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MODULE SYMMETRICALLY AMENABLE BANACH ALGEBRAS

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Abstract. In this article, we develop the concept of symmetric amenability for a Banach algebra $\mathcal{A}$ to the case that there is an extra $\mathfrak{A}$-module structure on $\mathcal{A}$. For every inverse semigroup S with the set E of idempotents, we find necessary and sufficient conditions for the $l^1(S)$ to be module symmetrically amenable (as a $l^1(E)$-module). We also present some module symmetrically amenable semigroup algebras to show that this new notion of amenability is different from the classical case introduced by Johnson.

1. Introduction

A Banach algebra $\mathcal{A}$ is *amenable* if every bounded derivation from $\mathcal{A}$ into any dual Banach $\mathcal{A}$-bimodule is inner, equivalently if $H^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-module X, where $H^1(\mathcal{A}, X^*)$ is the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in X^*. This concept was first introduced and studied by Johnson [9] in 1972. He also gave an alternative formulation of the notion of amenability in [10], and proved that a Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}$ has a bounded approximate diagonal; i.e. a bounded net $\{d_\alpha\}$ in the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\|\pi(d_\alpha)a - a\| \to 0 \quad \text{and} \quad \|a \cdot d_\alpha - d_\alpha \cdot a\| \to 0$$

for all $a \in \mathcal{A}$, where the operations on $\mathcal{A}\widehat{\otimes}\mathcal{A}$ are defined by $a \cdot (b \otimes c) = ab \otimes c$, $(b \otimes c) \cdot a = b \otimes ca$ and $\pi(b \otimes c) = bc$ for all $a, b, c \in \mathcal{A}$. The flip map on $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is defined by

$$(a \otimes b)^\circ = b \otimes a \qquad (a, b \in \mathcal{A}),$$

and an element $\mathfrak{E}$ of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is called *symmetric* if $\mathfrak{E}^\circ = \mathfrak{E}$. A Banach algebra $\mathcal{A}$ is called *symmetrically amenable* if $\mathcal{A}$ has a bounded approximate diagonal consisting of symmetric tensors. Symmetrically amenable Banach algebras were defined by Johnson in [11]. Using this concept, he found some hereditary

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properties and examples which are similar to those in [9] for amenable Banach algebras. However, unlike amenability, the proofs of that results do not depend on homological characterizations, because symmetric amenability has been considered only by the existence of a bounded (symmetric) approximate diagonal. The most important example in [11] asserts that the group algebra $L^1(G)$ of a locally compact group G is symmetrically amenable if and only if G is amenable.

In 2004, M. Amini [1] introduced the notion of module amenability for a class of Banach algebras which could be considered as a generalization of the Johnson's amenability [9]. He showed that for an inverse semigroup S with the set of idempotents E, the semigroup algebra $l^1(S)$ is module amenable, as a Banach module over $l^1(E)$, if and only if S is amenable. Other concepts of module amenability can be found in [3], [4], [5] and [13].

In this paper, we firstly define the concept of module symmetric amenability for a Banach algebra $\mathcal{A}$ which is a Banach module on another Banach algebra $\mathfrak{A}$ with compatible actions. Among many other things, we show that under some mild conditions, symmetric amenability of the quotient Banach algebra $\mathcal{A}/J$ implies module symmetric amenability of $\mathcal{A}$, where J is the closed ideal of $\mathcal{A}$ generated by $(a\cdot\alpha)b - a(\alpha\cdot b)$ for all $a \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. As a consequence of this result, we prove that for an inverse semigroup S with the set E of idempotents so that E satisfies the condition D_k [7] for some k, then $l^1(S)$ is module symmetrically amenable (as an $l^1(E)$-module) with trivial left action, if and only if S is amenable.

2. Module symmetric amenability for Banach algebras

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras such that $\mathcal{A}$ is a Banach $\mathfrak{A}$-bimodule with compatible actions as follows:

$$\alpha\cdot(ab) = (\alpha\cdot a)b, \quad (ab)\cdot\alpha = a(b\cdot\alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Furthermore, if $\alpha\cdot a = a\cdot\alpha$ for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$, then $\mathcal{A}$ is called a *commutative* $\mathfrak{A}$*-bimodule*.

Let X be a left Banach $\mathcal{A}$-module and a Banach $\mathfrak{A}$-bimodule with the following compatible actions:

$$\alpha\cdot(a\cdot x) = (\alpha\cdot a)\cdot x,\ a\cdot(\alpha\cdot x) = (a\cdot\alpha)\cdot x,\ a\cdot(x\cdot\alpha) = (a\cdot x)\cdot\alpha\ (a \in \mathcal{A}, \alpha \in \mathfrak{A}, x \in X).$$

Then, we say that X is a *left Banach* $\mathcal{A}$-$\mathfrak{A}$*-module*. Right Banach $\mathcal{A}$-$\mathfrak{A}$-modules and (two-sided) Banach $\mathcal{A}$-$\mathfrak{A}$-modules are defined similarly. If moreover, $\alpha\cdot x = x\cdot\alpha$ for all $\alpha \in \mathfrak{A}$ and $x \in X$, then X is called a *commutative* left (right or two-sided) Banach $\mathcal{A}$-$\mathfrak{A}$-module. If X is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module, then so is X^*, where the actions of $\mathcal{A}$ and $\mathfrak{A}$ on X^* are defined as usual [1]. Note that in general, $\mathcal{A}$ is not an $\mathcal{A}$-$\mathfrak{A}$-module because $\mathcal{A}$ does not satisfy the compatibility condition $a\cdot(\alpha\cdot b) = (a\cdot\alpha)\cdot b$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. But when $\mathcal{A}$ is a commutative $\mathfrak{A}$-module and acts on itself by multiplication from both sides, then it is also a Banach $\mathcal{A}$-$\mathfrak{A}$-module.

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach $\mathfrak{A}$-bimodules with compatible actions. Then, a $\mathfrak{A}$-*module map* is a bounded mapping $T\colon \mathcal{A} \longrightarrow \mathcal{B}$ with

$$T(a \pm b) = T(a) \pm T(b),\ T(\alpha \cdot a) = \alpha \cdot T(a) \text{ and } T(a \cdot \alpha) = T(a) \cdot \alpha$$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. Note that h is not necessarily linear, so it is not necessarily a $\mathfrak{A}$-module homomorphism.

Let $\mathcal{A}$ and $\mathfrak{A}$ be as above and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A ($\mathfrak{A}$-)*module derivation* is a bounded $\mathfrak{A}$-bimodule map $D\colon \mathcal{A} \longrightarrow X$ satisfying

$$D(ab) = D(a) \cdot b + a \cdot D(b)$$

for all $a, b \in \mathcal{A}$. One should note that D is not necessarily linear, but its boundedness (defined as the existence of $M > 0$ such that $\|D(a)\| \leq M\|a\|$, for all $a \in \mathcal{A}$) still implies its continuity, as it preserves subtraction. When X is commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, each $x \in X$ defines a module derivation $D_x(a) = a \cdot x - x \cdot a\,(a \in \mathcal{A})$. Module derivations of this kind are called *inner*. A derivation $D\colon \mathcal{A} \longrightarrow X$ is said to be *approximately inner* if there exists a net $(x_i) \subseteq X$ such that $D(a) = \lim_i(a \cdot x_i - x_i \cdot a)$ for all $a \in \mathcal{A}$.

Consider the module projective tensor product $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ which is isomorphic to the quotient space $(\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}}$, where $I_{\mathcal{A}}$ is the closed linear span of $\{a \cdot \alpha \otimes b - a \otimes \alpha \cdot b : \alpha \in \mathfrak{A}, a, b \in \mathcal{A}\}$. Also consider the closed ideal $J_{\mathcal{A}}$ of $\mathcal{A}$ generated by elements of the form $(a \cdot \alpha)b - a(\alpha \cdot b)$ for $\alpha \in \mathfrak{A}, a, b \in \mathcal{A}$. We shall denote $I_{\mathcal{A}}$ and $J_{\mathcal{A}}$ by I and J, respectively, if there is no risk of confusion. Then, I and J are $\mathcal{A}$-submodules and $\mathfrak{A}$-submodules of $\mathcal{A}\widehat{\otimes}\mathcal{A}$ and $\mathcal{A}$, respectively, and the quotients $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ and $\mathcal{A}/J$ are $\mathcal{A}$-modules and $\mathfrak{A}$-modules. Also, $\mathcal{A}/J$ is a Banach $\mathcal{A}$-$\mathfrak{A}$-module when $\mathcal{A}$ acts on $\mathcal{A}/J$ canonically. Also, let $\omega_{\mathcal{A}}\colon \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ be the product map, i.e., $\omega_{\mathcal{A}}(a \otimes b) = ab$, and let $\widetilde{\omega}_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} = (\mathcal{A}\widehat{\otimes}\mathcal{A})/I \longrightarrow \mathcal{A}/J$ be its induced product map, i.e., $\widetilde{\omega}_{\mathcal{A}}(a \otimes b + I) = ab + J$ and extended by continuity and linearity.

Recall that a *module approximate diagonal* for $\mathcal{A}$ is a bounded net $\{\widetilde{u}_j\}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ such that

$$(a + J)\widetilde{w}_{\mathcal{A}}(\widetilde{u}_j) \to a + J \tag{2.1}$$

and

$$\lim_j \|\widetilde{u}_j \cdot a - a \cdot \widetilde{u}_j\| = 0 \tag{2.2}$$

for each $a \in \mathcal{A}$ [1]. We define the *module flip map* on $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ by

$$(a \otimes b + I)^\circ = b \otimes a + I \qquad (a, b \in \mathcal{A}).$$

We say an element u of $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is *module symmetric* if $u^\circ = u$.

Definition 2.1. A Banach algebra $\mathcal{A}$ is *module symmetrically amenable* if $\mathcal{A}$ has a module approximate diagonal $\{\widetilde{u}_j\}$ such that all the elements of the net $\{\widetilde{u}_j\}$ are module symmetric.

The opposite algebra $\mathcal{A}^{op}$ is the Banach space $\mathcal{A}$ with product $a \circ b = ba$. Now we rewrite the above definitions for $\mathcal{A}^{op}$ in the module version. The bounded net $\{\widetilde{u}_j\}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is a module approximate diagonal for $\mathcal{A}^{op}$ if

$$\widetilde{w}^{\circ}_{\mathcal{A}}(\widetilde{u}_j)(a+J) \to a+J \tag{2.3}$$

and

$$\lim_j \|\widetilde{u}_j \circ a - a \circ \widetilde{u}_j\| = 0 \tag{2.4}$$

for all $a \in \mathcal{A}$, where $a \circ (b \otimes c) = b \otimes ac$, $(b \otimes c) \circ a = ba \otimes c$ and $\widetilde{w}^{\circ}_{\mathcal{A}}(b \otimes c + I) = cb + J$.

The following proposition is the module version of [11, Proposition 2.2].

Proposition 2.2. *A Banach algebra $\mathcal{A}$ is module symmetrically amenable if and only if there is a bounded net $\{\widetilde{u}_j\}$ in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ which satisfies* (2.1), (2.2), (2.3) *and* (2.4).

Proof. Let $\mathcal{A}$ be module symmetrically amenable. Then, $\mathcal{A}$ has a module approximate diagonal $\{\widetilde{u}_j\}$ which satisfies (2.1) and (2.2). Since $\widetilde{u}_j = \widetilde{u}^{\circ}_j$, we know that $\{\widetilde{u}_j\}$ also satisfies (2.3) and (2.4).

Conversely, if the bounded net $\{\widetilde{u}_j\}$ satisfies (2.1), (2.2), (2.3) and (2.4), so does $\{\widetilde{u}^{\circ}_j\}$. Hence, $\{\frac{1}{2}(\widetilde{u}_j + \widetilde{u}^{\circ}_j)\}$ is a net of symmetric tensors in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ satisfying (2.1) and (2.2). This means that $\mathcal{A}$ is module symmetrically amenable. □

Corollary 2.3. *If $\mathcal{A}$ is a commutative module amenable Banach algebra, then it is module symmetrically amenable.*

Recall that a (*bounded*) *left approximate identity* in a Banach algebra $\mathcal{A}$ is a (bounded) net $\{e_l\}_{l\in\mathcal{L}}$ in $\mathcal{A}$ such that $\lim_l e_l a = a$ for all $a \in \mathcal{A}$. Similarly, a (bounded) right approximate identity can be defined in $\mathcal{A}$. A (bounded) approximate identity in $\mathcal{A}$ is both a (bounded) left approximate identity and a (bounded) right approximate identity.

It is easy to see that $K = \ker \widetilde{w}_{\mathcal{A}}$ is an $\mathcal{A}$-$\mathfrak{A}$-submodule of $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. In fact, K is a left ideal in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}^{op}$. Aghababa and Bodaghi [15, Theorem 4.4] have shown that, if $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-bimodule, then $\mathcal{A}$ is module amenable if and only if $\mathcal{A}$ has a bounded approximate identity and $K = \ker \widetilde{w}_{\mathcal{A}}$ has a bounded right approximate identity, where $\widetilde{w}_{\mathcal{A}} \colon \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}^{op} \longrightarrow \mathcal{A}$ is the usual multiplication map. Similarly, one can show that if $\mathcal{A}$ is a commutative Banach $\mathfrak{A}$-bimodule, then $\mathcal{A}$ is module symmetrically amenable if and only if $\mathcal{A}$ has a bounded approximate identity and the subalgebra $\ker \widetilde{w}_{\mathcal{A}} \cap \ker \widetilde{w}^{\circ}_{\mathcal{A}}$ of $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}^{op}$ has a bounded two sided approximate identity.

Now, we give some hereditary properties of module symmetrically amenable for Banach algebras.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\mathcal{I}$ be a closed two sided ideal and $\mathfrak{A}$-submodule of A. If $\mathcal{A}$ is module symmetrically amenable and $\mathcal{I}$ has a bounded approximate identity, then $\mathcal{I}$ is module symmetrically amenable.*

Proof. Let $\{\widetilde{u}_i\}$ be a module symmetric approximate diagonal for $\mathcal{A}$, where $\widetilde{u}_i = \sum\limits_k a_k^i \otimes b_k^i + I_{\mathcal{A}}$ is in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. Assume that $\{e_j\}$ is the bounded approximate identity of $\mathcal{I}$. For each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we have

$$\begin{aligned}(((a\cdot\alpha)\otimes b - a\otimes(\alpha\cdot b))\circ e_j)e_j &= e_j((a\cdot\alpha)\otimes b)e_j - e_j(a\otimes(\alpha\cdot b))e_j\\ &= e_j(a\cdot\alpha)\otimes be_j - e_ja\otimes(\alpha\cdot b)e_j\\ &= (e_ja)\cdot\alpha\otimes be_j - e_ja\otimes\alpha\cdot(be_j) \in I_{\mathcal{I}},\end{aligned}$$

where $I_{\mathcal{I}}$ is the corresponding ideal of $\mathcal{I}\widehat{\otimes}\mathcal{I}$. Put $\widetilde{d}_{ij} = (\widetilde{u}_i \circ e_j)e_j$. Then, $\widetilde{d}_{ij} = \sum\limits_k e_j a_k^i \otimes b_k^i e_j + I_{\mathcal{I}} \in \mathcal{I}\hat{\otimes}_{\mathfrak{A}}\mathcal{I}$ is a bounded symmetric subnet of $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. For $x \in \mathcal{I}$, we get

$$\widetilde{d}_{ij}\cdot x - x\cdot\widetilde{d}_{ij} = [(\widetilde{u}_i\cdot x - x\cdot\widetilde{u}_i)\circ e_j]e_j + (\widetilde{u}_i\circ e_j)(e_jx - xe_j).$$

Since $\{\widetilde{u}_i\}$ is a module symmetric approximate diagonal for $\mathcal{A}$, we have $\widetilde{u}_i\cdot x - x\cdot\widetilde{u}_i \to 0$. On the other hand, $\{e_j\}$ is a bounded approximate identity for $\mathcal{I}$. So, $e_jx - xe_j \to 0$. Hence, $\lim_{i,j}(\widetilde{d}_{ij}\cdot x - x\cdot\widetilde{d}_{ij}) = 0$. Also,

$$\begin{aligned}(x+J_{\mathcal{I}})\cdot\widetilde{w}_{\mathcal{I}}(\widetilde{d}_{ij}) &= (xe_j - x + J_{\mathcal{I}})\cdot\widetilde{w}_{\mathcal{I}}(\widetilde{u}_i\circ e_j) + (x+J_{\mathcal{I}})\cdot\widetilde{w}_{\mathcal{I}}(\widetilde{u}_i\circ e_j)\\ &\to (x+J_{\mathcal{I}})\cdot\widetilde{w}_{\mathcal{I}}(\widetilde{u}_i).\end{aligned}$$

Thus, $\lim_i\lim_j(x+J_{\mathcal{I}})\cdot\widetilde{w}_{\mathcal{I}}(\widetilde{d}_{ij}) = x + J_{\mathcal{I}}$. Therefore, $\{\widetilde{d}_{ij}\}$ becomes a module symmetric approximate diagonal for $\mathcal{I}$. □

Theorem 2.5. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and Banach $\mathfrak{A}$-bimodules. If $\mathcal{A}$ is module symmetrically amenable and $\phi\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a continuous module homomorphism with dense range, then $\mathcal{B}$ is module symmetrically amenable.*

Proof. Let $\{\widetilde{u}_i\}$ be a module symmetric approximate diagonal in $\mathcal{A}$ such that $\widetilde{u}_i = \sum\limits_k a_k^i \otimes b_k^i + I_{\mathcal{A}}$ is in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. Define the map $\widetilde{\phi}\colon \mathcal{A}/J_{\mathcal{A}} \longrightarrow \mathcal{B}/J_{\mathcal{B}}$ by $\widetilde{\phi}(a+J_{\mathcal{A}}) = \phi(a) + J_{\mathcal{B}}$. For each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we obtain

$$\phi((a\cdot\alpha)b - a(\alpha\cdot b)) = (\phi(a)\cdot\alpha)\phi(b) - \phi(a)(\alpha\cdot\phi(b)) \in J_{\mathcal{B}}.$$

So, the map $\widetilde{\phi}$ is well-defined. Put $\widetilde{v}_i = \sum\limits_k \phi(a_k^i) \otimes \phi(b_k^i) + I_{\mathcal{B}}$. For each $a \in \mathcal{A}$, we have

$$\begin{aligned}
\lim_i(\phi(a) + J_{\mathcal{B}}) \cdot \widetilde{w}_{\mathcal{B}}(\widetilde{v}_i) &= \lim_i(\phi(a) + J_{\mathcal{B}}) \cdot \left(\sum_k \phi(a_k^i)\phi(b_k^i) + J_{\mathcal{B}}\right) \\
&= \lim_i \left(\sum_k \phi(a a_k^i b_k^i) + J_{\mathcal{B}}\right) \\
&= \lim_i \widetilde{\phi}\left((a + J_{\mathcal{A}}) \cdot \left(\sum_k a_k^i b_k^i + J_{\mathcal{A}}\right)\right) \\
&= \lim_i \widetilde{\phi}((a + J_{\mathcal{A}}) \cdot \widetilde{w}_{\mathcal{B}}(\widetilde{u}_i)) = \widetilde{\phi}(a + J_{\mathcal{A}}) = \phi(a) + J_{\mathcal{B}}.
\end{aligned}$$

Also, we get

$$\begin{aligned}
\widetilde{w}_{\mathcal{B}}^{\circ}(\widetilde{v}_i) \cdot \lim_i(\phi(a) + J_{\mathcal{B}}) &= \lim_i \left(\sum_k \phi(b_k^i)\phi(a_k^i) + J_{\mathcal{B}}\right) \cdot (\phi(a) + J_{\mathcal{B}}) \\
&= \lim_i \left(\sum_k \phi(b_k^i a_k^i a) + J_{\mathcal{B}}\right) \\
&= \lim_i \widetilde{\phi}\left(\left(\sum_k b_k^i a_k^i + J_{\mathcal{A}}\right) \cdot (a + J_{\mathcal{A}})\right) \\
&= \lim_i \widetilde{\phi}(\widetilde{w}_{\mathcal{B}}^{\circ}(\widetilde{u}_i) \cdot (a + J_{\mathcal{A}})) = \widetilde{\phi}(a + J_{\mathcal{A}}) = \phi(a) + J_{\mathcal{B}}.
\end{aligned}$$

Since the range of ϕ is dense and ϕ is continuous, we get $\lim\limits_i(b + J_{\mathcal{B}}) \cdot \widetilde{w}_{\mathcal{B}}(\widetilde{v}_i) = b + J_{\mathcal{B}}$ and $\widetilde{w}_{\mathcal{B}}^{\circ}(\widetilde{v}_i) \cdot \lim\limits_i(b + J_{\mathcal{B}}) = b + J_{\mathcal{B}}$ for all $b \in \mathcal{B}$. Now, we consider the map $\overline{\phi}$: $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A} \cong (\mathcal{A}\widehat{\otimes}\mathcal{A})/I_{\mathcal{A}} \longrightarrow \mathcal{B}\widehat{\otimes}_{\mathfrak{A}}\mathcal{B} \cong (B\widehat{\otimes}\mathcal{B})/I_{\mathcal{B}}$ defined through $\overline{\phi}(a \otimes b + I_{\mathcal{A}}) = \phi(a) \otimes \phi(b) + I_{\mathcal{B}}$, $(a, b \in \mathcal{A})$. The map $\overline{\phi}$ is well-defined because for each $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we have

$$(\phi \otimes \phi)((a \cdot \alpha) \otimes b - a \otimes (\alpha \cdot b)) = (\phi(a) \cdot \alpha) \otimes \phi(b) - \phi(a)(\alpha \cdot \phi(b)) \in I_{\mathcal{B}}.$$

It is easily to chek that $\overline{\phi}$ is a module homomorphism. For each $a \in \mathcal{A}$, we find

$$\begin{aligned}
\lim_i(\widetilde{v}_i \cdot \phi(a) - \phi(a) \cdot \widetilde{v}_i) &= \lim_i \left(\sum_k (\phi(a_k^i) \otimes \phi(b_k^i a) - \phi(a a_k^i) \otimes \phi(b_k^i)) + I_{\mathcal{B}}\right) \\
&= \overline{\phi}\left(\lim_i \left(\sum_k (a_k^i \otimes b_k^i a - a a_k^i \otimes b_k^i) + I_{\mathcal{A}}\right)\right) \\
&= \overline{\phi}(\lim_i(\widetilde{u}_i \cdot a - a \cdot \widetilde{u}_i)) = 0,
\end{aligned}$$

On the other hand,

$$\begin{aligned}\lim_i(\widetilde{v}_i \circ \phi(a) - \phi(a) \circ \widetilde{v}_i) &= \lim_i \left(\sum_k (\phi(a_k^i a) \otimes \phi(b_k^i) - \phi(a_k^i) \otimes \phi(ab_k^i)) + I_{\mathcal{B}} \right) \\ &= \overline{\phi} \left(\lim_i \left(\sum_k (a_k^i a \otimes b_k^i - a_k^i \otimes ab_k^i) + I_{\mathcal{A}} \right) \right) \\ &= \overline{\phi}(\lim_i(\widetilde{u}_i \circ a - a \circ \widetilde{u}_i)) = 0.\end{aligned}$$

Hence, for each $b \in \mathcal{A}$, we arrive at $\lim_i(\widetilde{v}_i \cdot b - b \cdot \widetilde{v}_i) = 0$ and $\lim_i(\widetilde{v}_i \circ b - b \circ \widetilde{v}_i) = 0$. So, $\{\widetilde{v}_i\}$ is a module symmetric approximate diagonal in $\mathcal{B}$. This finishes the proof. □

Corollary 2.6. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule and $\mathcal{I}$ be a closed ideal in $\mathcal{A}$. If $\mathcal{A}$ is module symmetrically amenable, then so is $\mathcal{A}/\mathcal{I}$.*

Proof. If $q\colon \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ is the natural $\mathfrak{A}$-module map and $\{\widetilde{u}_i\}$ is a module symmetric approximate diagonal for $\mathcal{A}$, then $\{(q \otimes q)\widetilde{u}_i\}$ is a module symmetric approximate diagonal for $\mathcal{A}/\mathcal{I}$. □

Lemma 2.7. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with compatible actions. If $\mathcal{A}$ is module symmetrically amenable and X is a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then every module derivation from $\mathcal{A}$ into X, is approximately inner.*

Proof. Let $\{\widetilde{u}_j\} \subseteq \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be a module symmetric approximate diagonal for $\mathcal{A}$ such that $\widetilde{u}_j = \sum\limits_k a_k^j \otimes b_k^j + I$ and $D\colon \mathcal{A} \longrightarrow X$ be a module derivation. It is clear that $J \cdot X = X \cdot J = \{0\}$. Obviously, X becomes a Banach $\mathcal{A}/J$-bimodule with the following module actions

$$(a + J) \cdot x := a \cdot x \ , \quad x \cdot (a + J) := x \cdot a \quad (x \in X, a \in \mathcal{A}).$$

Define $\widetilde{D}\colon \mathcal{A}/J \longrightarrow X$ by $\widetilde{D}(a + J) = D(a)$ for $a \in \mathcal{A}$. Hence, $\widetilde{D}$ is a module derivation. Let $x_j = \sum\limits_k \widetilde{D}(a_k^j + J) \cdot b_k^j$. For each $\varphi \in X^*$, we have

$$\begin{aligned}\langle \varphi, x_j \cdot (a + J)\rangle &= \langle \varphi, (\sum_k \widetilde{D}(a_k^j + J) \cdot b_k^j) \cdot (a + J)\rangle = \langle \varphi, \sum_k \widetilde{D}(aa_k^j + J) \cdot b_k^j\rangle \\ &= \langle \varphi, \widetilde{D}(a + J) \cdot (\sum_k a_k^j b_k^j + J)\rangle + \langle \varphi, (a + J) \cdot \sum_k \widetilde{D}(a_k^j + J) \cdot b_k^j\rangle \\ &= \langle \varphi, \widetilde{D}(a + J) \cdot \widetilde{w}_A(\widetilde{u}_j)\rangle + \langle \varphi, (a + J) \cdot x_j\rangle.\end{aligned}$$

Then, $\widetilde{D}(a + J) = \lim\limits_j x_j \cdot (a + J) - (a + J) \cdot x_j$ for all $a \in \mathcal{A}$. Therefore, $\widetilde{D}$ is approximately inner and thus D is an approximately inner module derivation. □

Theorem 2.8. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with bounded approximate identity and $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ be a commutative $\mathfrak{A}$-bimodule such that each net of $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is* *bounded. Suppose that $\mathcal{I}$ is a closed ideal and $\mathfrak{A}$-submodule of $\mathcal{A}$. If $\mathcal{I}$ and $\mathcal{A}/\mathcal{I}$ are module symmetrically amenable, then so is $\mathcal{A}$.*

Proof. Let X be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module with compatible actions and $D\colon \mathcal{A} \longrightarrow X$ be a module derivation. Since $\mathcal{I}$ is module symmetrically amenable, the restriction of D to $\mathcal{I}$, i.e. $D|_{\mathcal{I}}$, is approximately inner by Lemma 2.7. Thus, the map $\widetilde{D} = D - D|_{\mathcal{I}}$ vanishes on $\mathcal{I}$. This map induces a module derivation from $A/\mathcal{I}$ into X defined via $\widetilde{D}(a+\mathcal{I}) = \widetilde{D}(a)$. Due to the module symmetric amenability of $\mathcal{A}/\mathcal{I}$, $\widetilde{D}$ is also approximately inner by Lemma 2.7. It follows from that D is an approximately inner module derivation. Let $\{e_j\}$ be a bounded approximate identity for $\mathcal{A}$. Then, passing to a subnet we may assume that $e_j \otimes e_j + I$ is w^*-convergent to T in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. By the continuity of $\widetilde{w}_{\mathcal{A}}$ and $\widetilde{w}^{\circ}_{\mathcal{A}}$, we have

$$\begin{aligned}
\widetilde{w}_{\mathcal{A}}(D_T(a)) &= \widetilde{w}_{\mathcal{A}}(\lim_j a\cdot T - T\cdot a) = \lim_j \widetilde{w}_{\mathcal{A}}(a\cdot T - T\cdot a)\\
&= \lim_j \widetilde{w}_{\mathcal{A}}(ae_j\otimes e_j - e_j\otimes e_j a + I)\\
&= \lim_j (ae_j^2 - e_j^2 a + J) = J,
\end{aligned}$$

$$\begin{aligned}
\widetilde{w}^{\circ}_{\mathcal{A}}(D_T(a)) &= \widetilde{w}^{\circ}_{\mathcal{A}}(\lim_j a\cdot T - T\cdot a)\\
&= \lim_j \widetilde{w}^{\circ}_{\mathcal{A}}(ae_j\otimes e_j - e_j\otimes e_j a + I)\\
&= \lim_j (e_j a e_j - e_j a e_j + J) = J
\end{aligned}$$

for all $a \in \mathcal{A}$. So, both $\widetilde{w}_{\mathcal{A}}$ and $\widetilde{w}^{\circ}_{\mathcal{A}}$ vanishes on the range of D_T, and D_T could be regarded as a module derivation from $\mathcal{A}$ into $K = \ker \widetilde{w}_{\mathcal{A}} \cap \ker \widetilde{w}^{\circ}_{\mathcal{A}}$. Since $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ is commutative $\mathfrak{A}$-bimodule, there is a (bounded) net $\{N_j\} \in K$ such that

$$D_T(a) = \lim_j a\cdot N_j - N_j\cdot a = D_{N_j}(a) \tag{2.5}$$

for all $a \in A$. Letting $\widetilde{u}_j = T - N_j \in \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$, we get

$$\begin{aligned}
(a+J)\widetilde{w}_{\mathcal{A}}(\widetilde{u}_j) &= (a+J)(\widetilde{w}_{\mathcal{A}}(T) - \widetilde{w}_{\mathcal{A}}(N_j))\\
&= (a+J)(e_j^2 + J)\\
&= ae_j + J \to a + J
\end{aligned}$$

for all $a \in \mathcal{A}$. The relation (2.5) implies that $a\cdot \widetilde{u}_j - \widetilde{u}_j\cdot a \to 0$. Similarly, we can obtain that $\widetilde{w}^{\circ}_{\mathcal{A}}(\widetilde{u}_j)(a+J) \to a+J$ and $a\circ \widetilde{u}_j - \widetilde{u}_j \circ a \to 0$. Hence, $\{\widetilde{u}_j\}$ is a module symmetric approximate diagonal in $\mathcal{A}$. This completes the proof. $\square$

We say the Banach algebra $\mathfrak{A}$ acts trivially on $\mathcal{A}$ from left (right) if there is a continuous linear functional f on $\mathfrak{A}$ such that $\alpha\cdot a = f(\alpha)a$ $(a\cdot\alpha = f(\alpha)a)$ for all $\alpha \in \mathfrak{A}$ and $a \in \mathcal{A}$.

The following result is main key to achieve our purpose of this paper.

Proposition 2.9. *Let $\mathcal{A}$ be a Banach $\mathfrak{A}$-bimodule with trivial left action and $\mathcal{A}$ has a bounded approximate identity. If $\mathcal{A}/J$ is symmetrically amenable, then $\mathcal{A}$ is module symmetrically amenable.*

Proof. Suppose that $\{d_i\}$ is a bounded approximate diagonal for $\mathcal{A}/J$, that is $d_i = \sum_k (a_k^i + J) \otimes (b_k^i + J) \in (\mathcal{A}/J)\widehat{\otimes}(\mathcal{A}/J)$. Define the map $\phi\colon (\mathcal{A}/J)\widehat{\otimes}(\mathcal{A}/J) \longrightarrow (\mathcal{A}\widehat{\otimes}\mathcal{A})/I \cong \mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$ via $\phi((a+J)\otimes(b+J)) := (a\otimes b) + I$. Assume that $\{e_j\}$ is a bounded approximate identity for $\mathcal{A}$. For each $a, b, c \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$, we obtain

$$\begin{aligned}
[(a\cdot\alpha)b - a(\alpha\cdot b)]\otimes c &= (a\cdot\alpha)b\otimes c - a(\alpha\cdot b)\otimes c \\
&= \lim_j [((a\cdot\alpha)b\otimes ce_j) - (a(\alpha\cdot b)\otimes e_j c)] \\
&= \lim_j [((a\cdot\alpha)\otimes c)(b\otimes e_j) - (a\otimes e_j)((\alpha\cdot b)\otimes c)] \\
&= \lim_j [((a\cdot\alpha)\otimes c)(b\otimes e_j) - (a\otimes(\alpha\cdot c))(b\otimes e_j) \\
&\quad +(a\otimes(\alpha\cdot c))(b\otimes e_j) - (a\otimes e_j)((\alpha\cdot b)\otimes c) \\
&\quad +(a\otimes e_j)(b\otimes(\alpha\cdot c)) - (a\otimes e_j)(b\otimes(\alpha\cdot c))] \\
&= \lim_j [((a\cdot\alpha)\otimes c - a\otimes(\alpha\cdot c))(b\otimes e_j) \\
&\quad +(a\otimes(\alpha\cdot c))(b\otimes e_j) - (a\otimes e_j)((\alpha\cdot b)\otimes c \\
&\quad -b\otimes(\alpha\cdot c)) - (a\otimes e_j)(b\otimes(\alpha\cdot c))] \\
&= \lim_j [((a\cdot\alpha)\otimes c - a\otimes(\alpha\cdot c))(b\otimes e_j) \\
&\quad +(ab\otimes(\alpha\cdot c)e_j) - (a\otimes e_j)(f(\alpha)b\otimes c \\
&\quad -b\otimes f(\alpha)c) - (ab\otimes e_j(\alpha\cdot c))] \\
&= \lim_j [((a\cdot\alpha)\otimes c - a\otimes(\alpha\cdot c))(b\otimes e_j) \\
&\quad +(ab\otimes(\alpha\cdot c)e_j) - (a\otimes e_j)f(\alpha)(b\otimes c - b\otimes c) \\
&\quad -(ab\otimes e_j(\alpha\cdot c))] \\
&= \lim_j [((a\cdot\alpha)\otimes c - a\otimes(\alpha\cdot c))(b\otimes e_j) \in I.
\end{aligned}$$

Similarly, $c\otimes[(a\cdot\alpha)b - a(\alpha\cdot b)] \in I$. Hence, ϕ is well-defined. Also, ϕ is a module homomorphism. It is easily verified that $\{\phi(d_i)\}$ is a bounded symmetric net in $\mathcal{A}\widehat{\otimes}_{\mathfrak{A}}\mathcal{A}$. Put $\widetilde{u}_i = \phi(d_i) = \sum\limits_k a_k^i \otimes b_k^i + I$. By [11, Proposition 2.2], we have

$$\begin{aligned}
\lim_i (a+J)\cdot\widetilde{w}_{\mathcal{A}}(\widetilde{u}_i) &= \lim_i (a+J)\cdot\left(\sum_k a_k^i b_k^i + J\right) \\
&= \lim_i (a+J)\cdot\left(\sum_k (a_k^i + J)(b_k^i + J)\right) \\
&= \lim_i (a+J)\cdot w^{\circ}_{\mathcal{A}/J}(d_i) = a + J
\end{aligned}$$

for each $a \in A$. Also,

$$\begin{aligned}\lim_i(\widetilde{u}_i \cdot a - a \cdot \widetilde{u}_i) &= \lim_i \left(\sum_k (a_k^i \otimes b_k^i a - a a_k^i \otimes b_k^i) + I \right) \\ &= \phi \left(\lim_i \left(\sum_k (a_k^i + J) \otimes (b_k^i a + J) - (a a_k^i + J) \otimes (b_k^i + J) \right) \right) \\ &= \phi(\lim_i (a \cdot d_i - d_i \cdot a)) = 0.\end{aligned}$$

Thus, $\{\widetilde{u}_i\}$ is a module symmetric approximate diagonal for A. This shows that $\mathcal{A}$ is module symmetrically amenable. □

3. Application to semigroup algebras

By an inverse semigroup S we shall mean a discrete semigroup such that for any $s \in S$ there is a unique element $s^* \in S$ with $ss^*s = s$ and $s^*ss^* = s^*$. An element $e \in S$ is called an idempotent if $e^2 = e^* = e$. Here and subsequently, S will always denote an inverse semigroup with the set of idempotents E_S (or E), where the order of E is defined by

$$e \leq d \Leftrightarrow ed = e \quad (e, d \in E).$$

Since E is a (commutative) subsemigroup of S (see [8, Theorem V.1.2]) and a semilattice, the algebra $l^1(E)$ could be regarded as a commutative subalgebra of $l^1(S)$. Hence, $l^1(S)$ is a Banach algebra and a Banach $l^1(E)$-module with compatible actions [1]. We impose the following actions of $l^1(E)$ on $l^1(S)$:

$$\delta_e \cdot \delta_s = \delta_s, \quad \delta_s \cdot \delta_e = \delta_{se} = \delta_s * \delta_e \quad (e \in E, s \in S).$$

With these actions, we consider $l^1(S)$ as a Banach $l^1(E)$-module. In this case, the ideal J (see section 2) is the closed linear span of $\{\delta_{set} - \delta_{st} \mid e \in E, s, t \in S\}$.

We consider an equivalence relation on S as follows:

$$s \approx t \Longleftrightarrow \delta_s - \delta_t \in J \qquad (s, t \in S).$$

In this case the quotient $S/\approx$ is a discrete group (see [2] and [13]). In fact, $S/\approx$ is homomorphic to the maximal group homomorphic image $\mathcal{G}_S$ [12] of S [14]. In particular, S is amenable if and only if $S/\approx = \mathcal{G}_S$ is amenable [7, 12]. As in [16, Theorem 3.3], we may observe that $l^1(S)/J \cong l^1(\mathcal{G}_S)$. With the notations of the previous section, $l^1(S)/J$ is a commutative $l^1(E)$-bimodule with the following actions

$$\delta_e \cdot \delta_{[s]} = \delta_{[s]}, \ \delta_{[s]} \cdot \delta_e = \delta_{[se]} \quad (s \in S, e \in E),$$

where $[s]$ denotes the equivalence class of s in $\mathcal{G}_S$.

Suppose that $k \in \mathbb{N}$. If there exist $e \in E$ and $i, j \in \mathbb{N}$ such that

$$1 \leq i < j \leq k+1, \quad f_i e = f_i, \quad f_j e = f_j \quad (f_1, f_2, \ldots, f_{k+1} \in E),$$

then we say that E satisfies condition D_k [7]. In [7, Theorem 16], the authors proved that for any inverse semigroup S, $l^1(S)$ has a bounded approximate identity if and only if E satisfies condition D_k for some k.

Theorem 3.1. *Let S be an inverse semigroup with the set of idempotents E and $l^1(S)$ be a Banach $l^1(E)$-module with trivial left action. If E satisfies condition D_k for some k, then $l^1(S)$ is module symmetrically amenable if and only if S is amenable.*

Proof. Firstly, we assume that $l^1(S)$ is module symmetrically amenable. Then, it is module amenable. Now, Theorem 3.1 from [1] necessities that S is amenable.

Conversely, suppose that S is amenable. Then, the (discrete) group $\mathcal{G}_S$ is amenable by [7, Theorem 1], and so $l^1(\mathcal{G}_S)$ is symmetrically amenable by [11, Theorem 4.1]. The result follows from Proposition 2.9 with $\mathcal{A} = l^1(S)$ and $\mathfrak{A} = l^1(E)$. $\square$

In the following we bring two examples to show that there are some module symmetrically amenable semigroup algebras which are not symmetrically amenable.

Example 3.2. Let G be a group with identity e, and let $\mathfrak{I}$ be a non-empty set. Then, the Brandt inverse semigroup corresponding to G and $\mathfrak{I}$, denoted by $S = \mathcal{M}(G, \mathfrak{I})$, is the collection of all $\mathfrak{I} \times \mathfrak{I}$ matrices $(g)_{ij}$ with $g \in G$ in the $(i,j)^{\text{th}}$ place and 0 (zero) elsewhere and the $\mathfrak{I}\times\mathfrak{I}$ zero matrix 0. Multiplication in S is given by the formula

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \qquad (g, h \in G,\ i, j, k, l \in \mathfrak{I}),$$

and $(g)^*_{ij} = (g^{-1})_{ji}$ and $0^* = 0$. The set of all idempotents is $E_S = \{(e)_{ii} : i \in \mathfrak{I}\} \bigcup \{0\}$. It is shown in [13, Example 3.2] that $\mathcal{G}_S$ is the trivial group, and so $l^1(S)$ is module symmetrically amenable by Theorem 3.1. Note that if G is not amenable or $\mathfrak{I}$ is not finite, then $l^1(S)$ is not amenable by Theorems 7 and 12 from [7] and hence it is not symmetrically amenable.

Example 3.3. Let $\mathcal{C}$ be the bicyclic inverse semigroup generated by p and q, that is

$$\mathcal{C} = \{p^a q^b : a, b \geq 0\}, \ \ (p^a q^b)^* = p^b q^a.$$

The multiplication operation is defined by

$$(p^a q^b)(p^c q^d) = p^{a-b+max\{b,c\}} q^{d-c+max\{b,c\}}.$$

The set of idempotents of $\mathcal{C}$ is $E_{\mathcal{C}} = \{p^a q^a : a = 0, 1, \dots\}$ which is also totally ordered with the following order

$$p^a q^b \leq p^b q^b \Longleftrightarrow a \leq b.$$

Therefore, E satisfies condition D_1. It is shown in [2] that $\mathcal{G}_{\mathcal{C}}$ is isomorphic to the group of integers $\mathbb{Z}$, hence $l^1(\mathcal{C})$ is module symmetrically amenable by

Theorem 3.1. On the other hand, $l^1(\mathcal{C})$ is not symmetrically amenable since it is not amenable [7].

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