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NON-LINEAR η-JORDAN $*$-DERIVATION MAPS ON PRIME C^*-ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a prime C^*-algebra that contains a non-trivial projection and η be a non-zero complex number that $|\ \eta\ |\neq 1$ and ψ be the η-Jordan $*$-derivation map on $\mathcal{A}$, that is, for every $A, B \in \mathcal{A}$, $\psi(A \diamond_\eta B) = \psi(A) \diamond_\eta B + A \diamond_\eta \psi(B)$ where $A \diamond_\eta B = AB + \eta BA^*$, then we show that ψ is an additive $*$-derivation map.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say that the map $\Phi\colon \mathcal{R} \to \mathcal{R}'$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$. The question of when a product preserving or multiplicative map is additive was discussed by several authors, see [10] and references therein. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving Lie product $[A, B] = AB - BA$ or Jordan product $A \circ B = AB + BA$ (for example, see [1, 6, 9]). These results show that, in some sense, Jordan product or Lie product structure is enough to determine the ring or algebraic structure. Historically, many mathematicians devoted themselves to the study of additive or linear Jordan or Lie product preserver maps between rings or operator algebras. Such maps are always called Jordan homomorphism or Lie homomorphism. Here we only list several results [5, 10, 11].

Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [2, 4, 8, 12]).

Let define η-Jordan $*$-product by $A \diamond_\eta B = AB + \eta BA^*$. We say the map ψ with property of $\psi(A \diamond_\eta B) = \psi(A) \diamond_\eta B + A \diamond_\eta \psi(B)$ is a η-Jordan $*$-derivation map. It is clear that for $\eta = -1$ and $\eta = 1$, the η-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [3].

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We should mention here whenever we say ψ is a derivation map, it means $\psi(AB) = \psi(A)B + A\psi(B)$.

Recently, Yu and Zhang in [15] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation, i. e. for every $A, B \in \mathcal{A}$, we have $\psi(A + B) = \psi(A) + \psi(B)$, $\psi(A^*) = \psi(A)^*$ and $\psi(AB) = \psi(A)B + A\psi(B)$. In [14] we show that every non-linear $*$-Jordan derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [7] have investigated a non-linear ξ-Jordan $*$-derivation map. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and ξ is a non-zero scaler, then $\phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a non-linear ξ-Jordan $*$-derivation if and only if ϕ is an additive $*$-derivation.

Let $\mathcal{A}$ be a prime C^*-algebra that contains a non-trivial projection and η be a non-zero complex number that $\mid \eta \mid \neq 1$. In this paper we show that every η-Jordan $*$-derivation map on $\mathcal{A}$ is an additive $*$-derivation.

Note that an algebra $\mathcal{A}$ is called prime when for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$. We denote real and imaginary part of an operator A by A_1 and A_2, respectively i.e., $A_1 = \frac{A+A^*}{2}$ and $A_2 = \frac{A-A^*}{2i}$.

2. The statement of the main theorem

The statement of our main theorem is the following.

Theorem 2.1. *Let $\mathcal{A}$ be a prime C^*-algebra that contains a non-trivial projection and η be a non-zero complex number that $\mid \eta \mid \neq 1$ and $\psi\colon \mathcal{A} \to \mathcal{A}$ be the η-Jordan $*$-derivation map on $\mathcal{A}$ that is, for every $A, B \in \mathcal{A}$,*

$$\psi(A \diamond_\eta B) = \psi(A) \diamond_\eta B + A \diamond_\eta \psi(B) \tag{2.1}$$

where $A \diamond_\eta B = AB + \eta BA^*$, then ψ is additive $*$-derivation.

Before proving Theorem 2.1, we need the following lemma.

Lemma 2.2. *([14]) Let $T \in \mathcal{A}$ and η be a non-zero complex number such that $\mid \eta \mid \neq 1$. If $T + \eta T^* = 0$ then $T = 0$.*

Let $P_1 \in \mathcal{A}$ be a non-trivial projection and $P_2 = I - P_1$. By getting $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$ for $i, j = 1, 2$, we can write $\mathcal{A} = \sum_{i,j=1,2} \mathcal{A}_{ij}$ such that their pairwise intersections are $\{0\}$. In all that follows when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$.

Proof. We prove Theorem 2.1 in several Steps.

Step 1. $\psi(0) = 0$.

Indeed by getting $A = B = 0$, we have $\psi(0) = 0$.

Step 2. For $i, j \in \{1, 2\}$ with $i \neq j$, we have

(a) $\psi(A_{ii} + B_{ij}) = \psi(A_{ii}) + \psi(B_{ij})$,
(b) $\psi(A_{ii} + C_{ji}) = \psi(A_{ii}) + \psi(C_{ji})$,

(c) $\psi(A_{ii} + D_{jj}) = \psi(A_{ii}) + \psi(D_{jj})$,
(d) $\psi(B_{ij} + C_{ji}) = \psi(B_{ij}) + \psi(C_{ji})$.

(a) From $P_j \diamond_\eta A_{ii} = 0$ and $P_j \diamond_\eta (A_{ii}+B_{ij}) = P_j \diamond_\eta B_{ij}$, we have $\psi(P_j \diamond_\eta A_{ii}) = 0$ and

$$\psi(P_j \diamond_\eta (A_{ii} + B_{ij})) = \psi(P_j \diamond_\eta A_{ii}) + \psi(P_j \diamond_\eta B_{ij}).$$

So from Eq. (2.1) we can write

$$\begin{aligned}\psi(P_j)\diamond_\eta(A_{ii} + B_{ij}) &+ P_j \diamond_\eta \psi(A_{ii} + B_{ij})\\ &= \psi(P_j) \diamond_\eta A_{ii} + P_j \diamond_\eta \psi(A_{ii}) + \psi(P_j) \diamond_\eta B_{ij} + P_j \diamond_\eta \psi(B_{ij})\\ &= \psi(P_j) \diamond_\eta (A_{ii} + B_{ij}) + P_j \diamond_\eta (\psi(A_{ii}) + \psi(B_{ij})).\end{aligned}$$

Therefore $P_j \diamond_\eta K = 0$ where $K = \psi(A_{ii} + B_{ij}) - \psi(A_{ii}) - \psi(B_{ij})$ and so

$$P_j K + \eta K P_j = 0.$$

Multiplying above equation with P_i from the left side, and with P_i from the right side, and also with P_j from the two sides, respectively, we have $\eta P_i K P_j = 0$, $P_j K P_i = 0$ and $P_j K P_j + \eta P_j K P_j = 0$. This follows from the fact $\mid \eta \mid \neq 0, 1$ that $K_{ij} = K_{ji} = K_{jj} = 0$ and so $K = K_{ii}$.

On the other hand from $(\eta P_i - P_j) \diamond_\eta B_{ij} = 0$ and $(\eta P_i - P_j) \diamond_\eta (A_{ii} + B_{ij}) = (\eta P_i - P_j) \diamond_\eta A_{ii}$, we have

$$\psi[(\eta P_i - P_j) \diamond_\eta (A_{ii} + B_{ij})] = \psi((\eta P_i - P_j) \diamond_\eta A_{ii}) + \psi((\eta P_i - P_j) \diamond_\eta B_{ij}).$$

Hence we can write

$$\begin{aligned}\psi(\eta P_i - P_j)\diamond_\eta(A_{ii} + B_{ij}) &+ (\eta P_i - P_j) \diamond_\eta \psi(A_{ii} + B_{ij})\\ &= \psi(\eta P_i - P_j) \diamond_\eta A_{ii} + (\eta P_i - P_j) \diamond_\eta \psi(A_{ii}) + \psi(\eta P_i - P_j) \diamond_\eta B_{ij}\\ &\quad + (\eta P_i - P_j) \diamond_\eta \psi(B_{ij})\\ &= \psi(\eta P_i - P_j) \diamond_\eta (A_{ii} + B_{ij}) + (\eta P_i - P_j) \diamond_\eta (\psi(A_{ii}) + \psi(B_{ij})).\end{aligned}$$

Hence $(\eta P_i - P_j) \diamond_\eta K_{ii} = 0$ and from this, one can get $\eta(\overline{\eta} + 1)K_{ii} = 0$. Since $\mid \eta \mid \neq 0, 1$, we have $K_{ii} = 0$ and so $K = 0$. Thus

$$\psi(A_{ii} + B_{ij}) = \psi(A_{ii}) + \psi(B_{ij}).$$

(b) From $P_j \diamond_\eta A_{ii} = 0$ and $P_j \diamond_\eta (A_{ii}+C_{ji}) = P_j \diamond_\eta C_{ji}$, we have $\psi(P_j \diamond_\eta A_{ii}) = 0$ and

$$\psi(P_j \diamond_\eta (A_{ii} + C_{ji})) = \psi(P_j \diamond_\eta A_{ii}) + \psi(P_j \diamond_\eta C_{ji}).$$

Similar to part (a) and by a simple computing, one can give $P_j \diamond_\eta K = 0$ where $K = \psi(A_{ii} + C_{ji}) - \psi(A_{ii}) - \psi(C_{ji})$ and so

$$P_j K + \eta K P_j = 0.$$

Therefore $K_{ij} = K_{ji} = K_{jj} = 0$ and $K = K_{ii}$.

On the other hand from $(\eta P_j - P_i) \diamond_\eta C_{ji} = 0$ and $(\eta P_j - P_i) \diamond_\eta (A_{ii} + C_{ji}) = (\eta P_j - P_i) \diamond_\eta A_{ii}$, we have

$$\psi((\eta P_j - P_i) \diamond_\eta (A_{ii} + C_{ji})) = \psi((\eta P_j - P_i) \diamond_\eta A_{ii}) + \psi((\eta P_j - P_i) \diamond_\eta C_{ji}).$$

Similar to part (a), this yields $(\eta P_j - P_i) \diamond_\eta K_{ii} = 0$ and from it, one can get $-(\eta+1)K_{ii} = 0$. Thus $K_{ii} = 0$ and so $K = 0$, it means that

$$\psi(A_{ii} + C_{ji}) = \psi(A_{ii}) + \psi(C_{ji}).$$

(c) From $P_j \diamond_\eta A_{ii} = 0$ and $P_j \diamond_\eta (A_{ii} + D_{jj}) = P_j \diamond_\eta D_{jj}$ and similar to previous part we can easily see that $P_j \diamond_\eta K = 0$ where $K = \psi(A_{ii}+D_{jj}) - \psi(A_{ii}) - \psi(D_{jj})$ and so

$$P_j K + \eta K P_j = 0.$$

Therefore $K_{ij} = K_{ji} = K_{jj} = 0$ and $K = K_{ii}$.

Also by getting this way, from $P_i \diamond_\eta D_{jj} = 0$ we have $P_i \diamond_\eta K_{ii} = 0$. It follows that $K = K_{ii} = 0$ and so

$$\psi(A_{ii} + D_{jj}) = \psi(A_{ii}) + \psi(D_{jj}).$$

(d) From the fact $B_{ij} \diamond_\eta P_i = 0$ we can obtain $K \diamond_\eta P_i = 0$ where $K = \psi(B_{ij} + C_{ji}) - \psi(B_{ij}) - \psi(C_{ji})$ that is $KP_i + \eta P_i K^* = 0$. So $K_{ij} = K_{ji} = 0$ and $K_{ii} + \eta K_{ii}^* = 0$. This from Lemma 2.2 yields $K_{ii} = 0$

Also the relation $C_{ji} \diamond_\eta P_j = 0$ similarly yields that $K \diamond_\eta P_j = 0$ and so $K_{jj} = 0$, the result is derived.

Step 3. For $1 \leq i \neq j \leq 2$, we have $\psi(A_{ii} + A_{ij} + A_{ji} + A_{jj}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{jj})$.

First we show that

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}).$$

From $P_j \diamond_\eta A_{ii} = 0$ and $P_j \diamond_\eta (A_{ii} + A_{ij} + A_{ji}) = P_j \diamond_\eta (A_{ij} + A_{ji})$ we can write

$$\psi(P_j \diamond_\eta (A_{ii} + A_{ij} + A_{ji})) = \psi(P_j \diamond_\eta (A_{ij} + A_{ji})) + \psi(P_j \diamond_\eta A_{ii}),$$

and from part (d) of Step 2 we have

$$\begin{aligned}\psi(P_j) \diamond_\eta (A_{ii} + A_{ij} + A_{ji}) + P_j \diamond_\eta \psi(A_{ii} + A_{ij} + A_{ji}) &= \psi(P_j) \diamond_\eta (A_{ij} + A_{ji}) \\ &+ P_j \diamond_\eta (\psi(A_{ij}) + \psi(A_{ji})) + \psi(P_j) \diamond_\eta A_{ii} + P_j \diamond_\eta \psi(A_{ii}).\end{aligned}$$

Hence $P_j \diamond_\eta \psi(A_{ii} + A_{ij} + A_{ji}) = P_j \diamond_\eta [\psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{ii})]$ and so $P_j \diamond_\eta K = 0$ where $K = \psi(A_{ii} + A_{ij} + A_{ji}) - \psi(A_{ii}) - \psi(A_{ij}) - \psi(A_{ji})$. Then $P_j K + \eta K P_j = 0$ and from it, $K_{jj} = K_{ij} = K_{ji} = 0$.

Also, by this way from relation $A_{ij} \diamond_\eta P_i = 0$ we can derive $K \diamond_\eta P_i = 0$ and so $KP_i + \eta P_i K^* = 0$. Thus $K_{ii} + \eta K_{ii}^* = 0$, this from Lemma 2.2 yields $K_{ii} = 0$. Therefore $K = 0$ and so

$$\psi(A_{ii} + A_{ij} + A_{ji}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}).$$

In the general case, simillar to this way from $P_j \diamond_\eta A_{ii} = 0$ and $P_i \diamond_\eta A_{jj} = 0$ we can easily show that

$$\psi(A_{ii} + A_{ij} + A_{ji} + A_{jj}) = \psi(A_{ii}) + \psi(A_{ij}) + \psi(A_{ji}) + \psi(A_{jj}).$$

Step 4. $\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij})$ for every $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$ such that $i, j = 1, 2$.

Let $i \neq j$. From relation

$$A_{ij} + B_{ij} + \eta B_{ij}^* + \eta A_{ij} B_{ij}^* = (P_i + B_{ij}) \diamond_\eta (P_j + A_{ij})$$

we can write

$$\psi(A_{ij} + B_{ij} + \eta B_{ij}^* + \eta A_{ij} B_{ij}^*) = \psi((P_i + B_{ij}) \diamond_\eta (P_j + A_{ij})).$$

This follows from Eq. (2.1) and Steps 2 and 3 that

$$\begin{aligned}
&\psi(A_{ij} + B_{ij}) + \psi(\eta B_{ij}^*) + \psi(\eta A_{ij} B_{ij}^*)\\
&\quad = \psi(P_i + B_{ij}) \diamond_\eta (P_j + A_{ij}) + (P_i + B_{ij}) \diamond_\eta \psi(P_j + A_{ij})\\
&\quad = [\psi(P_i) + \psi(B_{ij})] \diamond_\eta (P_j + A_{ij}) + (P_i + B_{ij}) \diamond_\eta [\psi(P_j) + \psi(A_{ij})]\\
&\quad = \psi(P_i) \diamond_\eta P_j + P_i \diamond_\eta \psi(P_j) + \psi(P_i) \diamond_\eta A_{ij} + P_i \diamond_\eta \psi(A_{ij}) + \psi(B_{ij}) \diamond_\eta P_j\\
&\qquad + B_{ij} \diamond_\eta \psi(P_j) + \psi(B_{ij}) \diamond_\eta A_{ij} + B_{ij} \diamond_\eta \psi(A_{ij})\\
&\quad = \psi(P_i \diamond_\eta P_j) + \psi(P_i \diamond_\eta A_{ij}) + \psi(B_{ij} \diamond_\eta P_j) + \psi(B_{ij} \diamond_\eta A_{ij})\\
&\quad = \psi(0) + \psi(A_{ij}) + \psi(B_{ij} + \eta B_{ij}^*) + \psi(\eta A_{ij} B_{ij}^*)\\
&\quad = \psi(A_{ij}) + \psi(B_{ij}) + \psi(\eta B_{ij}^*) + \psi(\eta A_{ij} B_{ij}^*).
\end{aligned}$$

Therefore we have

$$\psi(A_{ij} + B_{ij}) = \psi(A_{ij}) + \psi(B_{ij}). \tag{2.2}$$

Now, for $i = 1, 2$ we will show that

$$\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii}).$$

From $P_j \diamond_\eta A_{ii} = 0$, as previous contents, one can see that $P_j \diamond_\eta K = 0$ where $K = \psi(A_{ii} + B_{ii}) - \psi(A_{ii}) - \psi(B_{ii})$. This clearly yields $K_{ij} = K_{jj} = K_{ji} = 0$.

Let $C_{ij} \in \mathcal{A}_{ij} (i \neq j)$. From $(A_{ii} + B_{ii}) \diamond_\eta C_{ij} = A_{ii} C_{ij} + B_{ii} C_{ij}$ we can write

$$\psi[(A_{ii} + B_{ii}) \diamond_\eta C_{ij}] = \psi(A_{ii} C_{ij} + B_{ii} C_{ij}).$$

This from Eq. (2.2) yields

$$\begin{aligned}
&\psi(A_{ii} + B_{ii}) \diamond_\eta C_{ij} + (A_{ii} + B_{ii}) \diamond_\eta \psi(C_{ij})\\
&\quad = \psi(A_{ii} C_{ij}) + \psi(B_{ii} C_{ij})\\
&\quad = \psi(A_{ii} \diamond_\eta C_{ij}) + \psi(B_{ii} \diamond_\eta C_{ij})\\
&\quad = \psi(A_{ii}) \diamond_\eta C_{ij} + A_{ii} \diamond_\eta \psi(C_{ij}) + \psi(B_{ii}) \diamond_\eta C_{ij} + B_{ii} \diamond_\eta \psi(C_{ij})\\
&\quad = (\psi(A_{ii}) + \psi(B_{ii})) \diamond_\eta C_{ij} + (A_{ii} + B_{ii}) \diamond_\eta \psi(C_{ij}).
\end{aligned}$$

Hence

$$\psi(A_{ii} + B_{ii}) \diamond_\eta C_{ij} = (\psi(A_{ii}) + \psi(B_{ii})) \diamond_\eta C_{ij}$$

and this implies that $K \diamond_\eta C_{ij} = 0$ and so $K C_{ij} + \eta C_{ij} K^* = 0$. From it since $K = K_{ii}$, we can derive $K_{ii} C_{ij} = 0$ for every $C_{ij} \in \mathcal{A}_{ij}$. Hence $K_{ii} \mathcal{A} P_j = 0$ and the primeness property of $\mathcal{A}$ yields that $K_{ii} = 0$.

So in the overall case we have $K = 0$, i. e.

$$\psi(A_{ii} + B_{ii}) = \psi(A_{ii}) + \psi(B_{ii}).$$

Step 5. ψ is additive on $\mathcal{A}$.

This is obtained from Steps 3 and 4.

Step 6. $\psi(I) = 0$.

From $P_2 \diamond_\eta P_1 = 0$ we have

$$\psi(P_2)P_1 + \eta P_1\psi(P_2)^* + P_2\psi(P_1) + \eta\psi(P_1)P_2 = 0. \tag{2.3}$$

Multyplying two sides of Eq. (2.3) with P_2, we derive $\psi(P_1)_{22} = 0$ and also by multyplying Eq. (2.3) with P_2 from the left side and P_1 from the right side we conclude $\psi(P_1)_{21} + \psi(P_2)_{21} = 0$ and from additivity of ψ we have $\psi(I)_{21} = 0$.
Similar to this way, one can obtain $\psi(P_2)_{11} = 0$ and $\psi(I)_{12} = 0$.
On the othere hand for every $A_{12} \in \mathcal{A}_{12}$, from $A_{12} = P_1 \diamond_\eta A_{12}$ we have

$$\psi(A_{12}) = \psi(P_1)A_{12} + \eta A_{12}\psi(P_1)^* + P_1\psi(A_{12}) + \eta\psi(A_{12})P_1.$$

From it we have $P_1\psi(P_1)A_{12} = -\eta A_{12}\psi(P_1)^*P_2$, and so

$$P_1\psi(P_1)A_{12} = -\eta A_{12}(\psi(P_1)_{22})^* = 0.$$

This yields $\psi(P_1)_{11} = 0$ as $\mathcal{A}$ is prime.
Similar to this way we can conclude $\psi(P_2)_{22} = 0$ and therefore from additivity of ψ, we get

$$\begin{aligned}\psi(I) &= \psi(I)_{12} + \psi(I)_{21} + \psi(I)_{11} + \psi(I)_{22}\\ &= \psi(I)_{11} + \psi(I)_{22}\\ &= \psi(P_1)_{11} + \psi(P_2)_{11} + \psi(P_1)_{22} + \psi(P_2)_{22}\\ &= 0.\end{aligned}$$

Step 7. ψ is $*-$preserving, i. e., $\psi(A^*) = \psi(A)^*$ for all $A \in \mathcal{A}$.

For all $A \in \mathcal{A}$ since $\psi(I) = 0$, we have

$$\psi(I \diamond_\eta A) = I \diamond_\eta \psi(A) \Longrightarrow \psi(A + \eta A) = \psi(A) + \eta\psi(A),$$

and from additivity of ψ, $\psi(\eta A) = \eta\psi(A)$.
On the othere hand from $\psi(A \diamond_\eta I) = \psi(A) \diamond_\eta I$, we derive $\psi(A) + \psi(\eta A^*) = \psi(A) + \eta\psi(A)^*$. Since $\eta \neq 0$, this yields $\psi(A^*) = \psi(A)^*$.

Step 8. ψ is a derivation map on $\mathcal{A}$, i. e., $\psi(AB) = \psi(A)B + A\psi(B)$ for all $A, B \in \mathcal{A}$.

This Step will be proved by 3 Stage.
Stage 1. Let A and B be self adjoint elements in $\mathcal{A}$. Since ψ is an additive and $*-$preserving map from equations

$$\psi(A \diamond_\eta B) = \psi(A) \diamond_\eta B + A \diamond_\eta \psi(B),$$
$$\psi(B \diamond_\eta A) = \psi(B) \diamond_\eta A + B \diamond_\eta \psi(A)$$

we have

$$\psi(AB) + \eta\psi(BA) = \psi(A)B + \eta B\psi(A) + A\psi(B) + \eta\psi(B)A,$$
$$\psi(BA) + \eta\psi(AB) = \psi(B)A + \eta A\psi(B) + B\psi(A) + \eta\psi(A)B.$$

By a simple computing since $\mid \eta \mid \neq 1$, these equations yield

$$\psi(AB) = \psi(A)B + A\psi(B).$$

Stage 2. We show that $\psi(iA) = i\psi(A)$ for all $A \in \mathcal{A}$. From $iI \diamond_\eta iI = -I + \eta I$ we have

$$\psi(iI) \diamond_\eta iI + iI \diamond_\eta \psi(iI) = -\psi(I) + \eta\psi(I)$$

$$\Rightarrow i\psi(iI) + \eta i\psi(iI)^* + i\psi(iI) - \eta i\psi(iI) = 0$$

$$\Rightarrow 2i(1-\eta)\psi(iI) = 0 \Rightarrow \psi(iI) = 0.$$

Also from $iI \diamond_\eta A = iA - i\eta A$ and $\psi(iI) = 0$ we can write

$$\psi(iI \diamond_\eta A) = \psi(iA - i\eta A)$$

$$\Rightarrow iI \diamond_\eta \psi(A) = \psi(iA) - \eta\psi(iA).$$

Since $\eta \neq 1$ this equation yields $\psi(iA) = i\psi(A)$.

Stage 3. Finally, we prove ψ is derivation on $\mathcal{A}$.

For each A and B in $\mathcal{A}$ let $A = A_1 + iA_2$ and $B = B_1 + iB_2$, where A_1, A_2, B_1 and B_2 are self adjoint elements in $\mathcal{A}$.

From additivity of ψ and two previous cases of this step we can write

$$\begin{aligned}
\psi(AB) &= \psi[(A_1 + iA_2)(B_1 + iB_2)] \\
&= \psi(A_1B_1 + iA_1B_2 + iA_2B_1 - A_2B_2) \\
&= \psi(A_1B_1) + i\psi(A_1B_2) + i\psi(A_2B_1) - \psi(A_2B_2) \\
&= \psi(A_1)B_1 + A_1\psi(B_1) + i\psi(A_1)B_2 + iA_1\psi(B_2) + i\psi(A_2)B_1 \\
&\quad + iA_2\psi(B_1) - \psi(A_2)B_2 - A_2\psi(B_2) \\
&= \psi(A_1)[B_1 + iB_2] + i\psi(A_2)[B_1 + iB_2] + A_1[\psi(B_1) + i\psi(B_2)] \\
&\quad + iA_2[\psi(B_1) + i\psi(B_2)] \\
&= \psi(A_1)B + i\psi(A_2)B + A_1\psi(B) + iA_2\psi(B) \\
&= \psi(A)B + A\psi(B).
\end{aligned}$$

Then ψ is an additive $*-$derivation map on $\mathcal{A}$ and the prove of Theorem 2.1 is completed. $\square$

As we mentioned in the introduction, for $\eta = -1$ Yu and Zhang in [15] and for $\eta = 1$ Taghavi, Rohi and Darvish in [14] proved that every η-Jordan $*$-derivation map on factor von Neumann algebra is an additive $*$-derivation. Also since every factor von Neumann algebra is prime, from Theorem 2.1 and [15] and [14] we can write the following Corollary.

Corollary 2.3. *Let $\mathcal{A}$ be a von Neumann algebra and η be a non-zero real number and $\psi\colon \mathcal{A} \to \mathcal{A}$ be the η-Jordan $*$-derivation, then ψ is an additive $*$-derivation map.*

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