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SET-VALUED MAPS FOR IMAGE SEGMENTATION

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Abstract. In the following we want to develop an approach to image segmentation that combines two properties : no regularity assumption about the contours and keeping close to the graphical notions. Hence all considered parts of the image are merely regarded as subsets of $\mathbb{R}^N$ and correspondingly their deformation is described as a set-valued map which ought to make some error functional decrease.
The theoretical concept leads to simple algorithms for both the (abstract) continuous segmentation problem and the (discrete) computer implementation.

Key words. Set-valued maps, differential inclusions, reachable set, contour detection, image segmentation

AMS subject classifications. 34A60, 54C60, 68U10, 93B03

1. Introduction.

The detection of image segments represents a basic problem of image processing.

Meanwhile many popular algorithms share the idea of approximations which are improved in some sense while time is increasing. For example, active contour models (so-called *snakes*) describe each contour as a Jordan curve that is deformed for minimizing some energy functional (see e.g. [5],[11]). But then the topology of the detected image segment is always simply connected. Moreover snakes (in their early form) are assumed to be twice differentiable and consequently edges can only be detected in some smoothed shape.

Such regularity assumptions are also a common weakness of other segmentation algorithms. Meanwhile some approaches have overcome these restrictions. These concepts follow former ideas and apply more abstract mathematical concepts. Level set methods are a very popular example which has been implemented efficiently (see [8], [9]). They use viscosity solutions of (generalized) Hamilton-Jacobi equations.

In the following we want to develop an approach to image segmentation that avoids both regularity assumptions about the contours and very abstract mathematical concepts. Instead our aim is to keep close to the graphical notions.
The image segments are regarded as subsets of $\mathbb{R}^N$. For a given compact set K_0 of the image the algorithm ought to detect the set (including K_0) which shows the object. Correspondingly to the idea of improving approximations, the set K_0 is "deformed" while time is increasing. Mathematically speaking, this deformation is described as a set-valued map $K : [0,T[\leadsto \mathbb{R}^N$ that maps each time $t \in [0,T[$ to a subset of $\mathbb{R}^N$.

For quantifying the "quality" of the approximations $K(t)$ an error functional Φ maps each subset of $\mathbb{R}^N$ to $\mathbb{R} \cup \{\infty\}$. Motivated by the variance of the grey-values $G|_M$ (restricted to a subset $M \subset \mathbb{R}^N$ with positive Lebesgue measure) we will assume that Φ has the form

$$\Phi(M) = \psi\Big(\mathcal{L}^N(M),\ \int_M G\,dx,\ \int_M G^2\,dx\Big) \qquad \text{for all } M \subset \mathbb{R}^N,\ 0 < \mathcal{L}^N(M) < \infty$$

and that $\psi \in C^2(]0,\infty[\times \mathbb{R} \times \mathbb{R})$ has the support in $[0,c] \times \mathbb{R} \times \mathbb{R}$ for some $c > 0$.

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Now the composition $\Phi \circ K : [0, T[\to I\!R$ is a usual real-valued function and our aim is to construct $K(\cdot)$ (with $K(0) = K_0$) such that $\Phi \circ K$ is decreasing until reaching some "critical set".

For implementing these notions the next steps concern the construction of $K(\cdot)$ and sufficient conditions for the decrease of $\Phi \circ K(\cdot)$.
Set-valued maps also provide an answer to the first problem : $K(t)$ is the reachable set $\vartheta_F(t, K_0)$ of a differentiable inclusion $x'(\tau) \in F(x(\tau), \tau)$ starting at K_0. In comparison with an ordinary differential equation the set-valued map $F : I\!R^N \times I\!R \rightsquigarrow I\!R^N$ tolerates more than one velocity of propagation.
With respect to the second step we examine the regularity of the above-mentioned composition. Conditions on F imply the absolute continuity of $\Phi \circ \vartheta_F(\cdot, K_0)$ and lead to its (weak) derivative. The sign of the latter provides a sufficient condition to guarantee the wanted decrease.
Afterwards a solution $K(t)$ is constructed and we present a simple implementation for computer images based on the same notion.

A first idea of applying set-valued maps to image segmentation has already been sketched briefly by J. Demongeot and F. Leitner ([4]). They suggest compact set-valued flows for various morphologies evolving in time to an asymptotic shape, which corresponds to the minimum of a potential function. In connection with image segmentation, snakes are regarded as (set-valued) potential flows. However, their survey is restricted to formal conclusions (without considering regularity, for example).
The following concept has been developed independently and is to avoid mathematical gaps. A more detailed presentation of these results is submitted to "Computing and Visualization in Science".

2. Deformation of sets. In this section we outline the basic mathematical tools (i.e. set-valued maps) and their application to the deformation of sets. This survey will serve as a basis for the precise formulation and the solution of the segmentation problem.

Set-valued maps represent the result of a trend typical of mathematics : give up unnecessary restrictions.
Functions have to fulfill the condition of a unique value (for each element of the domain). This uniqueness implies strong restrictions. Thus we want to tolerate more than one value for each element of the domain. Strictly speaking, this corresponds to replacing the range by its power set. But this roundabout way does not provide any advantage at all. Hence we return to the graph for a simple definition of set-valued maps :

DEFINITION 2.1 ([3]). *Let X, Y be sets and $M \subset X \times Y$. A* set-valued map *$F : X \rightsquigarrow Y$ is defined by M according to $F(x) := \{y \in Y \mid (x, y) \in M\} \quad \forall\ x \in X$. M is called the* graph *of F : $Graph(F) := M$. $F(x)$ is the* image *of the* value *of F at $x \in X$. The* domain *of F consists of all $x \in X$ with nonempty image, i.e. $Dom(F) := \{x \in X \mid F(x) \neq \emptyset\}$. F is said to be* strict *if the value $F(x)$ for each $x \in X$ is nonempty, i.e. $Dom(F) = X$.*

With respect to image segmentation we will restrict ourselves to set-valued maps with nonempty compact values. Then it is quite easy to extend the definition of continuity.

DEFINITION 2.2. *Let X, Y be metric spaces and $F : X \rightsquigarrow Y$ be strict with compact values. $\mathcal{K}(Y)$ denotes the set of all nonempty compact subsets of Y. $\boldsymbol{d} = \boldsymbol{d}_Y$ is the abbreviation of the* Hausdorff metrics *on $\mathcal{K}(Y)$.*
Then F is called continuous *if the corresponding function $X \to (\mathcal{K}(Y), \boldsymbol{d}),\ x \mapsto F(x)$ is continuous. Moreover F is said to be* Lipschitz continuous *if this function $X \to (\mathcal{K}(Y), \boldsymbol{d})$ is Lipschitz continuous.*

For implementing the deformation of sets mathematically, the graphical notion is based on the prescription of velocities. Traditionally vector fields and the corresponding ordinary differential equations describe the evolution according to

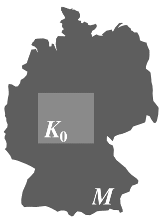
$$\begin{aligned} \vartheta_v(t,K) \quad := \quad \{\, x(t) \quad | \quad & x(\cdot) \in C^1([0,t], I\!R^N), \quad x(0) \in K, \\ & x'(\tau) = v(x(\tau),\tau) \qquad \forall\ \tau \in [0,t]\,\}. \end{aligned}$$

for a vector field $v \in C^{0,1}(I\!R^N \times [0,T[,\ I\!R^N)$ and $K \subset I\!R^N$.
The application of this concept often requires additional assumptions about the regularity. The prescription of the normal velocity, for example, is only possible if the unit normal vector is defined adequately. So the boundary has to be sufficiently smooth and edges are prohibited at all.
These restrictions are unnecessary if we admit more than one velocity at each point (and at every time). Generally speaking the vector field is replaced by a set-valued map. In addition, tolerating all continuous trajectories with well-defined velocities is more important than insisting on their smoothness. So the condition $x(\cdot) \in C^1([0,t], I\!R^N)$ is weakened to absolutely continuous functions $x(\cdot) \in AC([0,t], I\!R^N)$ and their weak derivatives $x'(\cdot) \in L^1([0,t], I\!R^N)$. This generalization provides useful advantages concerning the existence and convergence of solutions (see [2], [1]), but we do not need these details here explicitly.

DEFINITION 2.3 ([2]). *Suppose that $T \in]0,\infty]$, $t \in [0,T[$, $F : I\!R^N \times [0,T[\rightsquigarrow I\!R^N$ and $K \subset I\!R^N$. The* reachable set *of F and K at time t is defined according to*

$$\begin{aligned} \vartheta_F(t,K) \quad := \quad \{\, x(t) \quad | \quad & x(\cdot) \in AC([0,t], I\!R^N), \quad x(0) \in K, \\ & x'(\cdot) \in F(x(\cdot),\cdot) \quad \textit{almost everywhere in}\ [0,t]\,\}. \end{aligned}$$

3. Segmentation problem.
For a given image segment $K_0 \in \mathcal{K}(I\!R^N)$ we want to detect the superset M that shows the corresponding object in the image. This section is dedicated to the mathematically precise formulation of this rather heuristic problem.
Every grey-valued image determines a function $G : I\!R^N \to I\!R$ that maps each point (of the N-dimensional image) to the grey-value there. The fundamental idea is based on a set-valued map $K : [0,T[\rightsquigarrow I\!R^N$ with $K(0) = K_0$ that is "approximating" the wanted set M while time is increasing. So $K(\cdot)$ ought to satisfy the following 4 conditions :



1. $K(t)$ is compact for every $t \in [0,T[$,
2. $K(0) = K_0$,
3. $K(t_1) \subset K(t_2)$ for all $0 \le t_1 \le t_2 < T$,
4. $K(\cdot) : [0,T[\to (\mathcal{K}(I\!R^N), \boldsymbol{d})$ is continuous.

Then $K(\cdot)$ induces the resulting set M in a very simple way : $\quad M := \bigcup_{0 \le t < T} K(t).$

In the previous paragraph we specified a concept for the construction of $K(\cdot)$: the reachable set $\vartheta_F(\cdot, K_0)$ of a differentiable inclusion $x'(\cdot) \in F(x(\cdot), \cdot)$ (a.e.).
Then condition (2) is trivial : $\vartheta_F(0, K_0) = K_0$. Moreover the assumption $0 \in F(x,t)$ (for all $x \in \mathbb{R}^N$, $t \in [0,T[$) guarantees property (3) because graphically speaking, we can stay at any reached point $x \in \vartheta_F(t_1, K_0)$ by means of a trajectory at speed 0 and hence $\vartheta_F(t_1, K_0) \subset \vartheta_F(t_2, K_0)$ for all $0 \le t_1 \le t_2 < T$.
Assume that $\vartheta_F(t, K_0)$ is closed for every $t \in [0,T[$. Then we still need an additional condition on F ensuring the boundedness of the reachable sets $\vartheta_F(t, K_0)$. Corresponding to the results of ordinary differential equations the linear growth condition (on the right hand side) excludes the explosion of any trajectory. This condition is easy to extend to set-valued maps (with $B := B_1(0) = \{y \in \mathbb{R}^N \mid |y| \le 1\}$) :

DEFINITION 3.1 ([2]). *For any $T \in]0,\infty]$ a set-valued map $F : \mathbb{R}^N \times [0,T[\rightsquigarrow \mathbb{R}^N$ has* linear growth *if there is a constant $c > 0$ with*

$$F(x,t) \subset c\,(1 + |x| + t)\, B \qquad \textit{for all } x \in \mathbb{R}^N,\ t \in [0,T[\,.$$

Gronwall's lemma implies immediately the existence of an increasing function $R : [0,T[\to [0,\infty[$ (depending on K_0) with $\vartheta_F(t, K_0) \subset R(t)\, B$ for all $t \in [0,T[$ and the resulting local boundedness of $F(x,t)$ leads to the local Lipschitz continuity of $\vartheta_F(\cdot, K_0)$.

Now the next step concerns a criterion of the "quality" of the deformed set. Mathematically speaking, the "error functional" is a function $\Phi : \mathcal{P}(\mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ of all subsets of $\mathbb{R}^N$. We concentrate on the search for subareas giving a uniform impression. For example, the variance of $G|_K$ measures the oscillation of the grey-values in K. It motivates the more general assumption that $\Phi(\cdot)$ has the form

$$\Phi(K) \;=\; \psi\Big(\mathcal{L}^N(K),\ \int_K G\,dx,\ \int_K G^2\,dx\Big) \qquad \text{for all } K \subset \mathbb{R}^N,\ 0 < \mathcal{L}^N(K) < \infty.$$

$\psi(\cdot) \in C^2(]0,\infty[\, \times \mathbb{R} \times \mathbb{R})$ is supposed to satisfy *supp* $\psi \subset [0,c] \times \mathbb{R} \times \mathbb{R}$ with some $c \in]0,\infty[$. The latter condition prevents "explosions" of the deformed sets because $\mathcal{L}^N(K(t)) \ge c$ implies $\Phi(K(t)) = 0$ and hence the composition $\Phi(K(\cdot))$ cannot decrease any longer by expanding $K(t)$.

These preparations lead to the following segmentation problem (with the definition of "critical" sets given below) :

Given : function of grey-values $G \in BC(\mathbb{R}^N)$,
error functional $\Phi : \mathcal{P}(\mathbb{R}^N) \to \mathbb{R} \cup \{\infty\}$ of the form
$$\Phi(K) = \psi\Big(\mathcal{L}^N(K),\ \int_K G\,dx,\ \int_K G^2\,dx\Big) \quad \forall\ K \subset \mathbb{R}^N,\ \mathcal{L}^N(K) \in]0,\infty[.$$
with $\psi(\cdot) \in C^2(]0,\infty[\, \times \mathbb{R}^2)$, $proj_1(supp\ \psi)$ bounded,
initial set $K_0 \in \mathcal{K}(\mathbb{R}^N)$ with $\mathcal{L}^N(K_0) > 0$.

Wanted $F : \mathbb{R}^N \times [0,T[\rightsquigarrow \mathbb{R}^N$ $(T \in]0,\infty])$ satisfying
(a) $\vartheta_F(t, K_0)$ is closed for all $t \in [0,T[$,
(b) $0 \in F(x,t)$ for all $x \in \mathbb{R}^N$, $t \in [0,T[$,
(c) F has linear growth,
(d) $\Phi \circ \vartheta_F(\cdot, K_0) : [0,T[\to \mathbb{R}$ is nonincreasing,
(e) $M := \bigcup_{0 \le t < T} \vartheta_F(t, K_0)$ is "critical".

4. The absolute continuity of $\Phi(\vartheta_F(\cdot, K_0))$. Property (d) is of central interest. Hence we examine the regularity of the composition $\Phi(\vartheta_F(\cdot, K_0))$ now and investigate its derivative (if it exists). Then the sign of the latter provides a sufficient condition for the wanted nonincreasing.
Thanks to $\psi \in C^2(]0,\infty[\times I\!R^2)$, the chain rule allows us to concentrate on the function $t \mapsto \int_{\vartheta_F(t,K_0)} h(x)\ dx$ with $h \in L^1_{loc}(I\!R^N)$ (e.g. 1, G, G^2).
The evolution along vector fields has already been investigated in detail :

THEOREM 4.1 ([10]). *Let $K \subset I\!R^N$ be a Lebesgue measurable set satisfying the assumptions of Gauss' theorem and $v \in C^1_c(I\!R^{N+1}, I\!R^N)$. Then*

$$\lim_{t\to 0} \frac{\mathcal{L}^N(\vartheta_v(t,K)) - \mathcal{L}^N(K)}{t} = \int_{\partial K} v(x,0)\cdot \nu_K(x)\ d\omega_x$$

with $\nu_K(x)$ abbreviating the exterior unit normal vector of K at the point $x \in \partial K$.

This result is quite similar to the generalization to $\vartheta_F(t, K_0)$ although it cannot be extended immediately :

THEOREM 4.2. *Let $F : I\!R^N \times [0,T[\rightsquigarrow I\!R^N$ satisfy*
1. *$F(x,t)$ is compact, convex, $0 \in F(x,t)^\circ$ for all x, t,*
2. *F is continuous,*
3. *F has linear growth.*

Then the Lebesgue measure $\mathcal{L}^N(\vartheta_F(\cdot,K))$ is absolutely continuous for any $K \in \mathcal{K}(I\!R^N)$ and its (weak) derivative is

$$[0,T[\ \longrightarrow I\!R, \quad t \longmapsto \int_{\partial\, \vartheta_F(t,K)} \sup\ (F(x,t)\cdot \tilde{N}_{\vartheta_F(t,K)}(x))\ \ d\mathcal{H}^{N-1}x.$$

In comparison with Theorem 4.1, the unique normal vectors of the smooth boundary are replaced by the unit elements of the normal cone.

DEFINITION 4.3 ([3]). *Let $K \subset X$ be a subset of a normed vector space X and $x \in \overline{K}$ belong to the closure of K. The so-called* (Bouligand) contingent cone $T_K(x)$ *is defined by* $T_K(x) := \{ v \in X \mid \liminf_{h\downarrow 0} \frac{dist(x+hv,\,K)}{h} = 0 \}$,
the corresponding normal cone $N_K(x) := \{ p \in X^* \mid <p,v> \leq 0 \quad \forall\, v \in T_K(x) \}$,
in particular for a Hilbert space X $N_K(x) = \{ p \in X \mid p\cdot v \leq 0 \quad \forall\, v \in T_K(x) \}$.
Moreover, we add an abbreviation for the unit normal vectors :

$$\tilde{N}_K(x) := \begin{cases} N_K(x) \cap S^{N-1} & \text{if } N_K(x) \neq \{0\}, \\ S^{N-1} & \text{if } N_K(x) = \{0\}. \end{cases}$$

Assumption $0 \in F(x,t)^\circ$ implies that $\vartheta_F(\cdot, K)$ is even a "strict" expansion, i.e.

$$\overline{\vartheta_F(t_1,K)} \subset \vartheta_F(t_2,K)^\circ \qquad \text{for all } 0 \leq t_1 < t_2 < T.$$

This property is not useful for image segmentation because graphically speaking the deformed set needs the opportunity to stop when reaching the wanted contour. Hence we want to extend the previous result to the condition $0 \in F(x,t)^\circ \vee F(x,t) = \{0\}$. To guarantee that the trajectories can reach only points $x \notin K$ satisfying $0 \in F(x)^\circ$, we assume that F depends only on x and $F : I\!R^N \rightsquigarrow I\!R^N$ is Lipschitz continuous. (Then Gronwall's lemma implies $F(x) \neq \{0\}$ for any reached point $x \notin K$.)

Theorem 4.4. *Let the set-valued map $F : \mathbb{R}^N \rightsquigarrow \mathbb{R}^N$ be Lipschitz continuous and have compact, convex values satisfying $0 \in F(x)^\circ$ or $F(x) = \{0\}$.*
Then the Lebesgue integral $[0,\infty[\; \to \mathbb{R}, \quad t \mapsto \int_{\vartheta_F(\cdot,K)} h(x)\ dx$ is absolutely continuous for $K \in \mathcal{K}(\mathbb{R}^N)$, $h \in L^1_{loc}(\mathbb{R}^N)$.
Its (weak) derivative is $\int_{\partial\,\vartheta_F(t,K)} h(x)\ \sup\,(F(x) \cdot \tilde{N}_{\vartheta_F(t,K)}(x))\ d\mathcal{H}^{N-1}x$.

With respect to the segmentation problem we obtain the absolute continuity of $\Phi(\vartheta_F(\cdot,K))$ under the assumptions of Theorem 4.4 and its derivative has the form

$$\frac{d}{dt}\ \Phi(\vartheta_F(t,K_0)) \;=\; \int_{\partial\,\vartheta_F(t,K_0)} \varphi(x,\vartheta_F(t,K_0))\ \ \sup\,(F(x)\cdot \tilde{N}_{\vartheta_F(t,K_0)}(x))\quad d\mathcal{H}^{N-1}x$$

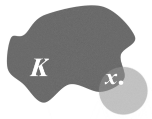
with the abbreviation $\varphi(x,M) \;:=\; \sum_{k=0}^{2}\ \partial_{k+1}\psi|_{\left(\mathcal{L}^N(M),\ \int_M G\ dy,\ \int_M G^2\ dy\right)} G(x)^k$
(for $x \in \mathbb{R}^N$ and $M \subset \mathbb{R}^N,\ 0 < \mathcal{L}^N(M) < \infty$).

5. Solution of the segmentation problem. Theorem 4.4 underlies a solution of the segmentation problem. As an approach for F, we assume $F(x) = r(x)\,B$ with a Lipschitz continuous function $r : \mathbb{R}^N \to [0,\infty[$, i.e. we prescribe only the speed, not the direction of each velocity. In comparison with morphological operators this expansion corresponds to a generalized dilation operator with its speed $r(\cdot)$ depending on the position x at the image (due to the symmetry of $B = -B$, the usual dilation operator leads to $\vartheta_{\varepsilon B}(t,K) = B_{\varepsilon t}(K) = \{y \in \mathbb{R}^N \,|\, \operatorname{dist}(y,K) \le \varepsilon\, t\}$ with a constant speed parameter $\varepsilon > 0$).
The ansatz $F(x) = r(x)\,B$ and Theorem 4.4 imply that $\Phi(\vartheta_F(\cdot,K))$ is nonincreasing if $\varphi(x,\vartheta_F(t,K_0)) \cdot r(x) \le 0$ for all $t \in [0,T[,\ x \in \partial\,\vartheta_F(t,K_0)$.

So far the assumption $G \in L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ would have sufficed. The main advantage of $G \in BC(\mathbb{R}^N)$ is now the continuity of

$$\mathbb{R}^N \times [0,T[\;\longrightarrow\; \mathbb{R}, \quad (x,t) \;\longmapsto\; \varphi(x,\ \vartheta_F(t,K))$$

if the set-valued map F satisfies the conditions of Theorem 4.2 or 4.4 and if $K \in \mathcal{K}(\mathbb{R}^N)$ has positive $\mathcal{L}^N$ measure.



As a consequence, whenever $\varphi(x,K) < 0$ for some $x \in \partial K$, then K can be expanded close to x for a short time such that $\Phi(\vartheta_{r\,B}(\cdot,K))$ is strictly decreasing. This method fails if $\varphi(x,K) \ge 0$ for all $x \in \partial K$. Hence we define a "critical" set :

Definition 5.1. *A set $K \subset \mathbb{R}^N$ with $0 < \mathcal{L}^N(K) < \infty$ is called* critical *if $\varphi(x,K) \ge 0$ for all $x \in \partial K$.*

The basic idea of the solution is to expand the set wherever φ is negative, i.e.

$$\varphi(x,\vartheta_{r\,B}(t,K_0)) > 0 \;\Longrightarrow\; r(x) = 0 \qquad \text{for all } t \in [0,T[\text{ and } x \in \mathbb{R}^N$$

(the latter instead of $x \in \partial\,\vartheta_{r\,B}(t,K_0)$ – for the sake of simplicity). Since the radius function $r(\cdot)$ (of $F(\cdot) = r(\cdot)\,B$) ought to depend only on x, it is very difficult to define it adequately for any future point of time.
Hence we construct it piecewise with respect to time : For each interval $[t_n, t_{n+1}]$ ($n \in \mathbb{N}$) the deformed set K_n at time t_n induces a Lipschitz continuous function $r_n : \mathbb{R}^N \to [0,\infty[$ and thus the deformed sets $\vartheta_{r_n\,B}(t - t_n,\ K_n)$ for $t \in [t_n, t_{n+1}]$ (including $K_{n+1} := \vartheta_{r_n\,B}(t_{n+1} - t_n,\ K_n)$).

The definition of $r_n(\cdot)$ is to guarantee finally for every $t \in [t_n, t_{n+1}]$

$$\forall\ x \in \partial\,\vartheta_{r_n\,B}(t-t_n,\,K_n): \quad r_n(x) > 0 \implies \varphi(x,\,\vartheta_{r_n\,B}(t-t_n,\,K_n)) < 0. \qquad (*)$$

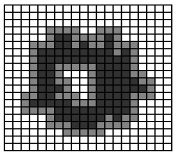
Hence we construct $r_n(\cdot)$ (with some $\delta > 0$) satisfying even

$$\forall\ x \in I\!\!R^N:\ \ r_n(x) > 0 \iff \varphi(x, K_n) < -\delta$$

and choose t_{n+1} then such that condition (*) is fulfilled until $t = t_{n+1}$.
After finitely many steps of this induction, all $x \in \partial K_n \cap (\frac{1}{\delta}B)$ satisfy $\varphi(x, K_n) > -2\delta$.
Then we replace δ by a smaller positive value (e.g. $\frac{\delta}{2}$) such that finally $\delta \longrightarrow 0$.

As a consequence, the continuity properties of φ imply that $M := \bigcup_{n\in I\!\!N} K_n$ is critical, i.e. $\varphi(x, M) \geq 0$ for every $x \in \partial M$.

6. Simple implementation for computer images. In comparison with the (continuous) segmentation problem, the main difference corresponds to discretizing : The smallest sensible unit of a computer image is one pixel or one voxel respectively. Furthermore the grey-values within each of these units are constant. Hence we concentrate on the decision if a pixel (or a voxel) belongs to the resulting set or not.



K_n denotes the finite union of pixels representing the deformed set at the n^{th} step (shown in dark-grey). The expansion to a neighbouring pixel $P \subset I\!\!R^N$ can again be regarded as reachable sets of a differential inclusion. Thus we underlie the same mathematical concept as in the previous section and use the criterion $\varphi(x, K_n) < -\delta$ with any point $x \in P^\circ$ (since $G|_{P^\circ}$ is constant).
If no further neighbouring voxel P (shown in light-grey) satisfies this condition then δ is replaced by a smaller positive value until reaching a given positive threshold.

Searching for areas with a uniform impression, we suggest the error functional

$$\begin{aligned}\Phi(K) &:= \beta\; Variance(G|_K) \;+\; \alpha\,\mathcal{L}^N(K)\\ &= \tfrac{\beta}{\mathcal{L}^N(K)} \int_K G^2\,dx \;-\; \beta\left(\tfrac{1}{\mathcal{L}^N(K)} \int_K G\,dx\right)^2 \;+\; \alpha\,\mathcal{L}^N(K)\end{aligned}$$

for all $K \subset I\!\!R^N$, $0 < \mathcal{L}^N(K) < \infty$, with the parameters $\alpha \geq 0$, $\beta > 0$.
Here the term $\alpha\,\mathcal{L}^N(K)$ allows tolerating small oscillations of grey-values while the deformed set keeps increasing. Then

$$\varphi(x, K_n) \;=\; \tfrac{\beta}{\mathcal{L}^N(K_n)}\; G(x)^2 \;-\; \tfrac{2\,\beta\,\int_{K_n} G\,dy}{\mathcal{L}^N(K_n)^2}\; G(x) \;-\; \tfrac{\beta\,\int_{K_n} G^2\,dy}{\mathcal{L}^N(K_n)^2} + \tfrac{2\,\beta\,(\int_{K_n} G\,dy)^2}{\mathcal{L}^N(K_n)^3} + \alpha.$$

is a quadratic polynomial of $G(x)$ whose coefficients are calculated merely once for all neighbouring pixels of K_n.

The resulting set depends mainly on three parameters : initial set K_0, quotient $\frac{\alpha}{\beta}$ and the threshold of δ (which represents the precision of calculation). In this connection the parameter $\beta > 0$ is only to scale the real values of the error functional Φ.

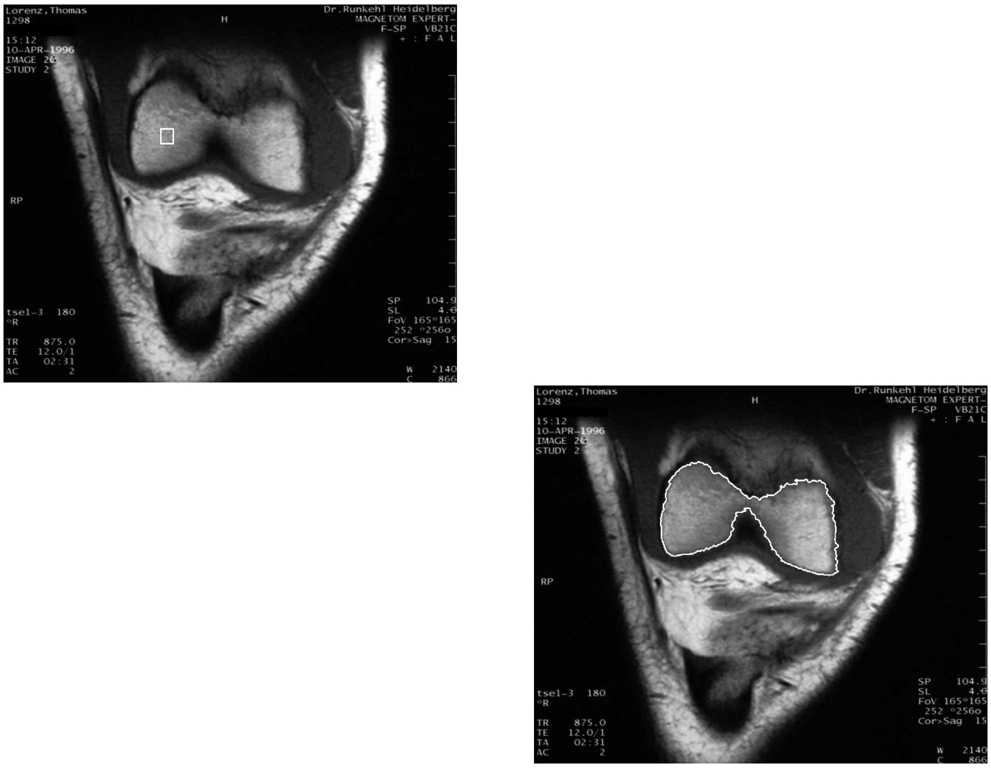


FIG. 1. *MR of right human knee.* (*Left : Initial set* K_0. *Right : resulting set*)

The two figures show examples of the computer implementation.
First, the bone inside a human knee is detected on MR images with 662 × 654 pixels and 250 grey levels (fig. 1, $\alpha = 1.6$, $\beta = 10^6$). Secondly, fractals in fig. 2 demonstrate the precise detection of boundaries. (This image has 570 × 445 pixels and 242 grey levels, $\alpha = 3.7$, $\beta = 10^7$.) Neither of these examples was prepared before applying the segmentation algorithm (i.e. no previous smoothing).

The computer method has three main advantages : It is very quick because the criterion applied to each neighbouring pixel is based on a quadratic polynomial of the grey value. Moreover it does not depend on the dimension of the image. Finally the extension to image sequences simply increases the dimension by 1 and restricts the set expansion to all (previous) space directions and the positive time direction.
The concept presented so far is isotropic. However the sequence of checking neighbouring pixels offers a simple opportunity of preferring given directions. This even leads to an anisotropic modification – with the same mathematical background.

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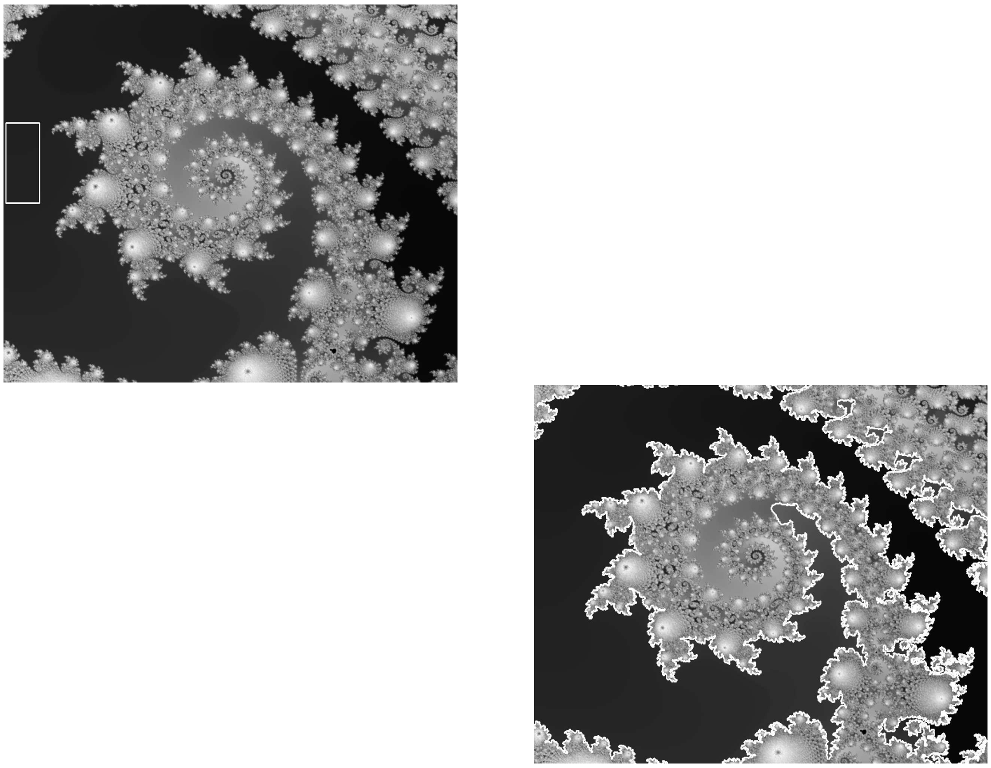

FIG. 2. *Fractal sets* . (*Left : Initial set* K_0. *Right : resulting set*)

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