Proceedings of ALGORITMY 2000
Conference on Scientific Computing, pp. 1–10

SIMULATION OF FREE SURFACES OF MAGNETIC LIQUIDS

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Abstract. The dynamic behaviour of magnetic liquids under the influence of magnetic fields can be described by a coupled system of nonlinear partial differential equations. A numerical solution strategy for determing the time-dependent free surfaces of magnetic liquids is proposed and applied to determine the shape of an oscillating drop in a uniform magnetic field.

Key words. Free Surface, Magnetic Liquids, Navier-Stokes Equations

AMS subject classifications. 65N30, 65N55, 76D05

1. Introduction. Ferrofluids (or magnetic liquids) are colloidal suspensions of ferromagnetic particles in a nonmagnetic carrier liquid (oil, water). Due to the small size of about 10nm the particles are magnetic monodomains and the fluids behave like super-paramagnetic material. The interaction between the magnetizable fluid and an external magnetic field gives rise to several interesting phenomena like new or modified instabilities, field dependent viscosities and viscoelastic effects. The possibility to modulate the hydrodynamic parameters of the fluid with an external magnetic field also opens the way for a variety of technical and medical applications. For more details on ferrofluids see [14] or [2].

The main objective of this paper is to describe the underlying equations for modelling magnetic liquids and to develop a numerical solution strategy for solving flow problems with free boundaries under the influence of external magnetic fields. The coupled system of nonlinear partial differential equations is splitted into smaller subproblems. Each of them has to be solved by a robust, reliable and accurate numerical method. In particular, we apply our solution strategy to simulate the dynamic behaviour of a ferrofluid drop under the influence of an external magnetic field.

The paper is organized as follows. In Section 2 we recall the Maxwell and Navier-Stokes equations specified to ferrofluids with a free surface. Then, Section 3 is devoted to an iterative decoupling strategy for the whole problem and the derivation of fast and accurate solvers for the corresponding subproblems. Finally, in Section 4 the proposed solution method will be applied to the case of an oscillating drop in an magnetic field.

2. Mathematical Modelling of Magnetic Liquids. Starting with the Maxwell equations applied to a nonconducting fluid the magnetostatic problem is given by

$$\nabla \times \mathbf{H} = 0, \qquad \nabla \cdot \mathbf{B} = 0 \tag{2.1}$$

with the magnetic field strength $\mathbf{H}$ and the magnetic induction $\mathbf{B}$. The first equation guarantees the existence of a magnetostatic potential Ψ which satisfies $\mathbf{H} = -\nabla\Psi$.

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The dependence of $\mathbf{H}$ and $\mathbf{B}$ is described by

$$\mathbf{B} = \mu_0(1+\chi(H))\mathbf{H} = \mu_0(\mathbf{H}+\mathbf{M}), \tag{2.2}$$

where $\mathbf{M}$ denotes the magnetization and μ_0 the permeability constant. The susceptibility function χ depends on the magnitude H of the magnetic field $\mathbf{H}$. This function is defined in different way inside and outside the magnetic liquid. As usual [14], inside the of the magnetic liquid the Langevin function is used which results in

$$\chi(H) = \begin{cases} 0 & \text{outside the fluid,} \\ \frac{M_S}{H}\left(\coth(\gamma H) - \frac{1}{\gamma H}\right) & \text{inside the fluid,} \end{cases} \tag{2.3}$$

where M_S denoted the magnetization of saturation. The parameter γ is fixed in such a way that $\gamma M_S/3$ corresponds to the slope of the susceptibility function in the linear regime. Let us suppose that the magnetic liquid occupies the domain $\Omega(t)$, $0 \le t \le T$, and that $\Omega(t) \subset \tilde{\Omega} \subset \mathbf{R}^d$. We further assume that for all $t \in [0,T]$ the distance $\mathrm{dist}(\partial\tilde{\Omega}, \Omega(t))$ is large compared to the diameter $\mathrm{diam}(\Omega(t))$. A uniform magnetic field $\mathbf{H}_0$ is applied on $\partial\tilde{\Omega}$ which corresponds to Dirichlet boundary conditions for the magnetostatic potential Ψ.

The hydrodynamical properties of the magnetic liquid are described by the time-dependent incompressible Navier-Stokes equations

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\sigma(\mathbf{u},p) = f + \alpha M \nabla H \quad \text{in } \Omega(t),\ t>0, \tag{2.4}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega(t),\ t>0, \tag{2.5}$$

where $\mathbf{u}$, p denote the velocity and pressure, respectively, Re the Reynolds number, α a dimensionless parameter describing the strength of influence of magnetic forces, $M = |\mathbf{M}|$,

$$\sigma_{ij} = \sigma(\mathbf{u},p)_{ij} = \frac{2}{\mathrm{Re}} D(\mathbf{u})_{ij} - p\delta_{ij}, \quad i,j = 1,\ldots,d$$

is the stress tensor and

$$D(\mathbf{u})_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad i,j=1,\ldots,d$$

the deformation tensor. We consider the case in which the whole boundary $\Gamma(t)$ of the fluid domain is a free boundary, thus we apply the following boundary and initial conditions:

$$\mathbf{n}\cdot\sigma(\mathbf{u},p)\,\mathbf{n} = \frac{2\mathcal{H}}{\mathrm{Ca}} + \frac{\alpha}{2}(\mathbf{M}\cdot\mathbf{n})^2 \quad \text{on } \Gamma(t) \tag{2.6}$$

$$\mathbf{t}\cdot\sigma(\mathbf{u},p)\,\mathbf{n} = 0 \quad \text{on } \Gamma(t) \tag{2.7}$$

$$\mathbf{u}\cdot\mathbf{n} = V_\Gamma \quad \text{on } \Gamma(t) \tag{2.8}$$

$$\mathbf{u}(0,.) = \mathbf{u}_0 \quad \text{in } \Omega(0). \tag{2.9}$$

Here, $\mathcal{H}$ denotes the mean curvature, Ca the capillary number, V_Γ the normal velocity of the free boundary, $\mathbf{n}$ and $\mathbf{t}$ are the unit outer normal and tangential vectors, respectively. Note that (2.1)-(2.9) is a coupled system of nonlinear partial differential equations with a free boundary $\Gamma(t)$, $t>0$.

3. Decoupling Strategies. A closer look to the structure of the system (2.1)-(2.9) suggests the following decoupling strategy. We split the time interval $[0, T]$ into $0 = t_0 < t_1 < \ldots < t_N = T$. For a given domain $\Omega(t_n)$ the solution of the magnetostatic problem (2.1)-(2.2) is determined. Then, having the magnetization $\mathbf{M}$ and the magnetic field $\mathbf{H}$, we solve the nonstationary Navier-Stokes equations (2.4)-(2.5) in the time interval (t_n, t_{n+1}) using the initial condition at $t = t_n$ from a previous calculation or (when $t_n = 0$) from the initial state and the normal stress boundary conditions (2.6) and (2.7). Now, from (2.8) we get an information how to move the free boundary $\Gamma(t_n) \to \Gamma(t_{n+1})$. In the corresponding new domain $\Omega(t_{n+1})$ we start again with calculating the solution of the magnetostatic problem (2.1)-(2.2). In the following sections we will describe the efficient solution of each subproblem in detail.

3.1. Fast Solution of Magnetic Fields. The fast and robust solution of the magnetostatic problem plays an important role in the whole simulation process since the magnetic field has to be calculated in each time step inside and outside the magnetic liquid.

The standard weak formulation of (2.1)-(2.2) is given by
Find $\Psi \in H^1_D(\tilde{\Omega})$ such that

$$(\mu(|\nabla\Psi|)\nabla\Psi, \nabla v) = 0, \quad \forall v \in H^1_0(\tilde{\Omega}), \tag{3.1}$$

where $H^1_D(\tilde{\Omega})$ is the space of those functions $v \in H^1(\tilde{\Omega})$ which satisfy the given boundary condition on $\partial\tilde{\Omega}$.

We want to discretize the problem by using a finite element method. To this end we take some finite element approximation spaces $V_{D,h} \approx H^1_D(\tilde{\Omega})$ and $V_h \approx H^1_0(\tilde{\Omega})$. Moreover, in order to solve the nonlinear system one has to linearize the problem. There are several techniques to do this. The first one is to use a Newton-like method. This is analysed in [8, 9]. We choose a fixed point iteration for linearization. Altogether we obtain the sequence of linear problems:

Given a current iterate Ψ_h^{n-1} one iteration step reads as follows
Find $\Psi_h^n \in V_{D,h}$ such that

$$\left(\mu(|\nabla\Psi_h^{n-1}|)\nabla\Psi_h^n, \nabla v_h\right) = 0, \quad \forall v_h \in V_h, \tag{3.2}$$

i.e., in each iteration step a Laplace-like problem with jumping coefficient has to be solved. As first approximation Ψ_h^0 for the magnetostatic potential we take the potential which corresponds to a homogenous magnetic field.

Since the magnetic field is a sensitive input data for the Navier-Stokes equation we use isoparametric finite elements of second order for the approximation of the magnetostatic potential Ψ, in contrast to [8, 9].

3.2. Incompressible Navier-Stokes Equation in Fixed Domains. Here, we describe how to get a stable discretization of the Navier-Stokes equations in a fixed domain Ω. Let $\mathcal{T}_h$ be a regular decomposition of Ω into shape regular triangles or quadrilaterals. It is well-known that the finite element spaces V_h and Q_h used to approximate velocity and pressure, respectively, cannot be chosen independently. They have to fulfill the Babuška-Brezzi stability condition

$$\sup_{\mathbf{v}_h \in V_h} \frac{(q_h, \nabla \cdot \mathbf{v}_h)}{|\mathbf{v}_h|_1} \geq \beta \|q_h\|_0, \quad \forall q_h \in Q_h, \tag{3.3}$$

with a positive constant $\beta > 0$ which is independent of the mesh size parameter h [7]. Note also, that due to the normal stress boundary conditions we have to work with the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \frac{2}{\mathrm{Re}} \sum_{i,j=1}^{d} \int_\Omega D_{ij}(\mathbf{u}) D_{ij}(\mathbf{v})\, dx \tag{3.4}$$

instead of the simpler form $\mathrm{Re}^{-1}(\nabla \mathbf{u}, \nabla \mathbf{v})$. This causes additional problems when using nonconforming elements since Korn's inequality is not automatically fulfilled for nonconforming approximations, such that the bilinear form (3.4) could violate the coerciveness. Indeed, the simplest nonconforming finite element pair of piecewise linear/constant approximation satisfies the Babuška-Brezzi condition but not Korn's inequality [1]. Thus, we will concentrate in the following on conforming finite elements only.

First, we consider quadrilateral finite elements. Let $(-1,1)^2$ be the reference square $\hat{K}$, and F_K the mapping from $\hat{K}$ onto an arbitrary quadrilateral K. In general, F_K is a bilinear mapping. Let $Q_k(\hat{K})$ and $P_k(\hat{K})$ be the following sets of polynomials on $\hat{K}$

$$Q_k(\hat{K}) := \left\{ \hat{q}(\hat{x}_1,\hat{x}_2) = \sum_{i,j=0}^{k} a_{ij} \hat{x}_1^i \hat{x}_2^j \right\},\ P_k(\hat{K}) := \left\{ \hat{p}(\hat{x}_1,\hat{x}_2) = \sum_{0 \le i+j \le k} a_{ij} \hat{x}_1^i \hat{x}_2^j \right\}.$$

Then, we define

$$Q_k(K) := \{ q = \hat{q} \circ F_K^{-1} : \hat{q} \in Q_k(\hat{K}) \}, \quad P_k(K) := \{ p = \hat{p} \circ F_K^{-1} : \hat{p} \in P_k(\hat{K}) \}$$

and

$$\begin{aligned} Q_k &:= \{ v \in H^1(\Omega) : v_{|K} \in Q_k(K) \}, \quad k \ge 1, \\ Q_0 &:= \{ v \in L^2(\Omega) : v_{|K} \in Q_0(K) \}, \\ P_k^{disc} &:= \{ v \in L^2(\Omega) : v_{|K} \in P_k(K) \}, \quad k \ge 0. \end{aligned}$$

We used in our computations the pairs of quadrilateral finite elements Q_2/P_1^{disc}, Q_2/Q_1 which fulfil (3.3). As usual, the facts that the velocity is a vector valued function is not indicated in these notations.

Now we describe the triangular finite elements which have been implemented in our code MooNMD. Let us denote for $k \ge 1$ by P_k the space of continuous, piecewise polynomials of degree k, by P_0 the space of piecewise constant functions. The reference triangle has the vertices $(0,0)$, $(1,0)$ and $(0,1)$. In particular the computations reported in Section 4 have been performed with the triangular finite element pair P_2/P_1. For more detailed descriptions of the finite element spaces we refer to [3, 6] and the literature cited there.

If we combine a discretization in space by using the finite element space V_h and Q_h with the simple backward Euler scheme with time step $\tau_n = t_{n+1} - t_n$ we get the following discretization corresponding (2.4)-(2.7):

For given $\mathbf{u}_h^n \in V_h$, find $\mathbf{u}_h^{n+1} \in V_h$ and $p_h^{n+1} \in Q_h$ such that

$$\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\tau_n}, \mathbf{v}_h\right) + \frac{2}{\mathrm{Re}}(D(\mathbf{u}_h^{n+1}), D(\mathbf{v}_h)) + n(\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (p_h^{n+1}, \nabla \cdot \mathbf{u}_h^{n+1})$$

$$= \frac{2}{\mathrm{Ca}} \int_\Gamma \mathcal{H}\, \mathbf{v}_h \cdot \mathbf{n}\, d\gamma + (\mathbf{f}, \mathbf{v}_h) + (\mathbf{f}_m, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_h \tag{3.5}$$

$$(q_h, \nabla \cdot \mathbf{u}_h^{n+1}) = 0 \quad \forall q_h \in Q_h. \tag{3.6}$$

Here, the notation $\mathbf{u}_h^n$ stands for an approximation of $\mathbf{u}(t_n, .)$, $n(., ., .)$ denotes the convective term given by

$$n(\mathbf{w}, \mathbf{u}, \mathbf{v}) := ((\mathbf{w} \cdot \nabla)\mathbf{u}, \mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V_h,$$

and $(\mathbf{f}_m, .)$ abbreviates

$$(\mathbf{f}_m, \mathbf{v}_h) := \alpha(M\nabla H, \mathbf{v_h}) + \frac{\alpha}{2} \int_\Gamma (\mathbf{M} \cdot \mathbf{n})^2 \mathbf{v_h} \cdot \mathbf{n}\, d\gamma$$

The backward Euler scheme is a very stable time discretization but, unfortunately, it is only of first order and rather dissipative. A nice alternative to the backward Euler scheme is the fractional θ-scheme which we describe for simplicity on the following model problem

$$\begin{aligned} u_t + A\,u &= 0 \\ u(0) &= u_0 \end{aligned}$$

in a Banach space.

Let $\theta = 1 - \frac{\sqrt{2}}{2}$ and $\alpha, \beta \in (0,1)$ with $\alpha + \beta = 1$ and $\alpha > 1/2$. We split each time interval (t_n, t_{n+1}) into three subintervals $(t_n, t_{n+\theta})$, $(t_{n+\theta}, t_{n+\theta'})$ and $(t_{n+\theta'}, t_{n+1})$ where $t_{n+\theta} = t_n + \theta\tau_n$ and $t_{n+\theta'} = t_n + \theta'\tau_n$ with $\theta' = 1 - \theta$.

The fractional θ-scheme for the time interval (t_n, t_{n+1}) is defined by:

$$\begin{array}{ccccccc} \dfrac{u^{n+\theta} - u^n}{\theta\tau_n} & + & \alpha A\,u^{n+\theta} & + & \beta A\,u^n & = & 0 \\[2ex] \dfrac{u^{n+\theta'} - u^{n+\theta}}{(1-2\theta)\tau_n} & + & \beta A\,u^{n+\theta'} & + & \alpha A\,u^{n+\theta} & = & 0 \\[2ex] \dfrac{u^{n+1} - u^{n+\theta'}}{\theta\tau_n} & + & \alpha A\,u^{n+1} & + & \beta A\,u^{n+\theta'} & = & 0 \end{array}$$

If one chooses $\alpha = (1-2\theta)/\theta'$ all implicit operators on the left hand side are equal. The fraction θ-scheme is of second order, strongly A-stable and nearly non-dissipative which makes it attractive for the considered type of problems . A theoretical analysis of the fractional θ-scheme can be found in [11].

Originally [4], the fractional θ-scheme has been applied to the Navier-Stokes equations as an operator splitting method to solve separately in each substep either a generalized Stokes problems or a transport problem. Since in the meantime fast multilevel solvers have been developed the splitting approach is no longer essential. We follow this new techniques and solve in each substep the whole Navier-Stokes equations.

The space-time discretizations described above correspond to a nonlinear algebraic system of equations of the form

$$\begin{pmatrix} A(u) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{3.7}$$

which has to be solved in each time step. Now, u and p represent the nodal vectors for velocity and pressure, respectively, g_1 and g_2 are the given right hand side vectors. The formats of the matrices $A(u)$, B depend on the dimension of the finite element spaces V_h and Q_h. A simple linearization method can be obtained by replacing $A(u)$ by $A(u^{old})$, where u^{old} is a previous iterate or the initial guess from the previous time level. Thus, it remains to solve a large coupled linear system with the same structure in each nonlinear iteration step. This can be done efficiently by a multilevel approach as described in [10]. The essential ingredients of this multilevel approach are Gauss-Seidel smoother and prolongation operators defined by evaluating nodal functionals. The latter opens the possibility to use different discretization methods on different or even on the same level of mesh refinement. The different convergence behaviour of multilevel methods for low and higher order discretizations gives an attractive alternative to combine the higher discretization scheme on the finest mesh with robust and fast multigrid methods based on low order nonconforming discretization schemes, like the space of nonconforming, mean value oriented, rotated piecewise bilinears Q_1^{rot} defined and analysed in [13, 15]. For a detailed discussion of the design of multigrid methods and their efficiency we refer to [10].

3.3. Arbitrary Lagrangian Eulerian Approach for Time-Dependent Domains. In order to solve the coupled free boundary problem mentioned at the beginning, we still have to take into consideration that the domain Ω is time-dependent. As a consequence the nodes where the solutions are evaluated change their position in time. Thus, we need a special treatment of this problem. One possibility is to use the *Arbitrary Lagrangian Eulerian method* (ALE) which has been successfully applied, e.g. to fluid-structure interaction problems [12]. The ALE approach is a generalization of both the Eulerian and Lagrangian approaches and focusses neither on material points nor on a fixed spatial region. The view point is directed at special points, say $Y \in \Omega_Y$ at time $t = 0$, which will change there position with the continuum but independently from the motion of material points. Usually, the domain Ω_Y is called reference domain. Note that Ω_Y has not necessarily to be the domain $\Omega(0)$. The special case of interest for the considered class of problems is the situation where Y represents the grid coordinates. Let $\Phi(t,.) : \Omega_Y \to \Omega(t)$ be the bijection between the reference domain Ω_Y and the actual domain $\Omega(t)$ at time t. Then, $y(t) = \Phi(t, Y)$ are the mesh coordinates at time $t > 0$ and $y(t) - Y$ the mesh displacement. The time discretization has to be expressed with respect to the ALE coordinates. Taking into consideration that for a scalar function $\lambda(t, y) = \lambda(t, \Phi(t, Y))$

$$\left.\frac{\partial \lambda}{\partial t}\right|_Y = \left.\frac{\partial \lambda}{\partial t}\right|_y + \left.\frac{\partial y}{\partial t}\right|_Y \cdot \nabla_y \lambda \tag{3.8}$$

holds we get

$$\left.\frac{\partial \lambda}{\partial t}\right|_y = \left.\frac{\partial \lambda}{\partial t}\right|_Y - \mathbf{w} \cdot \nabla \lambda \tag{3.9}$$

with the mesh velocity $\mathbf{w}$. Consequently, with respect to the ALE coordinates the substantial derivative will be

$$\frac{d\lambda}{dt} = \frac{\partial\lambda}{\partial t} + (\mathbf{u} - \mathbf{w}) \cdot \nabla\lambda. \tag{3.10}$$

Applying this approach to the Navier-Stokes equation in case of the backward Euler time discretization, we get

$$\left(\frac{\bar{\mathbf{u}}_h^{n+1} - \mathbf{u}_h^n}{\tau_n}, \mathbf{v}_h\right)_{\Omega(t_n)} + \int_{\Omega(t_n)} ((\bar{\mathbf{u}}_h^{n+1} - \mathbf{w}) \cdot \nabla)\bar{\mathbf{u}}_h^{n+1}\, \mathbf{v}_h \, dx \tag{3.11}$$

as the discrete version of the substantial derivative. Here, $\bar{\mathbf{u}}_h^{n+1}$ is defined by

$$\bar{\mathbf{u}}_h^{n+1}(x) = \mathbf{u}_h^{n+1}(\Phi(t_{n+1}, Y)) \quad \text{with } x = \Phi(t_n, Y).$$

Furthermore, we have to deal with the curvature term in (3.5). Using the Laplace-Beltrami operator $\underline{\triangle} f$, the tangential derivatives $\underline{\nabla} f$ and the identity

$$\underline{\triangle} id_\Gamma = 2\mathcal{H}\mathbf{n},$$

we get by integration by parts

$$\int_\Gamma 2\mathcal{H}\mathbf{n} \cdot \mathbf{v}\, d\gamma = \int_\Gamma (\underline{\triangle} id_\Gamma) \cdot \mathbf{v}\, d\gamma = -\int_\Gamma \underline{\nabla} id_\Gamma : \underline{\nabla}\mathbf{v}\, d\gamma. \tag{3.12}$$

For more details on the underlying differential geometry see e.g. [5].

The identities in (3.12) allow us to derive the following relation

$$\begin{aligned} &\frac{2}{\mathrm{Re}} \int_\Omega D(\mathbf{u}) : D(\mathbf{v})\, dx - \int_\Omega p\, \nabla \cdot \mathbf{v}\, dx + \frac{1}{\mathrm{Ca}} \int_\Gamma \underline{\nabla} id_\Gamma : \underline{\nabla}\mathbf{v}\, d\gamma \\ = &\int_\Omega (\nabla \cdot \sigma(\mathbf{u}, p)) \cdot \mathbf{v}\, dx \end{aligned}$$

for a sufficiently smooth, vector-valued function $\mathbf{v}$, i.e., compared to standard weak formulation for the Navier-Stokes equations with Dirichlet boundary condition we get an additional term. This term can be treated in different ways, explicit or semi-implicit. We will use the semi-implicit way where $\underline{\nabla} id_{\Gamma(t_n)}$ is replaced by $\underline{\nabla}(id_{\Gamma(t_n)} + \tau_n \bar{\mathbf{u}}_h^{n+1})$. This treatment of the curvature term gives more stability since we get an additional term to the bilinear form which is symmetric and positive semi-definite. Once $\bar{\mathbf{u}}_h^{n+1}$ have been calculated we update the position of the boundary $\Gamma(t)$ by

$$x^{n+1} = x^n + \tau_n\, \bar{\mathbf{u}}_h^{n+1}(x^n).$$

The new positions of the inner mesh points are obtained by elliptic regularization.

4. Numerical Testexample. We consider a drop of magnetic liquid which is in magnetic saturation, i.e., the magnetization M inside the drop is constant. Furthermore, we assume that the drop is in a zero gravity field.

Fig. 4.1 shows the considered drop of magnetic liquid between two coils which generate the external magnetic field.

As initial domain we take an ellipse with different radii of the main axes, $r_x = 1.0$, $r_y = 1.2$, the reference domain is the unit circle. Furthermore, we assume that the

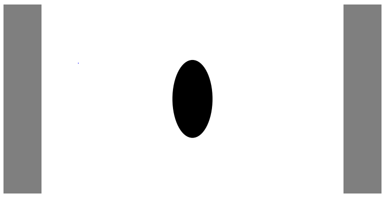

FIG. 4.1. *Drop of magnetic liquid between two coils*

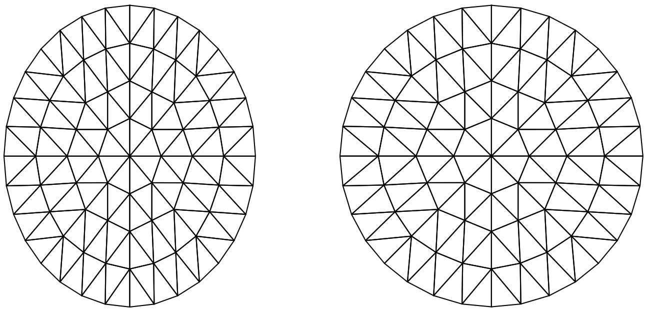

FIG. 4.2. *Grid of initial domain (left)and domain at time $t = T/2$ (right)*

fluid is in rest at time $t = 0$, i.e., $\mathbf{u}(0, .) = \mathbf{u}_0 = 0$. Surface tension and magnetic pressure act as driving forces which cause an onset of drop motion towards a shape of lower energy. Due to inertia the drop will not stay in the shape which has the minimal energy. But the drop will oscillate around its shape of minimal energy. The length of the oscillation period will be denoted by T.

The triangulations of the initial domain Ω_0 and the domain Ω at time $t = T/2$ are shown in Fig. 4.2.

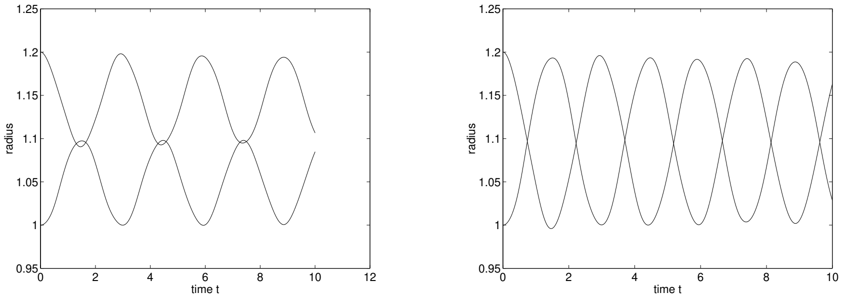

FIG. 4.3. *Trajectories of tips of drop as function of time, left with field, right without field*

Fig. 4.3 shows the trajectories of the tips of the drop as function of time for two cases, with and without an external magnetic field. We see that the horizontal and vertical radii oscillate around their equilibrium values. The equilibrium shape in the absence of a field is a circle. If a field is applied the equilibrium shape changes to an ellipse. There is a slightly decreasing of oscillation amplitude due to damping effects. The damping increases with a decreasing Reynolds number Re.

Fig. 4.4 illustrated the velocity inside the drop at different times during one oscillation period. It is clearly to see that the drop is a state of rest if it reaches the

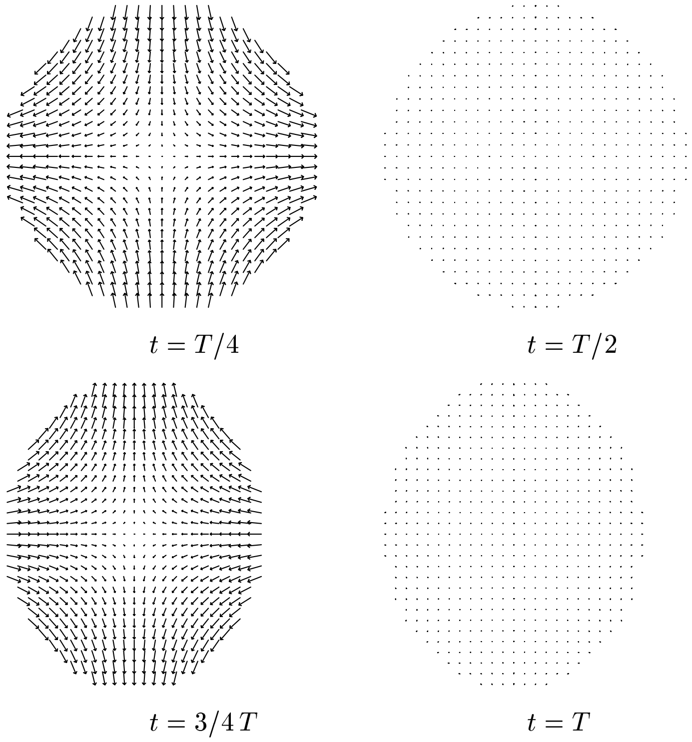

FIG. 4.4. *Flow field inside to drop at $t = T/4$, $t = T/2$, $t = 3/4\,T$, $t = T$*

turning points of oscillation.

Acknowledgement. The authors like to thank the DFG for supporting the research through grant En278/2-1. The calculations have been done by the software package MooNMD. The visualization package GRAPE enabled us to produce most of the figures.

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