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ALGORITHMS FOR OPTIMAL SPLINE INTERPOLATION ‡

J. KOBZA

Abstract. Splines interpolating some function, mean or derivative values can have some free parameters which are usually used to fulfil some boundary conditions prescribed. We can use them also to find the splines with minimal value of some norm or quadratic form on the linear space of splines on the given knotset. The problem can be solved explicitly in simple cases, or with several special LSQ or difference equations techniques mentioned below. In the general case we can use the algorithms of the quadratic programming. Some overview of such results for the low order polynomial splines on the knotset with points of interpolation between spline knots is presented.

Key words. polynomial spline, minimal norm interpolation

AMS subject classifications. 41A15, 65D05

1. Statement of the problem. Let us have given the monotone spline knotset $\mathbf{x} = \{x_i;\ i = 0(1)n+1\}$ on the real axis with stepsizes $h_i = x_{i+1} - x_i$ and the data $\mathbf{g} = \{g_i,\ i = 0(1)n\}$. *Polynomial spline of the degree* l (with the defect one) is a piecewise polynomial function $s(x) = s_{l1}(x) \in C^{l-1}$, which is a polynomial of the l-th degree on each interval $[\ x_i, x_{i+1}]$. We will consider the cases where the prescribed values $g_i,\ i = 0(1)n$ are

- function values $\quad g_i = s(t_i)$ in the points $t_i \in (x_i, x_{i+1})$ (points of interpolation between spline knots), (FVI)
- the local mean values $\quad g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x)dx,$ (MVI)
- the derivative values $\quad g_i = s'(t_i).$ (DVI)

Polynomial splines of the given degree with knots $\mathbf{x}$ form a linear space. The conditions of interpolation mentioned do not determine interpolating spline uniquely - there are some free parameters, which are usually used for some boundary conditions prescribed (see e.g. Spline Toolbox in Matlab, [8]) or for some optimization purposes. In this contribution we restrict the optimization process to the linear space of splines on the given knotset and with the given low degree (1-4). Spline free parameters will be used to minimize some norm (discrete or continuous) of the spline derivative chosen. We will discuss the minimization of functionals (with various choice of k)

$$J_{kd} = \sum_{i=0}^{n+1}[s^{(k)}(x_i)]^2, \quad J_k(s) = \int_{x_0}^{x_{n+1}}[s^{(k)}(x)]^2 dx. \tag{1.1}$$

The case of points of interpolation coinciding with spline knots is discussed for quadratic splines separately in [11]. The detailed discussion and proofs of statements are given in [10]–[14].

2. Interpolating polygon. Let us have given the mesh ($\mathbf{x}$,$\mathbf{t}$) of spline knots x_i and points of interpolation $t_i,\ x_i < t_i < x_{i+1}$ with prescribed function values $g_i = s(t_i)$. We shall denote the local parameters of the *FVI polygon* $s(x) \in C^0$ with

*Department of MAaAM, FS UP Olomouc, Tomkova 40, 77900 Olomouc, Czech Republic
†e-mail address: kobza@risc.upol.cz
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the knotset geometry local parameters $d_i = (t_i - x_i)/h_i$ as

$$(2.1) \qquad s_i = s(x_i), \quad \mathbf{s} = [s_i]_{i=0}^{n+1}, \quad m_i = s'(t_i), \quad \mathbf{m} = [m_i]_{i=0}^{n}, \quad \mathbf{g} = [g_i]_{i=0}^{n} .$$

It is easy to show the following recurrence relations (*continuity conditions – CC*) for unknown parameters s_j, m_j :

$$(2.2) \quad (1-d_j)s_j + d_j s_{j+1} = g_j , \quad h_j(1-d_j)m_j + d_{j+1}h_{j+1}m_{j+1} = g_{j+1} - g_j .$$

From these recurrences we can obtain by induction the explicit expressions for components of **s, m** dependent on initial values s_0, m_0 and data **g** and also the explicit expressions for initial values of s_0 or m_0 corresponding to polygons with minimal values of $J_{kd}(s), k = 0, 1$ (see [10]). Functionals $J_0(s), J_1(s)$ we can write as

$$(2.3) \qquad J_0(s) = \frac{1}{3}\sum_{i=0}^{n} h_i(s_i^2 + s_i s_{i+1} + s_{i+1}^2), \qquad J_1(s) = \sum_{i=0}^{n} h_i m_i^2$$

and their minimization under conditions 2.2 is an quadratic programming problem. The components of the vector **m** corresponding to the minimum of the functional $J_1(s)$ can be computed also from the second part of continuity conditions 2.2 completed to the regular system of linear equations with necessary condition of minima

$$(2.4) \qquad \sum_{j=0}^{n+1} \frac{1}{d_j} c_0^{j-1} m_j = 0; \quad c_0^{-1} = 1, \quad c_0^j = \prod_{k=0}^{j}(1 - \frac{1}{d_k}).$$

Vectors **s, m** of local parameters minimizing functionals $J_{0d}(s), J_{1d}(s)$ can be computed in the most simple way as pseudoinverse solution of the underdetermined system of corresponding continuity conditions.

Similar approach can be applied (see [10]) to the MVI and DVI problems. For the *MVI problem* the CC can be written as recurrences (with $m_i = s'(x_i + 0)$)

$$(2.5) \qquad s_i + s_{i+1} = 2g_i; \quad h_i m_i + h_{i+1}m_{i+1} = 2(g_{i+1} - g_i)$$

and we can obtain explicit expression e.g. for the value of s_0 minimizing norm of **s** –

$$(2.6) \qquad s_0 = \frac{1}{n+2}\sum_{j=0}^{n}(-1)^{n+j}(j+1)g_{n-j}.$$

The most simple way how to compute vector **s (m)** with minimal norm is again the pseudoinverse solution of the corresponding system of CC 2.5.

In the *DVI problem* with given values $g_i = m_i = s'(t_i)$ the initial value s_0 giving the minimum to $J_{od}(s)$ can be computed as

$$(2.7) \qquad (n+2)s_0 = -\sum_{j=0}^{n}(n+1-j)h_j m_j .$$

Some special algorithms for computing optimal solutions of similar problems will be mentioned in the last section. Generally, all functionals $J_k(s), J_{kd}(s)$ mentioned till now and in the following sections can be expressed as quadratic forms in proper local spline parameters. When we write the spline continuity conditions as equations

(recursions, difference equations) in the corresponding parameters, we obtain special quadratic programming problem with equality constrains (with full rank matrix) and we can solve it with standard QP algorithms (e.g. **qp** in Matlab Optimization Toolbox). According to the optimization theory (see e.g. [4]), the positive definiteness of the matrix of the quadratic form is sufficient condition for uniqueness of the solution (in case of semidefiniteness we have to use more involved technique of nullspaces to prove the uniqueness.)
The results of this section we can summarize in the following theorem.

THEOREM 2.1. *For the FVI and MVI problems there exist unique polygons minimizing the functionals $J_{kd}(s), J_k(s),\ k = 0, 1$. The problem DVI (with $g_i = s'(t_i)$) has unique solution with minimal value of $J_{0d}(s)$.*

3. Quadratic splines. Optimal interpolation with quadratic splines and points of interpolation $t_i = x_i$ is discussed in [11]; here we shall deal with the more interesting case of FVI with $t_i \neq x_i$ only and $g_i = s(t_i)$ (the details can be found in [13]).
Let us denote $s(x) = s_{21}(x) \in C^1$ a quadratic interpolatory spline on the knotset $(\mathbf{x},\mathbf{t})$, $M_i = s''(t_i),\ i = 0(1)n$. The local spline representation and continuity conditions can be written in different local parameters ($\mathbf{s},\ \mathbf{m},\ \mathbf{M}$ – see [8]) and used for computing (with two free parameters) of the whole vector of optimal local parameters giving minimum to the functionals $J_{kd}(s), J_k(s),\ k = 0, 1, 2$. As a rule we cannot write now the explicit formulas for optimal values of spline parameters, but we can describe algorithms for their computing and to prove the existence of such optimal splines.

3.1. Function and mean values interpolation. The local representation of FVI quadratic spline with parameters $\mathbf{g},\ \mathbf{m}$ we can write as

$$s(x) = g_i + \frac{1}{2}h_i(u - d_i)[(2 - u - d_i)m_i + (u + d_i)m_{i+1}],\ d_i = (t_i - x_i)/h_i \tag{3.1}$$

(with $u = (x - x_i)/h_i,\ m_i = s'(x_i)$) and similarly for MVI problem and another local parameters. The continuity conditions with given g_i and different local parameters M_i can be written for FVI problem as

$$\frac{1}{8}(M_{i-1} + 6M_i + M_{i+1}) = 2[t_{i-1}, t_i, t_{i+1}]g \tag{3.2}$$

for equidistant knotset $\mathbf{x}$ with all $d_i = \frac{1}{2}$ (for the general case see [11]), with local parameters M_i, or

$$h_{i-1}m_{i-1} + 3(h_{i-1} + h_i)m_i + h_i m_{i+1} = 8(g_i - g_{i-1}), \tag{3.3}$$

$$\frac{1}{h_{i-1}}s_{i-1} + 3\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)s_i + \frac{1}{h_i}s_{i+1} = 4\left(\frac{g_{i-1}}{h_{i-1}} + \frac{g_i}{h_i}\right) \tag{3.4}$$

on the general knotset $(\mathbf{x},\mathbf{t})$ with $d_i = \frac{1}{2}$, with local parameters m_i, s_i.
For functionals considered in FVI problem we obtain

$$J_{2d}(s) = \sum_{i=0}^{n} M_i^2, \quad J_2(s) = \sum_{i=0}^{n} h_i M_i^2; \tag{3.5}$$

$$J_{1d}(s) = \sum_{i=0}^{n+1} m_i^2, \quad J_1(s) = \frac{1}{6}\mathbf{m}^{\mathbf{T}}\mathbf{R}\mathbf{m}, \tag{3.6}$$

where the tridiagonal symmetric positive definite matrix $\mathbf{R}$ has the main diagonal and subdiagonals dependent on the geometry of the knotset only –

$$(3.7)\qquad diag(\mathbf{R}) = [2h_0, 2(h_0+h_1), \cdots, 2(h_{n-1}+h_n), 2h_n],\ subdiag(\mathbf{R}) = \mathbf{h}.$$

For the knotset $(\mathbf{x},\mathbf{t})$ with $d_i = \frac{1}{2}$ we obtain

$$(3.8)\ J_{0d}(s) = \sum_{i=0}^{n+1} s_i^2,\quad J_0(s) = \sum_{i=0}^{n} h_i \left[8g_i^2 + 2g_i(s_i+s_{i+1}) + 2s_i^2 + 2s_{i+1}^2 - s_i s_{i+1}\right]$$

with symmetric positive definite M-matrices of quadratic forms.
For the corresponding MVI problems we obtain similar (but different) CC

$$(3.9)\qquad \frac{1}{6}(M_{i-1} + 4M_i + M_{i+1}) = \frac{1}{h^2}(g_{i-1} - 2g_i + g_{i+1})$$

$$(3.10)\qquad h_{i-1}m_{i-1} + 2(h_{i-1}+h_i)m_i + h_i m_{i+1} = 6(g_i - g_{i-1})$$

$$(3.11)\qquad \frac{1}{h_{i-1}}s_{i-1} + 2(\frac{1}{h_{i-1}} + \frac{1}{h_i})s_i + \frac{1}{h_i}s_{i+1} = 3(\frac{g_{i-1}}{h_{i-1}} + \frac{g_i}{h_i}).$$

The functionals $J_1(s), J_2(s)$ have identical expression as for FVI problem. For $J_0(s)$ we obtain quadratic form with symmetric positive definite matrix (see[13])

$$(3.12)\qquad J_0(s) = \frac{1}{15}\sum_{i=0}^{n} h_i[18g_i^2 - 3g_i(s_i+s_{i+1}) + 2s_i^2 + 2s_{i+1}^2 - s_i s_{i+1}].$$

According to the known theorems from quadratic programming theory (see [4]), we can prove the following theorem (details in [13]):

THEOREM 3.1. *In all mentioned cases of FVI and MVI problems there exist unique quadratic splines minimizing functionals* $J_{kd}(s), J_k(s),\ k = 0,1,2$.

Example 1. The results of discrete data interpolation with natural cubic spline and quadratic spline with minimal value of J_{1d} (spline knots in the midpoints of intervals) are plotted on Fig. 1. We can see nearly identical curves; also plots of the first derivatives are satisfactory close.

Example 2. The results of MVI interpolation (histopolation) with minimal values of J_{1d}, J_1, J_2 are plotted on Fig. 2. We can see small differences in the middle intervals and different behavior at the boundaries.

3.2. Interpolation of the derivative values. The DVI problem has some special features – e.g. the case when we prescribe the value $g_i = s'(t_i)$ in the midpoints of intervals $[x_i, x_{i+1}]$ needs a special treatment, the optimal solutions for parameters $k = 1,2$ are not unique (till to an arbitrary additive constant). Also the continuity conditions have more complex form in this case.
Using e.g. the local representation with parameters s_i, m_i, g_i we obtain the continuity conditions as the system

$$(3.13)\qquad \frac{s_{i+1}-s_i}{h_i} + \left(\frac{1}{2d_i} - 1\right)m_i = \frac{1}{2d_i}g_i\ ,$$

$$(3.14)\qquad (1-d_i)m_i + d_i m_{i+1} = g_i,\quad i = 0(1)n.$$

Only the parameters m_i appear in the second part and so we can minimize $J_{1d}(s), J_1(s)$ under this equality conditions only, then choose initial value s_0 and to compute remaining values s_i. The explicit form of necessary condition which completes this part

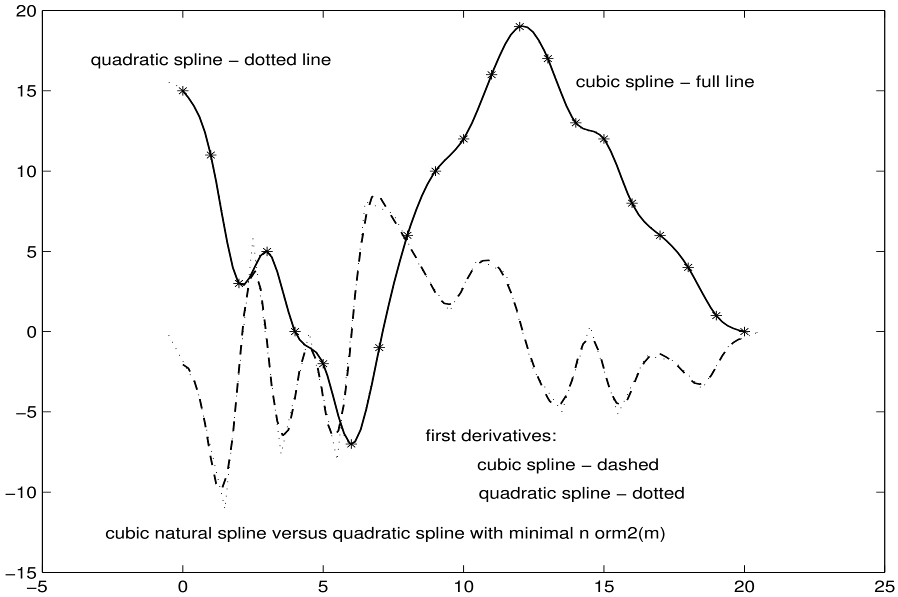

Fig. 1

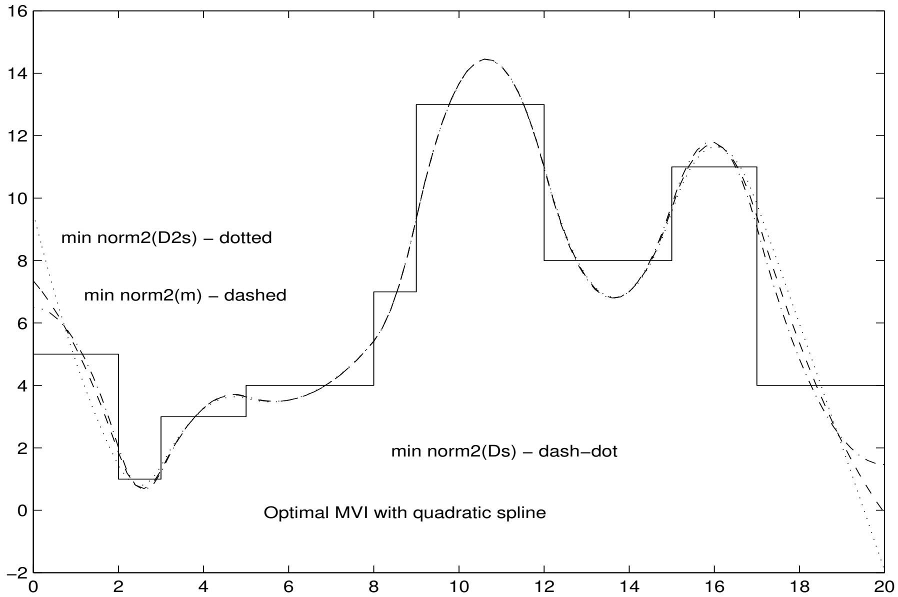

Fig. 2

of CC to the regular system of equations and enables to obtain optimal parameters m_i in this way is given in [13], where more detailed discussion can be found.
The results of our discussion concerning DVI problem can be summarized in the following theorem.

THEOREM 3.2. *There exist quadratic DVI splines which minimize the functionals* $J_{kd}(s), J_k(s),\ k = 0, 1, 2$. *In case of* $k = 0$ *there is the unique optimal solution. In* *cases of* $k = 1, 2$ *we obtain the optimal solution dependent on one free parameter* s_j *(additive constant).*

4. Cubic splines. We can apply our approach also to the cubic FVI splines $s(x) = s_{31}(x)$ (see [12]). For the spline local representation with local parameters **s,m** and local variable $u = (x - x_i)/h_i$, parameters $s_i = s(x_i),\ m_i = s'(x_i)$ we have

$$s(x) = (1-u)^2(1+2u)s_i + u^2(3-2u)s_{i+1} + h_i u(1-u)[(1-u)m_i - u m_{i+1}] \tag{4.1}$$

the continuity conditions can be written as recurrences

$$a_i m_{i-1} + 2m_i + c_i m_{i+1} = f_i\ , \quad i = 1(1)n, \tag{4.2}$$
$$a_i = \frac{h_i}{h_{i-1} + h_i};\ c_i = 1 - a_i;\ f_i = 3\left[c_i \frac{s_{i+1} - s_i}{h_i} + a_i \frac{s_i - s_{i-1}}{h_{i-1}}\right].$$

The matrix of CC has full rank and we can compute local parameters of the cubic spline with minimal l_2-norm of the vector **m** using pseudoinverse approach. We can compute expressions for functionals $J_k(s)$ and we obtain e.g.

$$J_1(s) = \sum_{i=0}^{n}[\frac{h_i}{15}(2m_i^2 - m_i m_{i+1} + 2m_{i+1}^2) \tag{4.3}$$
$$+\frac{1}{5}(s_i - s_{i+1})(m_i + m_{i+1}) + \frac{6}{5h_i}(s_i - s_{i+1})^2];$$

$$J_2(s) = 4[\sum_{i=0}^{n}\frac{1}{h_i}(m_i^2 + m_i m_{i+1} + m_{i+1}^2) \tag{4.4}$$
$$+\sum_{i=0}^{n}\frac{3}{h_i^2}(m_i + m_{i+1})(s_i - s_{i+1}) + \sum_{i=0}^{n}\frac{3}{h_i^3}(s_i - s_{i+1})^2].$$

We can recognize the positive definiteness of quadratic forms $J_k(s),\ k = 0, 1, 2$ (but singular matrix appears in $J_3(s)$). The quadratic programming algorithms can be used to compute optimal values of unknown parameters m_i from known data s_i. More details are given in [12].

THEOREM 4.1. *For given spline knots* **x** *and function values* **s** *in knots there exists unique cubic interpolatory spline with minimal value of functional* $J_{1d}(s)$ *or* $J_{2d}(s)$. *Its local parameters* $m_i = s'(x_i)$ *or* $M_i = s''(x_i)$ *can be computed as pseudoinverse solution to the underdetermined system of corresponding continuity conditions. There exist also the unique cubic interpolatory splines with minimal values of functionals* $J_k(s),\ k = 0, 1, 2$. *Their local parameters* m_i *or* M_i *we can compute with quadratic programming algorithms as minimizer of* $J_k(s)$ *under corresponding continuity conditions.*

In [12] we can find also the statements for the optimal Hermite cubic interpolatory spline with prescribed values of the function and the first derivative in the midpoints $t_i = (x_i + x_{i+1})/2$ (two free parameters for optimization purposes again).

THEOREM 4.2. *On the spline knotset with points of interpolation in midpoints* t_i *there exist the unique Hermite interpolatory cubic spline* $s_{32}(x)$ *with minimal value of the functional* $J_k(s),\ k \in \{0, 1, 2\}$ *or* $J_{0d}(s), J_{1d}(s)$ *for the data given.*

We can use also the local parameters **s, M** to find the cubic spline with minimal norm of **M**. When we compute cubic splines with minimal values of norms of vectors **m, M** corresponding to the data from the Example 1, the plots of such splines are very similar to the result obtained for natural cubic spline.

5. Quartic splines. For the quartic splines on the knotset $(\mathbf{x})$ we have yet more variants for local representations and corresponding CC (given mean values g_i or function values $g_i = s(t_i)$, four unknown local parameters taken in knots properly from the values $s_i, m_i, M_i, T_i = s'''(x_i)$ - see [6],[7]). The local representation e.g. with parameters $\mathbf{g}$, $\mathbf{m}$, $\mathbf{M}$ can be written with the local variable $u = (x - x_i)/h_i$ as

$$s(x) = \psi(u)g_i + h_i[\varphi_0^1(u)m_i + \varphi_1^1(u)m_{i+1}] + h_i^2[\varphi_0^2(u)M_i + \varphi_1^2(u)M_{i+1}] \tag{5.1}$$

with basis functions ψ, φ_j^i different for FVI, MVI problems (described e.g. in [7]). A quartic FVI or MVI spline has under our conditions four free parameters. We can write the CC for *the FVI problem* on the equidistant knotset (with t_i in midpoints) as the recursions (found as the result of some not quite simple elimination process) e.g. for local parameters m_i (similarly for the another – see [6])

$$\frac{1}{384}(m_{i-2} + 76m_{i-1} + 230m_i + 76m_{i+1} + m_{i+2}) = \frac{1}{6h}(-g_{i-2} - 3g_{i-1} + 3g_i + g_{i+1}).$$

When we find optimal values of m_i (e.g. by pseudoinverse when minimizing $J_{1d}(s)$), then we have to use special formulas for the values M_i following from the whole system of CC (see [14]). Such form of CC we can use in case that we search e.g. for the spline with minimal norm of $\mathbf{m}$. When we use a spline local representations as 5.1, then the functionals $J_k(s), k = 0, 1, 2, 3$ are now some quadratic forms in local parameters used. In such cases we have to use also the CC written as recurrences in the same two parameters – e.g. m_i, M_i (and given values g_i). As the result we obtain for FVI optimization problem on the equidistant knotset $\mathbf{x}$ the equality constrains (see [6])

$$\frac{1}{32}(3m_{i-1} + 26m_i + 3m_{i+1}) + \frac{5}{192}h(M_{i-1} - M_{i+1}) = \frac{1}{h}(g_i - g_{i-1}), \tag{5.2}$$

$$\frac{1}{2h}(m_{i-1} - m_{i+1}) + \frac{1}{6}(M_{i-1} + 4M_i + M_{i+1}) = 0, \quad i = 1(1)n, \tag{5.3}$$

which contain all local parameters needed. We can use them to compute quartic spline with minimal norm of the vector $[\mathbf{m};\mathbf{M}]$ as the pseudoinverse solution of such system of equations.
For the MVI problem with similar parameters we obtain on equidistant knotset the CC as recurrences ($i = 1(1)n$)

$$\begin{aligned} \frac{9}{2}m_{i-1} + 21m_i + \frac{9}{2}m_{i+1} + h(M_{i-1} - M_{i+1}) &= \frac{30}{h}(g_i - g_{i-1}), \\ m_{i-1} - m_{i+1} + \frac{h}{3}(M_{i-1} + 4M_i + M_{i+1}) &= 0. \end{aligned} \tag{5.4}$$

We can compute the expressions for functionals $J_k(s)$ - e.g. for FVI problem and the knotset with $d_i = \frac{1}{2}$ and for MVI problem we obtain equal expressions for

$$\begin{aligned} J_1(s) = \sum_{i-0}^{n} \frac{h_i}{210}[&78m_i^2 + 54m_i m_{i+1} + 78m_{i+1}^2 + h_i^2(2M_i^2 - 3M_i M_{i+1} + 2M_{i+1}^2) \\ &+ h_i(22m_i M_i - 13m_i M_{i+1} + 13m_{i+1}M_i - 22m_{i+1}M_{i+1})], \end{aligned} \tag{5.5}$$

$$\begin{aligned} J_2(s) = \sum_{i=0}^{n} \frac{1}{15h_i}[&18(m_i - m_{i+1})^2 + h_i^2(2M_i^2 - M_i M_{i+1} + 2M_{i+1}^2) \\ &+ 3h_i(m_i - m_{i+1})(M_i + M_{i+1})], \end{aligned} \tag{5.6}$$

$$J_3(s) = \sum_{i=0}^{n} \frac{4}{h_i^3}[3(m_i - m_{i+1})^2 + h_i^2(M_i^2 + M_i M_{i+1} + M_{i+1}^2)$$
$$+3h_i(m_i - m_{i+1})(M_i + M_{i+1})]. \tag{5.7}$$

We can obtain also the different expressions for functionals $J_0(s)$ for FVI and MVI problems - more details are given in [14]. The block matrices of quadratic forms $J_0(s), J_1(s)$ are positive definite, but the matrices for $J_2(s), J_3(s)$ positive semidefinite only.

THEOREM 5.1. *The problems of optimal interpolation with quartic splines on the given knotset* $\mathbf{x}$
– have the unique solution for functionals $J_{kd}(s), k = 0, 1, 2, 3$ *for MVI problems on equidistant knotset* $\mathbf{x}$,
– have the unique solution for these functionals and FVI problem on equidistant knotset $\mathbf{x}$ *with* $d_i = \frac{1}{2}$,
– have the unique solution with minimal l_2*-norm of the vector* $[\mathbf{m}; \mathbf{M}]$,
– have the unique solution for the functionals $J_k(s), k = 0, 1, 2, 3.$.

Remarks. The last result for $k = 2, 3$ has been proved using local parameters **g,m,T** - the matrices of corresponding quadratic form are singular and more involved technique of matrix nullspaces has to be used - see [14].
Such result shall be valid also for slightly nonequidistant knotsets.

6. Computational algorithms. As we have recognized, all optimization problems discussed above can be stated as quadratic programming problems with equality constrains. Our equality constrains – the spline continuity conditions (CC) – have a special form of the linear difference equation written for some set of indexes. We have demonstrated in this contribution different possibilities how to compute the local parameters of the optimal spline we search for. In some special cases or generally we can choose some of the following approaches:

1. To find explicit formulas for optimal values of free parameters (linear, some quadratic cases);
2. To complete the equality constrains with the necessary condition of minima and find all optimal components of local parameters as the solution of such extended system of linear equations (linear, quadratic cases);
3. To compute the optimal values of unknown local parameters as the pseudoinverse solution of the system of CC (in the cases of functionals $J_{kd}(s)$);
4. In the cases of functionals $J_{kd}(s), J_k(s)$ of the form $J(s) = \mathbf{y}^{\mathbf{T}}\mathbf{R}\mathbf{y}$ with symmetric positive definite matrix $\mathbf{R}$ we can use the Boundary Value Method technique for the stable computing of the fundamental system $[u^i]$ of the homogeneous equation and partial solution v of the nonhomogeneous difference equation (see [3],[9]). We can find the coefficients of the solution $y = \sum c_i u_i + v$ of the difference equation with minimal norm defined by the scalar product $(u, v)_R = u^T \mathbf{R} v$ as the solution of the normal system of equations $\sum c_j(u^i, u^j)_R = -(u^i, v)^R$.
5. In more general cases we can use standard algorithms of quadratic programming (e.g. **qp** in Matlab Optimization Toolbox).
6. We can use the B-spline basis to express functional minimized as quadratic form in B-spline coefficients and to solve unconstrained quadratic programming problem (CC are fulfilled implicitly).

More details are given in [9]–[13]. The author and his coworkers have prepared many Matlab M-files for dealing with low degree splines in local representations – see e.g. [15]. For the optimal interpolation with quartic splines and local spline parameters a special M-file **s4opte.m** was worked out and can be obtained from the author (the algorithms using B-spline basis and algorithms for smoothing splines are under preparation). We mention only its syntaxis here :

function [pr1,pr2,val]=s4opte(x,g,p1,p2,w)
computes local parameters of the quartic FVI and MVI spline with minimal norm chosen with input parameters by the user.

Input parameters:
x ... vector of monotone growing sequence of spline knots (equidistant, general);
g ... vector of FV (in midpoints) or MV on the knotset **x**;
p1 =1 for equidistant knotset (used with **p2**(1)=1,2,3; **p2**=[5,1]);
p2 ... the vector [j,k] for choice of the problem and functional minimized;
[1,k], k=0:3 .. minimizes Jkd – the vector of FVI spline k-th derivative;
[2,k], k=0:3 .. minimizes Jkd – in MVI problem;
[3,k], k=0:3 .. minimizes Jk – the L2-norm, FVI problem;
[4,k], k=0:3 .. minimizes Jk in MVI problem;
[5,1],[5,2] min norm of [**m;M**] for FVI, MVI problem;
w ... vector of weighting coefficients in Jkd .

Output parameters:
pr1, pr2 ... vectors of optimal spline parameters;
val... minimal value of the functional.

Example 3. For the knotset **x**=0:2:20, **t**=1:2:29
and monotone data **g**=[1,10,10,10,11,13,14,15,15,15,16,17,18,20,20] the quartic FVI splines with minimal values of functionals $J_k(s)$, $k = 0, 1, 2, 3$ are plotted on Fig. 3.

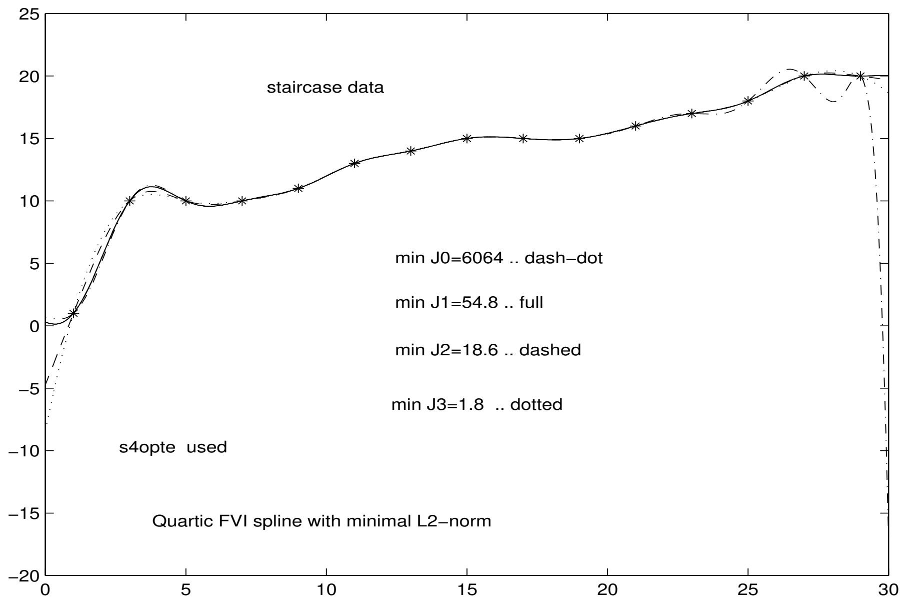

Fig. 3

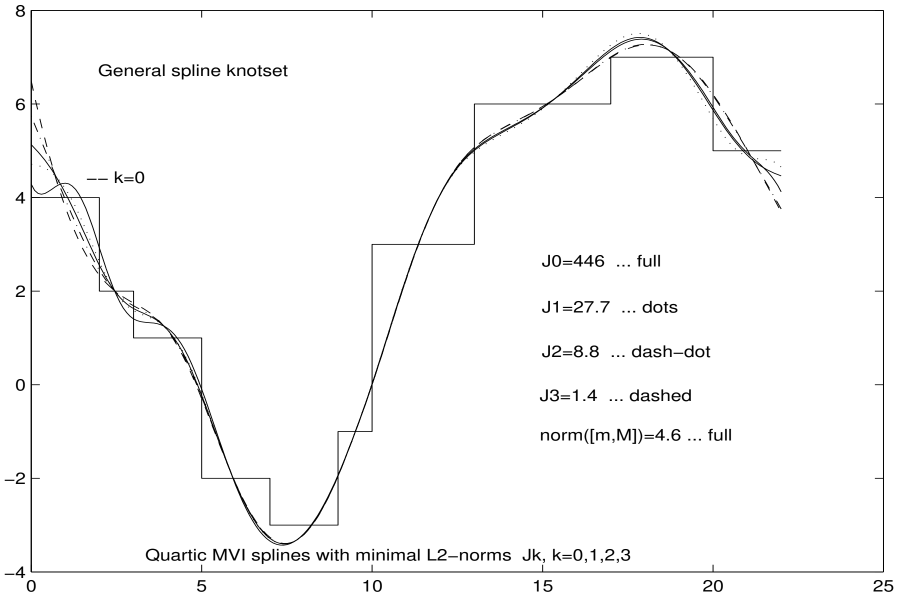

Fig. 4

Example 4. For the general knotset $\mathbf{x}$=[0,2,3,5,7,9,10,13,17,20,22] and data $\mathbf{g}$=[4,2,1,-2,-3,-1,3,6,7,5] the plots of MVI splines with minimal values of $J_k(s)$, $k = 0, 1, 2, 3$ and minimal value of the norm of [$\mathbf{m}$,$\mathbf{M}$] are plotted in Fig. 4.

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