*Acta Mathematica Academiae Paedagogicae Nyíregyháziensis*
**33** (2017), 367–372
www.emis.de/journals
ISSN 1786-0091

# AN ESTIMATE FOR THE ENTROPY OF NONPOSITIVELY CURVED FINSLER MANIFOLDS

CHANG-WAN KIM

*To my wife Yeon-Pyo Kim and my daughter Keun-Young Kim*

Abstract. In this paper, we expose to estimate the measure theoretic entropy of the geodesic flows for compact Finsler manifolds of nonpositive flag curvature.

## Introduction

Since the notion of Finsler manifolds is a generalization of Riemannian manifolds, it seems natural to consider the problem: To what extent can one extend results in Riemannian geometry to Finsler manifolds? The work of Foulon [6] is one of the first to extend the ideas of Hopf and Green to Finsler manifolds without conjugate points. Notably Foulon has developped the existence of a Riccati equation associated to the Jacobi equation and its connection to Lyapunov exponents and measure theoretic entropy. The measure theoretic entropy of a measure preserving flow is an asymptotic quantity associated with the flow. In this note we discuss an estimate of Foulon [6] for the measure theoretic entropy $h_\mu$ of the geodesic flows for compact Finsler manifolds with nonpositive flag curvature.

**Theorem.** *Let $M$ be a compact Finsler manifold with nonpositive flag curvature $K$. Then we have*

$$h_\mu \geq \int_{SM} \operatorname{tr} \sqrt{-K(v)}\, d\mu(v),$$

*where $SM$ is the unit tangent bundle of $M$ and $\mu$ is the normalized Liouville measure on $SM$. Equality holds if and only if the flag curvature is parallel along the geodesic flows.*

2010 *Mathematics Subject Classification.* 53C60, 53D25.
*Key words and phrases.* Finsler geometry, Riccati equations, measure theoretic entropy, nonpositive flag curvature.

A smooth Finsler metric is said to be parallel if the flag curvature is parallel along the geodesic flows. A Finsler manifold is locally symmetric if the geodesic reflection is a local isometry. It is well-known that a locally symmetric Finsler metric is parallel. In contrast to the Riemannian case, the converse is not true in general.

Our proof of main theorem is simplification of the proof of Foulon and it works under the weaker assumption of nonpositive flag curvature. We will be achieved by approximation of the flag curvature bounded above by a strictly negative constant, which does not mean that of the Finsler metric on $M$. This theorem was proved for the geodesic flows on Riemannian manifolds by Ballmann and Wojtkovski [1] and the Hamiltonian flows on symplectic manifolds by Chittaro [2].

## 1. Manifolds without conjugate points

In this section, we give a brief description of fundamental formulas in Finsler geometry, for more details the reader is referred to see [9]. A Finsler manifold $M$ is a smooth manifold for which a norm $F$ is prescribed on every tangent space $TM$. The unit sphere of this norm is assumed to be strictly convex in the sense that the Hessian

$$g_v(u,w) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(v + s\cdot u + t\cdot w)\Big]\Big|_{s=t=0}$$

is positive definite. For a tangent vector $v$ of $M$, denote by $\gamma_v(t)$ the geodesic with initial velocity $v$. The geodesic flow $\varphi_t$ of $M$ is defined $\varphi_t(v) = \gamma_v'(t)$. The geodesic flow acts on the unit tangent bundle $SM$ of $M$, and it leaves the normalized Liouville measure $\mu$ of $SM$ invariant (see [4]). Along the geodesic $\gamma_v(t)$, we have the osculating Riemannian metrics $g_{\gamma_v'(t)}(\cdot,\cdot)$ on $T_{\gamma_v(t)}M$. Define the flag curvature

$$K(\gamma_v'(t)) : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$$

by

$$K(\gamma_v'(t))(w) := R(w, \gamma_v'(t))\gamma_v'(t),$$

where $R$ is the Riemann curvature.

A Finsler manifold $M$ does not have conjugate points if for each pair of points in the universal covering space of $M$ there is a unique geodesic passing through the give pair of points. Finsler manifolds with nonpositive flag curvature are well-known examples such manifolds. We have the stable Jacobi tensors of Finsler manifolds without conjugate points as in the case of Riemannian manifolds (see [6]). There are stable Jacobi tensors, $J(\varphi_t(v))$, which satisfies the Jacobi equations

$$J''(\varphi_t(v)) + K(\varphi_t(v))\cdot J(\varphi_t(v)) = 0.$$

Making the change of variables

$$U(\varphi_t(v)) := \big(\ln J(\varphi_t(v))\big)' = J'(\varphi_t(v))\cdot J^{-1}(\varphi_t(v))$$

for $t$ values for which $\det J(\varphi_t(v)) \neq 0$. In fact, if $\gamma_v(t)$ has no points conjugate to $\gamma(0)$ on $(0,\infty)$, then $J(\varphi_t(v))$ is defined for all $t \in (0,\infty)$. Then the tensor $U(v)$ defined on the unit tangent bundle $SM$ such that for every $v \in SM$, $U(\varphi_t(v))$ is a self-adjoint linear operator on

$$\gamma_v'(t)^\perp := \{w \in T_{\gamma_v(t)}M \mid g_{\gamma_v'(t)}(\gamma_v'(t), w) = 0\}$$

and satisfies the *Riccati equation*

$$U'(\varphi_t(v)) + U^2(\varphi_t(v)) + K(\varphi_t(v)) = 0.$$

It is also known that $U(v)$ is the second fundamental form of the stable horosphere through $v$ in the universal covering space. We are in the position to state the result which is required to prove the main theorem.

**Theorem 1.1.** *(*[6, Théorème 2.5]*) The measure theoretic entropy $h_\mu$ of a compact Finsler manifold $M$ without conjugate points satisfies the following equality*

$$h_\mu = \int_{SM} \operatorname{tr}\big(U(v)\big)\, d\mu(v).$$

This theorem was proved for Riemannian metrics by Friere and Mañé [7], mechanical Lagrangians by Innami [8], and convex Hamiltonians by Contreras and Iturriaga [3]. Using Theorem 1.1 we can obtain an interesting upper bound for measure theoretic entropy $h_\mu$.

**Corollary.** *The measure theoretic entropy $h_\mu$ of the an $n$-dimensional compact Finsler manifold $M$ without conjugate points satisfies the following inequality*

$$h_\mu \leq (n-1)\Big( -\int_{SM} \operatorname{Ric}(v)\, d\mu(v)\Big)^{1/2},$$

*and equality holds if and only if the flag curvature is constant.*

*Proof.* The Cauchy-Schwarz's inequality implies

$$\int_{SM} \operatorname{tr}\big(U(v)\cdot \operatorname{Id}\big)\, d\mu(v) \leq \Big(\int_{SM} \operatorname{tr}\big(U^2(v)\big)\, d\mu(v)\Big)^{1/2} \Big(\int_{SM} \operatorname{tr}(\operatorname{Id}^2)\, d\mu(v)\Big)^{1/2}$$

and equality holds if and only if $U(v)$ is a scalar multiple of $\operatorname{Id}$. If equality holds, then by the Riccati equation, $K(v) = -U^2(v)$, and hence the flag curvature is constant.

The Riccati equation and Theorem 1.1 implies

$$h_\mu \leq \Big(\int_{SM} \operatorname{tr}\big(U^2(v)\big)\, d\mu(v)\Big)^{1/2} \cdot (n-1)^{1/2}.$$

The $\varphi_t$-invariant of the normalized Liouville measure $\mu$ implies

$$\int_{SM} \operatorname{tr}\big(U'(v)\big)\, d\mu(v) = 0.$$

Then, integration of the Riccati equation with respect to $\mu$ yields

$$\int_{SM} \operatorname{tr}\big(U^2(v)\big)\, d\mu(v) = -\int_{SM} \operatorname{tr}\big(K(v)\big)\, d\mu(v).$$

But tr $\big(K(v)\big)/(n-1)$ is the Ricci tensor $\operatorname{Ric}(v)$, and hence, we have proved. □

## 2. Manifolds with nonpositive flag curvature

Our proof of the main theorem is based on the following result.

**Lemma 2.1.** *Let $M$ be a compact Finsler manifold without conjugate points. If $U(v)$ is invertible for all $v \in SM$, then we have*

$$h_\mu = -\int_{SM} \operatorname{tr}\big(K(v)\cdot U^{-1}(v)\big)\, d\mu(v).$$

*Proof.* Since $U(v)$ is invertible for all $v \in SM$, we consider the trace of the Riccati equation

$$U'(\varphi_t(v))\cdot U^{-1}(\varphi_t(v)) + U(\varphi_t(v)) + K(\varphi_t(v))\cdot U^{-1}(\varphi_t(v)) = 0.$$

Since trace and derivative commute, we have

$$\big(\ln|\det U(\varphi_t(v))|\big)' + \operatorname{tr} U(\varphi_t(v)) + \operatorname{tr}\big(K(\varphi_t(v))\cdot U^{-1}(\varphi_t(v))\big) = 0.$$

Integrating it over $SM \times [0,s]$. First

$$\begin{aligned}
\int_{SM}\int_0^s &\quad \big(\ln|\det U(\varphi_t(v))|\big)'\, dt\, d\mu(v)\\
&= \int_{SM} \ln|\det U(\varphi_s(v))|\, d\mu(v) - \int_{SM} \ln|\det U(\varphi_0(v))|\, d\mu(v)\\
&= \int_{SM} \ln|\det U(\varphi_s(v))|\, d\mu(v) - \int_{SM} \ln|\det U(v)|\, d\mu(v)\\
&= 0
\end{aligned}$$

by Lioville's theorem. Applying Birkhoff's Ergodic Theorem to the trace of the Riccati equation,

$$\begin{aligned}
h_\mu &= \int_{SM} \operatorname{tr} U(v)\, d\mu(v)\\
&= \lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{SM} \operatorname{tr} U(\varphi_t(v))\, d\mu(v)\, dt\\
&= -\lim_{T\to\infty} \frac{1}{T}\int_0^T \int_{SM} \operatorname{tr}\big(K(\varphi_t(v)))\cdot U^{-1}(\varphi_t(v))\big)\, d\mu(v)\, dt\\
&= -\int_{SM} \operatorname{tr}\big(K(v)\cdot U^{-1}(v)\big)\, d\mu(v),
\end{aligned}$$

again, we have used the Liouville's theorem. □

The following lemma is the final ingredient needed for the main theorem.

**Lemma 2.2.** *(*[1, Lemma 3.5]*) For all linear symmetric linear operators $A, B$, and $C$ on a Euclidean space such that $B$ and $C$ are nonnegative definite and $A$ is strictly positive definite, we obtain that*

$$\frac{1}{2}\operatorname{tr}(B \cdot A + C \cdot A^{-1}) \geq \operatorname{tr}\sqrt{B}\sqrt{C},$$

*where the equality holds if and only if $\sqrt{B} \cdot A = \sqrt{C}$.*

The lower bound for $h_\mu$ is an improvement of Théorème 3.5 of Foulon ([6]), and is an immediate consequence of Lemma 2.1 and Lemma 2.2.

**Theorem.** *The measure theoretic entropy $h_\mu$ of the a compact Finsler manifold $M$ with nonpositive flag curvature $K$ satisfies the following inequality*

$$h_\mu \geq \int_{SM} \operatorname{tr}\sqrt{-K(v)}d\mu(v),$$

*and equality holds if and only if the flag curvature is parallel along the geodesic flows.*

*Proof.* The main difficulty in extending Foulon argument lie in the fact that $U$ is not necessarily invertible. We consider that $K_\varepsilon(v) = K(v) - \varepsilon \cdot \mathrm{Id}$ for $v \in SM, \varepsilon > 0$. Since $K_\varepsilon(v) \leq -\varepsilon \cdot \mathrm{Id}$, the second fundamental form $U_\varepsilon(v)$ is invertible and $U_\varepsilon(v) \to U(v)$ as $\varepsilon \to 0$. This proposition is due to Eschenburg and Heintze [5]. We can apply the Lemma 2.2 with $A = U_\varepsilon(v)$, $B = \mathrm{Id}$, and $C = -K_\varepsilon(v)$, obtaining

$$\frac{1}{2}\operatorname{tr}\big(U_\varepsilon(v) - K_\varepsilon(v) \cdot U_\varepsilon^{-1}(v)\big) \geq \operatorname{tr}\sqrt{\mathrm{Id}}\sqrt{-K_\varepsilon(v)}.$$

Taking $\varepsilon$ to zero we have

$$\frac{1}{2}\operatorname{tr}\big(U(v) - K(v) \cdot U^{-1}(v)\big) \geq \operatorname{tr}\sqrt{\mathrm{Id}}\sqrt{-K(v)}.$$

By Lemma 2.1, the measure theoretic entropy

$$h_\mu \geq \int_{SM} \operatorname{tr}\sqrt{-K(v)}\, d\mu(v).$$

The estimate is sharp(i.e., we have equality) if and only if $U_\varepsilon(v) = \sqrt{-K_\varepsilon(v)}$ for almost all $v \in SM$, which implies that $U_\varepsilon(\varphi_t(v)) = \sqrt{-K_\varepsilon(\varphi_t(v))}$ almost everywhere $v$ on $SM$ and all $t$. Then $U_\varepsilon(\varphi_t(v)) = \sqrt{-K_\varepsilon(\varphi_t(v))}$ convergence to $U(\varphi_t(v)) = \sqrt{-K(\varphi_t(v))}$ as $\varepsilon \to 0$. Hence, by continuity of $U(\varphi_t(v))$ on each geodesic flow, for every $v \in SM$,

$$U^2(\varphi_t(v)) + K(\varphi_t(v)) = 0.$$

The Riccati equation implies that $U'(\varphi_t(v)) = 0$ on $v \in SM$, and hence we conclude that

$$K'(\varphi_t(v)) = -\frac{d}{dt}\big(U^2(\varphi_t(v))\big) = -2U'(\varphi_t(v)) \cdot U(\varphi_t(v)) = 0.$$

Therefore the flag curvature $K$ is parallel along the geodesic flows. □

## References

[1] W. Ballmann and M. P. Wojtkowski, An estimate for the measure theoretic entropy of geodesic flows, *Ergodic Theory Dyn. Syst.* **9** (1989), 271–279.

[2] F. C. Chittaro, An estimate for the entropy of Hamiltonian flows, *J. Dyn. Control Syst.* **13** (2007), 55–67.

[3] G. Contreras and R. Iturriaga, Convex Hamiltonians without conjugate points, *Ergodic Theory Dyn. Syst.* **19** (1999), 901–952.

[4] P. Dazord, Tores finslériens sans points conjugués, *Bull. Soc. Math. France* **99** (1971), 171–192; erratum ibid. **99** (1971), 397.

[5] J.-H. Eschenburg and E. Heintze, Comparison theory for Riccati equations, *Manuscr. Math.* **68** (1990), 209–214.

[6] P. Foulon, Estimation de l'entropie des systèmes lagrangiens sans points conjugués, *Ann. Inst. Henri Poicaré* **57** (1992), 117–146.

[7] A. Friere and R. Mañé, On the entropy of the geodesic flow in manifolds without conjugate points, *Invent. Math.* **69** (1982), 375–392.

[8] N. Innami, Natural Lagrangian systems without conjugate points, *Ergodic Theory Dyn. Syst.* **14** (1994), 169-180.

[9] Z. Shen, Lectures on Finsler geometry, World Scientific. xiv, 307 pp. (2001).

*Received June 20, 2017.*

Division of Liberal Arts and Sciences,
Mokpo National Maritime University,
Mokpo 530-729 ,
Korea
*E-mail address*: cwkim@mmu.ac.kr