Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
33 (2017), 291–304
www.emis.de/journals
ISSN 1786-0091

ON $N(k)$-MIXED-SUPER QUASI-EINSTEIN MANIFOLDS SATISFYING SOME CONDITIONS

SANTU DEY AND ARINDAM BHATTACHARYYA

Abstract. In this paper $N(k)$-mixed super quasi Einstein manifold $N(k)-MS(QE)_n$ has been introduced and the existence of such manifold is proved. Here, we have studied the nature of Ricci curvature, Ricci symmetric, Ricci recurrent, Generalized Ricci recurrent $N(k)-MS(QE)_n$. Next we study when the curvature conditions $\tilde{C}(U,X).S=0$ and $\tilde{P}(U,X).S=0$ hold in $N(k)-MS(QE)_n$ where $\tilde{C}$ and $\tilde{P}$ are the concircular curvature tensor and Weyl projective curvature tensor. We also study the Ricci-pseudosymmetric $N(k)-MS(QE)_n$. Finally, we give an example of $N(k)-MS(QE)_n$.

1. Introduction

The notion of quasi-Einstein manifold was introduced by M. C. Chaki and R. K. Maity [3]. A non-flat Riemannian manifold (M^n, g), $(n \geq 3)$ is a *quasi-Einstein manifold* if its Ricci tensor S satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

and is not identically zero, where a, b are scalars, $b \neq 0$ and A is a non-zero 1-form such that

$$g(X,U) = A(X), \ \forall \ X \in \chi(M), \tag{1.2}$$

where $\chi(M)$ is the set of all differentiable vector fields on M and U being a unit vector field.

Here a and b are called the *associated scalars*, A is called the *associated 1-form* and U is called the *generator* of the manifold. Such an n-dimensional manifold will be denoted by $(QE)_n$.

2010 *Mathematics Subject Classification.* 53C25, 53B30, 53C15.

Key words and phrases. $N(k)$-Mixed super quasi-Einstein manifold $N(k)-MS(QE)_n$, $N(k)$-mixed quasi constant curvature, Ricci-pseudosymmetric manifold, concircular curvature tensor, Ricci curvature, Ricci symmetric manifold, Ricci recurrent manifold, generalized Ricci recurrent manifold, projective curvature tensor.

The first author is supported by DST/INSPIRE Fellowship/2013/1041, Govt. of India.

In [4], Authors have defined *generalized quasi-Einstein manifold*. A non-flat *Riemannian manifold* is called *generalized quasi-Einstein manifold* if its Ricci-tensor is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{1.3}$$

where a, b and c are non-zero scalars and A, B are two 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X), \tag{1.4}$$

U and V being unit vectors which are orthogonal, i.e.,

$$g(U,V) = 0. \tag{1.5}$$

The vector fields U and V are called the *generators* of the manifold. This type of manifold will be denoted by $G(QE)_n$.

In [2], Chaki introduced the *super quasi-Einstein manifold*, denoted by $S(QE)_n$, where the Ricci tensor is not identically zero and satisfies the condition

$$\begin{aligned} S(X,Y) = ag(X,Y) + bA(X)A(Y) + c[A(X)B(Y) \\ + A(Y)B(X)] + dD(X,Y), \end{aligned} \tag{1.6}$$

where a, b, c and d are scalars such that b, c, d are nonzero, A, B are two nonzero 1-forms defined as (1.4) and U, V are mutually orthogonal unit vector fields, D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition

$$D(X,U) = 0 \ \forall \ X \in \chi(M). \tag{1.7}$$

Here a, b, c, d are called the *associated scalars*, A, B are called the *associated main and auxiliary 1-forms* respectively, U, V are called the main and the *auxiliary generators* and D is called the *associated tensor* of the manifold.

The k-nullity distribution $N(k)$ [11] of a *Riemannian manifold* M is defined by

$$N(k) : p \to N_p(k) = \{Z \in T_pM / R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}$$

for all $X, Y \in \chi(M)$ and k is a smooth function.

M. M. Tripathi and Jeong-Sik Kim [12] introduced the notion of $N(k)$-*quasi Einstein manifold* which is defined as follows: If the generator U belongs to the k-*nullity distribution* $N(k)$, then a *quasi Einstein manifold* (M^n, g) is called $N(k)$-*quasi Einstein manifold*. In [9], Nagaraja introduced the notion of $N(k)$-*mixed quasi Einstein manifold*.

In [1], A. Bhattacharyya, M. Tarafdar and D. Debnath introduced the notion of $MS(QE)_n$. So, we define $N(k) - MS(QE)_n$ as follows:

Definition. Let (M^n, g) be a non flat Riemannian manifold. If the Ricci tensor S of (M^n, g) is non zero and satisfies

$$\begin{aligned} S(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) + d[A(X)B(Y) \\ + A(Y)B(X)] + eD(X,Y), \end{aligned} \tag{1.8}$$

where a, b, c, d, e are scalars of which $b \neq 0$, $c \neq 0$, $d \neq 0$, $e \neq 0$ and A, B are two non zero 1-forms such that

$$g(X,U) = A(X) \text{ and } g(X,V) = B(X) \ \forall\ X \in \chi(M), \tag{1.9}$$

D is a symmetric (0, 2) tensor with zero trace which satisfies the condition

$$D(X,U) = 0 \ \forall\ X \in \chi(M), \tag{1.10}$$

U and V being the orthogonal unit vector fields called *generators* of the manifold belong to $N(k)$, then we say that (M^n, g) is a $N(k)$-*mixed super quasi Einstein manifold* and is denoted by $N(k) - MS(QE)_n$.

2. Preliminaries

In $N(k) - MS(QE)n$, we have

$$R(X,Y)U = k\{A(Y)X - A(X)Y)\}. \tag{2.1}$$

From (1.8), we have

$$S(U,U) = a + b \tag{2.2}$$

$$S(V,V) = a + c + eD(V,V) \tag{2.3}$$

$$S(U,V) = d = S(V,U). \tag{2.4}$$

Now setting $X = Y = e_i$ in (1.8), where $\{e_i\}$, $i = 1, 2, ..., n$ be an orthonormal basis of vector fields in the manifold and taking summation over i, $1 \leq i \leq n$, we obtain

$$r = na + b + c, \tag{2.5}$$

where r is the scalar curvature of the manifold.

If X is a unit vector field, then $S(X,X)$ is the Ricci-curvature in the direction of X. Hence from (2.2) and (2.3) we can state that $a+b$ and $a+c+eD(V,V)$ are the Ricci curvature in the directions of U and V respectively.

Let Q be the Ricci operator, i.e.,

$$g(QX,Y) = S(X,Y) \ \forall\ X, Y \in \chi(M). \tag{2.6}$$

Here, we consider

$$g(lX,Y) = D(X,Y). \tag{2.7}$$

3. Existence theorem of a $N(k)$-mixed super quasi Einstein manifold $N(k) - MS(QE)_n$

Theorem 3.1. *If in a conformally flat Riemannian manifold (M^n, g), the Ricci tensor S satisfies the relation*

$$\begin{aligned} S(X,W)S(Y,Z) - S(Y,W)S(X,Z) = \mu_1[&S(Y,W)g(Z,X) + S(Z,X) \\ &g(Y,W)] + \beta_1[g(X,W)g(Y,Z) - g(Y,W)g(Z,X)] \\ &+ \gamma_1[g(Y,Z)D(X,W) - g(X,Z)D(Y,W) \\ &+ g(X,W)D(Y,Z) - g(Y,W)D(X,Z)], \end{aligned} \tag{3.1}$$

where μ_1, β_1, γ_1 are non-zero scalars and D is a symmetric $(0,2)$ tensor with zero trace which satisfies the condition $D(X,U) = 0$, $\forall X$ then the manifold is $N(k)$-mixed super quasi-Einstein manifold.

Proof. Existence theorem of a mixed super quasi Einstein manifold was proved in [1]. Now, we will prove the Existence Theorem of a $N(k)$-mixed super quasi Einstein manifold.
If (M^n, g) is conformally flat, then

$$\begin{aligned} R(X,Y)Z = &\frac{1}{n-1}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} \\ &- \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.2}$$

Taking $Z = U$ in (3.2), we obtain

$$\begin{aligned} R(X,Y)U = &\frac{1}{n-1}\{A(Y)QX - A(X)QY + S(Y,U)X - S(X,U)Y\} \\ &- \frac{r}{(n-1)(n-2)}\{A(Y)X - A(X)Y\}. \end{aligned} \tag{3.3}$$

Now taking $\mu_1 = \beta_1 = \gamma_1$ and $Z = U$ in (3.1), we obtain

$$\begin{aligned} S(X,W)[(a+b)A(Y) + dB(Y)] - S(Y,W)[(a+b)A(X) + dB(X)] \\ = S(Y,W)A(X) + [(a+b)A(X) + dB(X)]g(Y,W) \\ + g(X,W)A(Y) - g(Y,W)A(X) + D(X,W)A(Y) \\ - D(Y,W)A(X). \end{aligned} \tag{3.4}$$

Now taking $a + b = 1$ and $d = 1$ and using $S(X,W) = g(QX,W)$ and $D(X,W) = g(lX,W)$ in (3.4), we get

$$\begin{aligned} g((QX)A(Y) + (QX)B(Y) - (QY)A(X) - (QY)B(X) - (QY)A(X) \\ - A(X)Y - B(X)Y - A(Y)X + A(X)Y \\ - A(Y)lX + A(X)lY, W) = 0. \end{aligned} \tag{3.5}$$

$\forall W$, which implies

$$\begin{aligned}(QX)A(Y)+(QX)B(Y)-(QY)A(X)-(QY)B(X)-(QY)A(X)\\ -A(X)Y-B(X)Y-A(Y)X+A(X)Y\\ -A(Y)lX+A(X)lY=0.\end{aligned}\tag{3.6}$$

Or,

$$\begin{aligned}(QX)A(Y)-(QY)A(X)+[A(Y)+B(Y)]X-[A(X)+B(X)]Y\\ =A(Y)X-A(X)Y,\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}[A(Y)+B(Y)]X-[A(X)+B(X)]Y=[-A(Y)lX+B(Y)QX]\\ -[A(X)\{QY+Y-lY\}+B(X)(QY+Y)].\end{aligned}\tag{3.8}$$

Substituting (3.7) in (3.3), we get

$$R(X,Y)U=k[A(Y)X-A(X)Y],\tag{3.9}$$

where, $k=\frac{n(1-a)-b-c-1}{(n-1)(n-2)}$.
Therefore, $U\in N_p(k)$ for $k=\frac{n(1-a)-b-c-1}{(n-1)(n-2)}$.
Hence (M^n,g) is a $N(k)$-mixed super quasi Einstein manifold. □

As it is well known that a 3-dimensional Riemannian manifold is conformally flat.

Corollary. *A 3-dimensional manifold is $N(\frac{2-3a-b-c}{2})$ mixed super quasi Einstein manifold provided* (3.1) *holds.*

4. Ricci Curvature, Eigen Vectors and Associated Scalars of a $N(k)-MS(QE)_n$

From (1.8), we have $S(U,U)=a+b, S(V,V)=a+c+eD(V,V), S(U,V)=d$, $S(X,X)$ is the Ricci curvature in the direction of X. Now,

$$1=g(X,X)=g(\alpha U+\beta V,\alpha U+\beta V)=\alpha^2+\beta^2.\tag{4.1}$$

Since $g(U,V)=0$ and $g(U,U)=g(V,V)=1$.
Now,

$$S(X,X)=a+b\{A(X)\}^2+c\{B(X)\}^2+2dA(X)B(X)+eD(X,X).\tag{4.2}$$

Thus, we can state the following theorem:

Theorem 4.1. *In a $N(k)-MS(QE)_n$ manifold, the Ricci curvature in the direction of U is $a+b$ and in the direction of V is $a+c+eD(V,V)$ and the Ricci curvature in all other directions of the section of U and V is $a+b\{A(X)\}^2+c\{B(X)\}^2+2dA(X)B(X)+eD(X,X)$.*

Let (M^n, g) be $N(k)-MS(QE)_n$, then we get

$$S(U,U) = a+b,\ S(V,V) = a+c+eD(V,V),\ S(U,V) = d$$
$$g(QU,U) = a+b,\ g(QV,V) = a+c+eD(V,V).$$

Since $U, V \in N_p(k)$, we have

$$g(R(X,Y)U,W) = k\{g(Y,U)g(X,W) - g(X,U)g(Y,W)\}. \tag{4.3}$$

From (1.9),

$$g(R(X,Y)U,W) = k\{A(Y)g(X,W) - A(X)g(Y,W)\}. \tag{4.4}$$

Putting $X = W = e_i$ in (4.4) where $\{e_i\}$, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1 \leq i \leq n$, we obtain

$$S(Y,U) = k(n-1)A(Y), \tag{4.5}$$

and

$$S(Y,V) = k(n-1)B(Y). \tag{4.6}$$

Again from (1.8), we get

$$S(Y,U) = (a+b)A(Y) + dB(Y). \tag{4.7}$$

$$S(Y,V) = (a+c)B(Y) + dA(Y) + eD(Y,V). \tag{4.8}$$

Substracting (4.6) from (4.5), we obtain

$$S(Y,U) - S(Y,V) = k(n-1)[A(Y) - B(Y)]. \tag{4.9}$$

Substracting (4.8) from (4.7), we obtain

$$\begin{aligned} S(Y,U) - S(Y,V) &= (a+b)A(Y) + dB(Y) - (a+c)B(Y) \\ &\quad - dA(Y) - eD(Y,V). \end{aligned} \tag{4.10}$$

Equating (4.9) and (4.10), we get

$$\begin{aligned} k(n-1)[A(Y) - B(Y)] &= (a+b-d)A(Y) + B(Y)(d-a-c) \\ &\quad - eD(Y,V). \end{aligned} \tag{4.11}$$

Putting $Y = U$ in (4.11), we obtain

$$k = \frac{a+b-d}{n-1}. \tag{4.12}$$

And also putting $Y = V$ in (4.11), we obtain

$$k = \frac{a+c+\tilde{m}-d}{n-1}, \tag{4.13}$$

where, $eD(V,V)=\tilde{m}$(say). Equating (4.12) and (4.13), we get $b = c + \tilde{m}$. So, $k = \frac{a+b-d}{n-1}$. Now,

$$S(Y,U) = (a+b-d)g(Y,U), \tag{4.14}$$

and

$$S(Y,V) = (a+b-d)g(Y,V). \tag{4.15}$$

Therefore we can say that

Theorem 4.2. *In a $N(k)-MS(QE)_n$ manifold, the orthogonal vector fields U and V are the eigen vectors corresponding to the eigen value $(a+b-d)$.*

5. Ricci semi-symmetric $N(k)-MS(QE)_n$

A $N(k)-MS(QE)_n$ is said to be Ricci semi symmetric manifold [7] if it satisfy $R(X,Y).S = 0$, $\forall X, Y$ where $R(X,Y)$ denotes the curvature operator. Then we have,

$$S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = 0. \tag{5.1}$$

Putting $X = U$ in (5.1),

$$\begin{aligned} k[g(Y,Z)S(U,W) - A(Z)S(Y,W) + g(Y,W)S(Z,U) \\ - A(W)S(Z,Y)] = 0. \end{aligned} \tag{5.2}$$

Putting W = U in (5.2), we get

$$\begin{aligned} k[g(Y,Z)S(U,U) - A(Z)S(Y,U) + g(Y,U)S(Z,U) \\ - A(U)S(Z,Y)] = 0. \end{aligned} \tag{5.3}$$

That is,

$$\begin{aligned} &k\ [(a+b)g(Y,Z) - A(Z)\{ag(Y,U) + bA(Y)A(U) + cB(Y)(U) \\ &+ d(A(Y)B(U) + A(U)B(Y)) + eD(Y,U)\} + A(Y)\{ag(Z,U) \\ &+ bA(Z)A(U) + cB(Z)(U) + d(A(Z)B(U) + A(U)B(Z)) \\ &+ eD(Z,U)\} - S(Y,Z)] = 0. \end{aligned} \tag{5.4}$$

From the above equation, we obtain

$$k[(a+b)g(Y,Z) + d\{A(Y)B(Z) - A(Z)B(Y)\} - S(Y,Z)] = 0. \tag{5.5}$$

If $k \neq 0$, then (M^n, g) becomes $M(QE)_n$. Therefore, we must have $k = 0$.

Conversely suppose $k = 0$. Then we obtain $R(U,X)Y = 0$ which implies $R(U,X).S = 0$. Thus we have,

Theorem 5.1. *A $N(k)-MS(QE)_n$ manifold satisfies $R(U,X).S = 0$ if and only if $a+b-d = 0$.*

6. Ricci recurrent $N(k)-MS(QE)_n$

A $N(k)-MS(QE)_n$ manifold is called *Ricci recurrent* if it satisfies

$$(\nabla_X S)(Y,Z) = T(X)S(Y,Z), \tag{6.1}$$

where $T(X)$ is a non-zero 1-form. But,

$$(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - -S(Y, \nabla_X Z). \tag{6.2}$$

That is,

$$T(X)S(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z). \tag{6.3}$$

Putting $Y = Z = U$ in (6.3), we get

$$(a+b)T(X) = X(a+b) - S(\nabla_X U, U) - S(U, \nabla_X U), \tag{6.4}$$

or,

$$(a+b)T(X) = X(a+b) - 2[bA(\nabla_X U) + dB(\nabla_X U)], \tag{6.5}$$

or,

$$(a+b)T(X) = X(a+b) - 2[dB(\nabla_X U)]. \tag{6.6}$$

So,

$$(a+b)T(X) = X(a+b) \iff B(\nabla_X U) = 0.$$

But, $B(\nabla_X U) = 0$ implies either $\nabla_X U \perp V$ or U is a parallel vector field.

Similarly, if we put $Y = Z = V$ in (6.3), we obtain

$$\begin{aligned}(a+b+eD(V,V))T(X) = X(a+b+eD(V,V)) &- S(\nabla_X V, V) \\ &- S(V, \nabla_X V),\end{aligned} \tag{6.7}$$

or,

$$\begin{aligned}(a+b+eD(V,V))T(X) = X(a+b+eD(V,V)) &- 2[cB(\nabla_X V) \\ &+ A(\nabla_X V) + eD(\nabla_X V, V)].\end{aligned} \tag{6.8}$$

So, $(a+b+\tilde{m})T(X) = X(a+b+\tilde{m}) - 2eD(\nabla_X V, V)$ iff $A(\nabla_X V) = 0$. But, $A(\nabla_X V) = 0$ implies either $\nabla_X V \perp U$ or V is a parallel vector field, where, $eD(V,V) = \tilde{m}$.

Thus, we can say that

Theorem 6.1. *A $N(k) - MS(QE)_n$ manifold is Ricci recurrent, then either the vector field U(or V) is parallel or $\nabla_X U \perp V$(or $\nabla_X V \perp U$).*

7. Generalized Ricci recurrent $N(k) - MS(QE)_n$

A $N(k) - MS(QE)_n$ manifold is called *generalized Ricci recurrent* [5] if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z) + \beta(X)g(Y,Z), \tag{7.1}$$

where $\alpha(X)$ and $\beta(X)$ are two nowhere vanishing 1-forms such that $\alpha(X) = g(X,\rho)$ and $\beta(X) = g(X,\mu)$; ρ and μ being *associated vector fields* of the 1-forms α and β, respectively.

Definition. A Riemannian manifold is said to *admit cyclic parallel Ricci tensor* if

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0. \tag{7.2}$$

Now, we prove the following:

Theorem 7.1. *On a generalized Ricci recurrent $N(k)-MS(QE)_n$ with cyclic parallel Ricci tensor, the Ricci tensor is of the form*

$$\begin{aligned}\alpha(U)S(X,Y) = &-\beta(U)g(X,Y) - (a+b)[\alpha(X)A(Y) + \alpha(Y)A(X)]\\ &- d[\alpha(X)\beta(Y) + \alpha(Y)\beta(X)]\\ &- [A(X)\beta(Y) + A(Y)\beta(X)].\end{aligned} \tag{7.3}$$

Proof. Suppose that M is a generalized Ricci recurrent $N(k) - MS(QE)_n$ admitting cyclic parallel Ricci tensor. Then using (7.1) in (7.2), we get

$$\begin{aligned}&\alpha(X)S(Y,Z) + \beta(X)g(Y,Z) + \alpha(Y)S(Z,X)\\ &+\beta(Y)g(Z,X) + \alpha(Z)S(X,Y)\\ &+\beta(Z)g(X,Y) = 0.\end{aligned} \tag{7.4}$$

Setting $Z = U$ in (7.4), Using (1.9) and (4.7), we get the relation (7.3). Contracting (7.3) over X and Y , we get

$$\begin{aligned}\alpha(U)r = &-n\beta(U) - 2(a+b)\alpha(U) - 2\beta(U)\\ &- 2d\alpha(V).\end{aligned} \tag{7.5}$$

□

This leads to the following:

Corollary. *On a generalized Ricci recurrent $N(k)$-mixed super quasi Einstein manifold with cyclic parallel Ricci tensor, the scalar curvature is of the form* (7.5).

We now consider a *generalized Ricci recurrent* $N(k)-MS(QE)_n$ whose Ricci tensor is of Codazzi type. Then we have ([8], [10])

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(X,Y). \tag{7.6}$$

Using (7.1) in (7.6), we obtain

$$\alpha(X)S(Y,Z) + \beta(X)g(Y,Z) = \alpha(Z)S(X,Y) + \beta(Z)g(X,Y). \tag{7.7}$$

Setting $Z = U$ in (7.7), using (1.9) and (4.7), we get the relation

$$\begin{aligned}\alpha(U)S(X,Y) = &\beta(U)g(X,Y) - d\alpha(X)\beta(Y)\\ &+ [\alpha(X)(a+b) + \beta(X)]A(Y).\end{aligned} \tag{7.8}$$

This leads to the following:

Theorem 7.2. *On a generalized Ricci recurrent $N(k)-MS(QE)_n$ whose Ricci tensor is of Codazzi type, the Ricci tensor is of the form* (7.8).

Also from (7.8), we can state the following:

Corollary. *On a generalized Ricci recurrent $N(k) - MS(QE)_n$ whose Ricci tensor is of Codazzi type, the scalar curvature is given by*

$$\alpha(U)r = -(n-1)\beta(U) - d\alpha(V) + \alpha(U)(a+b). \tag{7.9}$$

8. $N(k)-MS(QE)_n$ SATISFYING THE CONDITION $\tilde{C}(U,X).S=0$

The *concircular curvature tensor* $\tilde{C}$ of type $(1,3)$ of n-dimentional *Riemannian manifold* (M^n,g), $(n\geq 3)$ is defined by [13]

$$\tilde{C}(X,Y)W = R(X,Y)W - \frac{r}{n(n-1)}[g(Y,W)X - g(X,W)Y] \tag{8.1}$$

for any vector fields $X,Y,Z \in \chi(M)$. Let us consider a $N(k)-MS(QE)_n$ $(n\geq 3)$ satisfying the condition $(\tilde{C}(U,X).S)(Y,Z)=0$.

Putting $Z=U$, we have

$$S(\tilde{C}(U,X)Y,U) + S(Y,\tilde{C}(U,X)U) = 0. \tag{8.2}$$

Now using the definition of k-nullity distribution in (8.1), we obtain

$$\tilde{C}(U,X)Y = [k - \frac{r}{n(n-1)}][g(X,Y)U - A(Y)X] \tag{8.3}$$

and

$$\tilde{C}(U,X)U = [k - \frac{r}{n(n-1)}][A(X)U - X]. \tag{8.4}$$

Now,

$$\begin{aligned} S(\tilde{C}(U,X)Y,U) = [k - \frac{r}{n(n-1)}][&g(X,Y)(a+b) - (a+b)A(X)A(Y) \\ &- dB(X)A(Y)] \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} S(Y,\tilde{C}(U,X)U) = [k - \frac{r}{n(n-1)}][&(a+b)A(X)A(Y) + dA(X)B(Y) \\ &- S(X,Y)]. \end{aligned} \tag{8.6}$$

From (8.2), we get

$$\begin{aligned} [k - \frac{r}{n(n-1)}][g(X,Y)(a+b) + d\{A(X)B(Y) - B(X)A(Y)\} \\ - S(X,Y)] = 0. \end{aligned} \tag{8.7}$$

So, either scalar curvature $r = kn(n-1)$ or (M^n,g) becomes $M(QE)_n$. But (M^n,g) is not $M(QE)_n$. So, we can state the following:

Theorem 8.1. *The $N(k)-MS(QE)_n$ satisfying the condition $\tilde{C}(U,X).S=0$, i.e., concircularly Ricci symmetric iff its scalar curvature is $n(a+b-d)$.*

9. $N(k)-MS(QE)_n$ SATISFYING THE CONDITION $\tilde{P}(U,X).S=0$

The *Weyl projective curvature tensor* $\tilde{P}$ of type $(1,3)$ of n-dimentional *Riemannian manifold* (M^n,g),$(n\geq 3)$ is defined by [13]

$$\tilde{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y] \tag{9.1}$$

for any vector fields $X, Y, Z \in \chi(M)$. Let us consider a $MS(QE)_n$ $(n \geq 3)$ satisfying the condition $(\tilde{P}(U, X).S)(Y, Z) = 0$.

Putting $Z = U$, we have

$$S(\tilde{P}(U, X)Y, U) + S(Y, \tilde{P}(U, X)U) = 0. \tag{9.2}$$

Now using the definition of k-nullity distribution in (9.1), we obtain

$$\begin{aligned} S(\tilde{P}(U, X)Y, U) = & k[g(X, Y)(a + b) - A(Y)\{(a + b)A(X) + dB(X)\}] \\ & - \frac{1}{n-1}[S(X, Y)(a + b) - S(Y, U)S(X, U)] \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} S(Y, \tilde{P}(U, X)U) = & k[A(X)\{(a + b)A(Y) + dB(Y)\} - S(X, Y)] \\ & - \frac{1}{n-1}[S(X, U)S(Y, U) - (a + b)S(X, Y)]. \end{aligned} \tag{9.4}$$

From (9.2),

$$\begin{aligned} k[g(X, Y)(a + b) + d\{A(X)B(Y) - B(X)A(Y)\} \\ - S(X, Y)] = 0. \end{aligned} \tag{9.5}$$

So, $k = 0$, otherwise (M^n, g) becomes $M(QE)_n$. Thus we have

Theorem 9.1. *The $N(k)-MS(QE)_n$ satisfying the condition $\tilde{P}(U, X).S = 0$, i.e., Weyl projectively Ricci symmetric iff $k = 0$.*

10. Ricci-pseudosymmetric $N(k) - MS(QE)_n$

An n-dimensional Riemannian manifold (M^n, g) is called *Ricci-pseudosymmetric* [6] if the tensors $R.S$ and $Q(g, S)$ are linearly dependent, where

$$(R(X, Y).S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W), \tag{10.1}$$

$$Q(g, S)(Z, W; X, Y) = -S((X \wedge Y)Z, W) - S(Z, (X \wedge Y)W) \tag{10.2}$$

and

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$$

for vector fields $X\ Y, Z, W$ on M^n, R denotes the curvature tensor of M^n. The condition of *Ricci-pseudosymmetry* is equivalent to the relation

$$(R(X, Y).S)(Z, W) = L_S Q(g, S)(Z, W; X, Y) \tag{10.3}$$

holding on the set

$$U_S = \{x \in M : S \neq \frac{r}{n} g \ at\ x\}, \tag{10.4}$$

where L_S is some function on U_S. If $R.S = 0$ then M^n is called *Ricci-semisymmetric.* Every Ricci-semisymmetric manifold is Ricci-pseudosymmetric but the converse is not true [6].

Assume that M^n is Ricci-pseudosymmetric. Then by the use of (10.1) to (10.4), we can obtain

$$\begin{aligned} S(R(X,Y)Z,W)+S(Z,R(X,Y)W) = L_S\{&g(Y,Z)S(X,W)\\ &-g(X,Z)S(Y,W)+g(Y,W)S(X,Z)\\ &-g(X,W)S(Y,Z)\}. \end{aligned} \tag{10.5}$$

Since M^n is also $N(k)-MS(QE)_n$, using the properties of the curvature tensor R we get

$$\begin{aligned} &b[A(R(X,Y)Z)A(W)+A(Z)A(R(X,Y)W)]+c[B(R(X,Y)Z)B(W)\\ &+B(Z)B(R(X,Y)W)]+d[A(R(X,Y)Z)B(W)+A(W)B(R(X,Y)Z)\\ &+A(Z)B(R(X,Y)W)+A(R(X,Y)W)B(Z)]+e[D(R(X,Y)Z,W)\\ &+D(Z,R(X,Y)W)]=L_S\{b[g(Y,Z)A(X)A(W)-g(X,Z)A(Y)A(W)\\ &+g(Y,W)A(X)A(Z)-g(X,W)A(Y)A(Z)]+c[g(Y,Z)B(X)B(W)\\ &-g(X,Z)B(Y)B(W)+g(Y,W)B(X)B(Z)-g(X,W)B(Y)B(Z)]\\ &+d[g(Y,Z)A(X)B(W)+g(Y,Z)A(W)B(X)-g(X,Z)A(Y)B(W)\\ &-g(X,Z)A(W)B(Y)+g(Y,W)A(X)B(Z)+g(Y,W)A(Z)B(X)\\ &-g(X,W)A(Y)B(Z)-g(X,W)A(Z)B(Y)]+e[g(Y,Z)D(X,W)\\ &-g(X,Z)D(Y,W)+g(Y,W)D(X,Z)-g(X,W)D(Y,Z)]\}. \end{aligned} \tag{10.6}$$

Putting $Y=Z=U$ in (10.6), we get

$$\begin{aligned} kb[A(X)A(W)-g(X,W)]+kc[B(X)B(W)]&+2kd[A(W)B(X)]\\ +ke[D(X,W)]=L_S\{&b[A(X)A(W)-g(X,W)]\\ +&c[B(X)B(W)]+2d[A(W)B(X)]+eD(X,W)\}. \end{aligned} \tag{10.7}$$

Putting $X=W=e_i$ in (10.7) where $\{e_i\}$, $i=1,2,....,n$ be an orthonormal basis of the tangent space at any point of the manifold and taking summation over i, $1\le i\le n$, we obtain

$$L_S=k=\frac{a+b-d}{n-1}. \tag{10.8}$$

Thus we can state the following:

Theorem 10.1. *The Ricci-pseudosymmetric $N(k)-MS(QE)_n$ is Ricci semi symmetric manifold iff $a+b-d=0$.*

11. Example of a 4-dimensional $N(k)-MS(QE)_n$

Here we construct a nontrivial concrete example of a $N(k)-MS(QE)_n$. Let us consider a Riemannian metric g on the 4-dimensional real number space M^4 by

$$ds^2=g_{ij}dx^idx^j=(e^{x^2})(dx^1)^2+(x^1x^3)^2(dx^2)^2+(dx^3)^2+(dx^4)^2, \tag{11.1}$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4. Then the only non vanishing components of the Christoffel symbols, the curvature tensors and the Ricci tensor are

$$[12, 1] = \frac{e^{x^2}}{2},\ [21, 2] = x^1(x^3)^2,\ [23, 2] = (x^1)^2x^3 \tag{11.2}$$

$$R_{1223} = R_{3221} = x^1x^3,\ R_{1221} = \frac{e^{x^2}}{4},\ R_{2113} = R_{3112} = -\frac{e^{x^2}}{2x^3} \tag{11.3}$$

$$R_{11} = \frac{e^{x^2}}{4(x^1x^3)^2},\ R_{22} = \frac{1}{4},\ R_{13} = \frac{1}{x^1x^3},\ R_{23} = -\frac{1}{2x^3} \tag{11.4}$$

and the components which can be obtained from these by symmetric properties. So, M^4 is a *Riemannian manifold* of non-vanishing scalar curvature. We shall now show that this manifold is an $N(k) - MS(QE)_4$. Let us now define

$$a = \frac{1}{8(x^1x^3)^2},\ b = \frac{1}{(x^1x^3)^2},\ c = \frac{1}{4(x^1x^3)^2},\ d = -\frac{\sqrt{e^{x^2}}}{(x^1)^3(x^3)^4},\ e = -\frac{1}{(x^1x^3)^2}$$

and the 1-forms are

$$A_i(x) = \begin{cases} \frac{(x^1x^3)^2}{8}, & \text{if } i = 1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$B_i(x) = \begin{cases} \sqrt{e^{x^2}}, & \text{if } i = 1 \\ \frac{(x^1x^3)^2}{8}, & \text{if } i = 3 \\ 0, & \text{otherwise} \end{cases}$$

and the associated tensor as

$$D_{ij}(x) = \begin{cases} \frac{e^{x^2}}{4}, & \text{if } i = j = 1 \\ 0, & \text{otherwise} \end{cases}$$

then we have

(i) $R_{11} = ag_{11} + bA_1A_1 + cB_1B_1 + 2dA_1B_1 + eD_{11}$,
(ii) $R_{22} = ag_{22} + bA_2A_2 + cB_2B_2 + 2dA_2B_2 + eD_{22}$,
(iii) $R_{13} = ag_{13} + bA_1A_3 + cB_1B_3 + d(A_1B_3 + A_3B_1) + eD_{13,}$
(iv) $R_{23} = ag_{23} + bA_2A_3 + cB_2B_3 + d(A_2B_3 + A_3B_2) + eD_{23}$.

Since all the cases other than $(i) - (iv)$ are trivial, we can say

$$R_{ij} = ag_{ij} + bA_iA_j + cB_iB_j + d(A_iB_j + A_jB_i) + eD_{ij}, for\ i, j = 1, 2, 3, 4.$$

So, we can say that the manifold under consideration is an $N(k)-MS(QE)_4$, where $k = \frac{9x^1(x^3)^2+8\sqrt{e^{x^2}}}{24(x^1)^3(x^3)^4}$. Thus if (M^4, g) is a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^idx^j = (e^{x^2})(dx^1)^2 + (x^1x^3)^2(dx^2)^2 + (dx^3)^2 + (dx^4)^2,$$

where $i, j = 1, 2, 3, 4$ and x^1, x^2, x^3, x^4 are the standard coordinates of M^4, then it is an $N(k) - MS(QE)_4$ with nonzero and nonconstant scalar curvature.

References

[1] A. Bhattacharyya, M. Tarafdar, and D. Debnath. On mixed super quasi-Einstein manifolds. *Differ. Geom. Dyn. Syst*, 10:44–57, 2008.

[2] M. C. Chaki. On super quasi Einstein manifolds. *Publ. Math. Debrecen*, 64(3-4):481–488, 2004.

[3] M. C. Chaki and R. K. Maity. On quasi-Einstein manifolds. *Publ. Math. Debrecen*, 57:297–306, 2000.

[4] U. C. De and G. C. Ghosh. On generalized quasi-Einstein manifolds. *Kyungpook Math. J*, 44:607–615, 2004.

[5] U. C. De, N. Guha, and D. Kamilya. On generalized Ricci-recurrent manifolds. *Tensor. New series*, 56(3):312–317, 1995.

[6] R. Deszcz. On pseudosymmetric spaces. *Bull. Soc. Math. Belg. Sér. A*, 44(1):1–34, 1992.

[7] R. Deszcz and M. Hotlos. Remarks on Riemannian manifolds satisfying certain curvature condition imposed on the Ricci tensor. *Prace Nauk. Pol. Szczec*, 11:23–34, 1989.

[8] D. Ferus. A remark on codazzi tensors in constant curvature spaces. In *Global differential geometry and global analysis*, pages 257–257. Springer, 1981.

[9] H. Nagaraja. On N(k)-mixed quasi Einstein manifolds. *European Journal of Pure and Applied Mathematics*, 3(1):16–25, 2009.

[10] J. A. Schouten. Ricci-calculus: An introduction to tensor analysis and its geometrical applications. *Verlag Berlin Heidelberg Gmbh, Springer*, 1954.

[11] S. Tanno. Ricci curvatures of contact Riemannian manifolds. *Tohoku Mathematical Journal, Second Series*, 40(3):441–448, 1988.

[12] M. M. Tripathi and J. Kim. On N(k)-quasi Einstein manifolds. *Communications-Korean Mathematical Society*, 22(3):411, 2007.

[13] K. Yano and M. Kon. *Structures on manifolds*, volume 3. World scientific, 1985.

Received November 9, 2015.

Department of Mathematics,
Jadavpur University,
Kolkata-700032, India
E-mail address: santu.mathju@gmail.com
E-mail address: bhattachar1968@yahoo.co.in