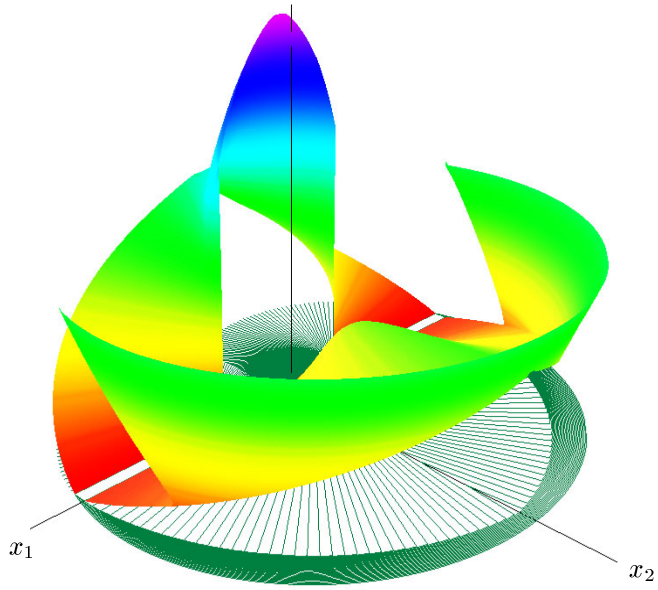

FIG. 5.6. *The graph of the value function for $w_e = 0.4$.*

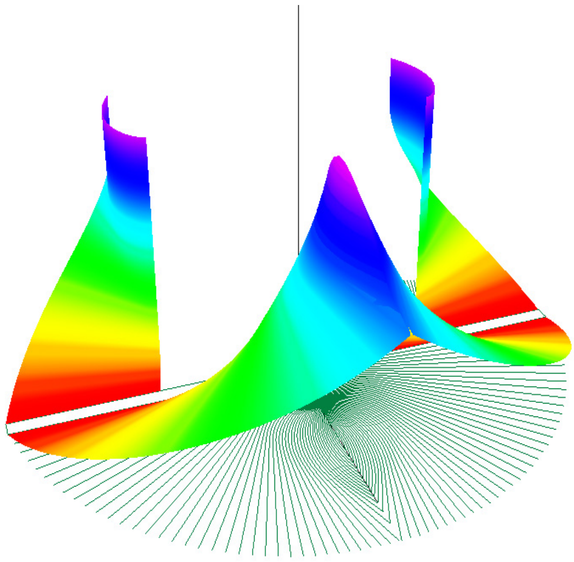

FIG. 5.7. *The graph of the value function for $w_e = 1.1$.*

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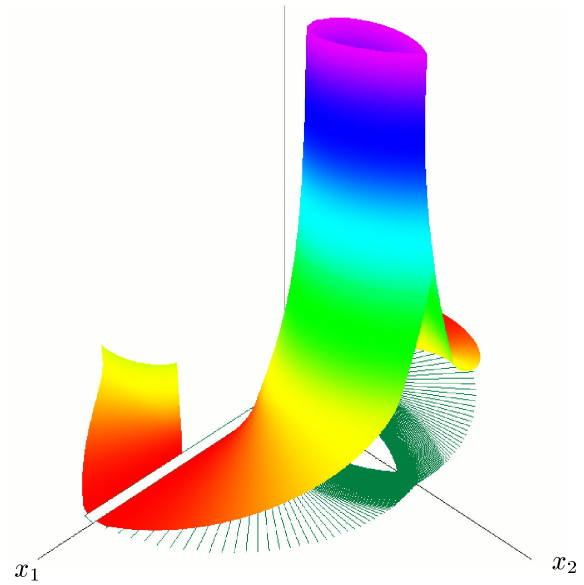

FIG. 5.8. *The graph of the value function for* $w_e = 1.5$.

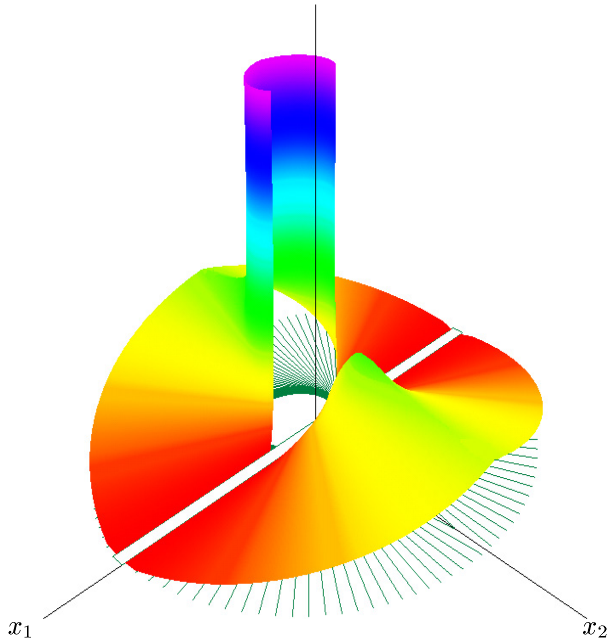

FIG. 5.9. *The graph of the value function for* $w_e = 1.4$, $R = 0.3$.

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