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ON A HYPERBOLIC KAEHLERIAN SPACE

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ABSTRACT. The object of the present paper is to study some curvature properties in a hyperbolic Kaehlerian manifold equipped with quarter-symmetric metric connection.

1. Introduction

Hyperbolic Kaehlerian manifold has been studied by different differential geometer through different approaches. Nevena Pušić [5] studied hyperbolic Kaehlerian space equipped with quarter-symmetric metric connection. In 1985, G. Ganchev and A. Borisov [3] discussed the isotropic sections and curvature properties of hyperbolic Kaehlerian manifolds. Nevena Pušić [6] discussed HB-parallel hyperbolic Kaehlerian spaces. In 2013, Arif Salimov and S. Turanli [1] has been discussed some curvature properties of anti-Kaehler-codazzi manifold. Recently, hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection has been studied by B.B. Chaturvedi and B.K. Gupta [2] in 2015. In the consequences of these studies, in this paper we have studied some curvature properties of a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection.

If F_i^h satisfies the relation

$$F_j^i\, F_i^h = \delta_j^h, \tag{1.1}$$

$$F_{i\,j} = -F_{j\,i}, \quad (F_{i\,j} = g_{j\,k}\, F_i^k), \tag{1.2}$$

and

$$F_{i,j}^h = 0, \tag{1.3}$$

then the manifold is called hyperbolic Kaehlerian (space) manifold.

Where F_i^h is a tensor field of type (1,1) and $F_{i,j}^h$ is a covariant derivative of F_i^h with respect to Riemannian connection.

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In 1975 S. Global [7] defined

Definition 1.1. A linear connection ∇ on a n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection if the torsion tensor T, defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \tag{1.4}$$

of the connection ∇, satisfies

$$T(X,Y) = \eta(X)\phi Y - \eta(Y)\phi X, \tag{1.5}$$

where η is a 1-form and ϕ is a tensor field of type (1,1) .

A quarter-symmetric connection ∇ is said to be a quarter-symmetric metric connection if the covariant derivative of metric g vanishes otherwise it is called a quarter-symmetric non-metric connection.

Yano and Imai [4] considered a quarter-symmetric metric connection ∇ and Riemannian connection D with coefficients Γ^h_{ij} and $\{^{\,h}_{ij}\}$ respectively. According to them if the torsion tensor T of the connection ∇ on (M^n, g), $(n > 2)$, satisfies

$$T^i_{jk} = p_j A^i_k - p_k A^i_j, \tag{1.6}$$

then the relation between the coefficients of quarter-symmetric metric connection ∇ and Riemannian connection D is given by

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} - p_k U^i_j + p_j V^i_k - p^i V_{jk}, \tag{1.7}$$

where

$$U_{ij} = \frac{1}{2}(A_{ij} - A_{ji}), \quad V_{ij} = \frac{1}{2}(A_{ij} + A_{ji}), \tag{1.8}$$

where $\nabla g = 0$ and p_i are the components of 1-form . Also A^i_j denotes the components of a tensor of type (1,1). U_{ij} and V_{ij} are covariant skew symmetric and symmetric tensors respectively.

Equation (1.8) implies

$$A_{ij} = U_{ij} + V_{ij}. \tag{1.9}$$

Nevena Pušić [6] found a relation between Γ^h_{ij} and $\{^{\,h}_{ij}\}$ by putting $V_{ij} = g_{ij}$ and $U_{ij} = F_{ij}$ in (1.7), given by

$$\Gamma^i_{jk} = \{^{\,i}_{jk}\} - p_k F^i_j + p_j \delta^i_k - p^i g_{jk}. \tag{1.10}$$

where $\omega^h = \omega_t g^{th}$, ω^h is a contravariant components of generating vector w_h.

Also, Nevena Pušić [6] found a relation between curvature tensor with respect to a quarter-symmetric metric connection ∇ and a Riemannian connection D given by

$$\begin{aligned}\overline{R}_{ijkh} =& R_{ijkh} - g_{ih}p_{kj} + g_{ik}\, p_{hj} - g_{jk}\, p_{hi} + g_{hj}\, p_{ki} \\ &+ p_j\, p_h\, F_{ik} + p_i\, p_k\, F_{jh} - p_j\, p_k\, F_{ih} - p_i\, p_h\, F_{jk},\end{aligned} \tag{1.11}$$

where

$$p_{jk} = \nabla_j p_k - p_j\, p_k + p_k\, q_j + \frac{1}{2} p_s\, p^s\, g_{jk}. \tag{1.12}$$

Taking covariant derivative of F_i^h with respect to quarter-symmetric metric connection ∇ and Riemannian connection D, we have

$$\nabla_k\, F_i^h = \partial_k\, F_i^h + F_i^r\, \Gamma_{rk}^h - F_r^h \Gamma_{ik}^r, \tag{1.13}$$

and

$$D_k\, F_i^h = \partial_k\, F_i^h + F_i^r\, \{{}^{\,h}_{rk}\} - F_r^h\, \{{}^{\,r}_{ik}\}. \tag{1.14}$$

Subtracting (1.14) from (1.13), we have

$$\nabla_k\, F_i^h - D_k\, F_i^h = F_i^r (\Gamma_{rk}^h - \{{}^{\,h}_{rk}\}) - F_r^h (\Gamma_{ik}^r - \{{}^{\,r}_{ik}\}). \tag{1.15}$$

Using (1.10) in (1.15), we have

$$\nabla_k\, F_i^h - D_k\, F_i^h = F_i^r(-p_k F_r^h + p_r \delta_k^h - p^h g_{rk}) - F_r^h(-p_k F_i^r + p_i \delta_k^r - p^r g_{ik}). \tag{1.16}$$

Using (1.1) and (1.2) in (1.16), we have

$$\nabla_k\, F_i^h = D_k\, F_i^h. \tag{1.17}$$

Again taking covariant derivative of (1.13) with respect to quarter-symmetric metric connection, we get

$$\begin{aligned} \nabla_j \nabla_k\, F_i^h =& \partial_j \partial_k\, F_i^h - \partial_r\, F_i^h \Gamma_{jk}^r - \partial_k\, F_r^h \Gamma_{ij}^r \\ &+ \partial_k\, F_i^r \Gamma_{rj}^h + (\partial_j\, F_i^r + F_i^m \Gamma_{mj}^r - F_m^r \Gamma_{ij}^m)\Gamma_{rk}^h \\ &+ F_i^r \nabla_j \Gamma_{rk}^h - (\partial_j\, F_r^h + F_r^m \Gamma_{mj}^h - F_m^h \Gamma_{rj}^m)\Gamma_{ik}^r - F_r^h \nabla_j \Gamma_{ik}^r. \end{aligned} \tag{1.18}$$

Interchanging j and k in equation (1.18), we get

$$\begin{aligned} \nabla_k \nabla_j\, F_i^h =& \partial_k \partial_j\, F_i^h - \partial_r\, F_i^h \Gamma_{jk}^r - \partial_j\, F_r^h \Gamma_{ik}^r \\ &+ \partial_j\, F_i^r \Gamma_{rk}^h + (\partial_k\, F_i^r + F_i^m \Gamma_{mk}^r - F_m^r \Gamma_{ik}^m)\Gamma_{rj}^h \\ &+ F_i^r \nabla_k \Gamma_{rj}^h - (\partial_k\, F_r^h + F_r^m \Gamma_{mk}^h - F_m^h \Gamma_{rk}^m)\Gamma_{ij}^r - F_r^h \nabla_k \Gamma_{ij}^r. \end{aligned} \tag{1.19}$$

Subtracting (1.18) from (1.19), we get

$$\begin{aligned} \nabla_k \nabla_j\, F_i^h - \nabla_j \nabla_k\, F_i^h =& F_i^m (\Gamma_{mk}^r \Gamma_{rj}^h - \Gamma_{mj}^r \Gamma_{rk}^h + \nabla_k \Gamma_{mj}^h - \nabla_j \Gamma_{mk}^h) \\ &- F_r^h (\Gamma_{mk}^r\, \Gamma_{ij}^m - \Gamma_{mj}^r\, \Gamma_{ik}^m + \nabla_j \Gamma_{ik}^r - \nabla_k \Gamma_{ij}^r). \end{aligned} \tag{1.20}$$

Equation (1.20) implies

$$\nabla_k \nabla_j\, F_i^h - \nabla_j \nabla_k\, F_i^h = \overline{R}_{kjm}^{\,h}\, F_i^m - \overline{R}_{kji}^{\,r}\, F_r^h. \tag{1.21}$$

2. Twin anti-Hermitian metric

A skew symmetric tensor ω defined by

$$\omega(Y,Z)=g(FY,Z), \tag{2.1}$$

is said to be a killing-yano tensor if

$$(D_X\omega)(Y,Z)+(D_Y\omega)(X,Z)=0. \tag{2.2}$$

The twin anti-Hermitian metric G defined by

$$G(Y,Z)=g(FY,Z), \tag{2.3}$$

where $G(Y,Z)=G(Z,Y)$.

Since G is symmetric but 2-form ω is not symmetric so the killing-yano equation (2.2) has no immediate meaning. Therefore, we can change the killing-yano equation by Codazzi equation

$$(D_X\,G)(Y,Z)-(D_Y\,G)(X,Z)=0. \tag{2.4}$$

Equation (2.4) equivalent to

$$(D_X\,F)Y-(D_Y\,F)X=0. \tag{2.5}$$

3. Curvature properties

We know that for Riemannian connection D

$$D_kD_j\,F_i^h-D_jD_k\,F_i^h=R^h_{kjm}\,F_i^m-R^r_{kji}\,F_r^h. \tag{3.1}$$

Now subtracting (1.21) from (3.1), we get

(3.2)
$$\begin{aligned}(D_kD_j\,F_i^h-\nabla_k\nabla_j\,F_i^h)-(D_jD_k\,F_i^h-\nabla_j\nabla_k\,F_i^h)=&R^h_{kjm}\,F_i^m-R^r_{kji}\,F_r^h\\&-\overline{R}^h_{kjm}\,F_i^m+\overline{R}^r_{kji}\,F_r^h.\end{aligned}$$

After contracting (3.2) by h and k and using (1.3) and (1.17), we get

$$(D_hD_j\,F_i^h-\nabla_h\nabla_j\,F_i^h)=S_{jm}\,F_i^m-R^r_{hji}\,F_r^h-\overline{S}_{jm}\,F_i^m+\overline{R}^r_{hji}\,F_r^h. \tag{3.3}$$

Equation (3.3) can be written as

$$(D_hD_j\,F_i^h-\nabla_h\nabla_j\,F_i^h)=S_{jm}\,F_i^m-R_{hjil}\,g^{rl}\,F_r^h-\overline{S}_{jm}\,F_i^m+\overline{R}_{hjil}\,g^{rl}\,F_r^h. \tag{3.4}$$

Using $g^{rl}\,F_r^h=G^{hl}$ in (3.4), we get

$$(D_hD_j\,F_i^h-\nabla_h\nabla_j\,F_i^h)=S_{jm}\,F_i^m-R_{hjil}\,G^{hl}-\overline{S}_{jm}\,F_i^m+\overline{R}_{hjil}\,G^{hl}. \tag{3.5}$$

In 2013, Arif Salimov and S. Turanli [6] defined

$$H_{ji}=R_{hjil}\,G^{hl}. \tag{3.6}$$

Now we are taking

$$\overline{H}_{ji}=\overline{R}_{hjil}\,G^{hl}, \tag{3.7}$$

using (3.6) and (3.7) in (3.5), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = S_{jm} F_i^m - H_{ji} - \overline{S}_{jm} F_i^m + \overline{H}_{ji}, \tag{3.8}$$

where S_{jm} and $\overline{S}_{jm}$ are Ricci tensors with respect to Riemannian connection and quarter-symmetric metric connection respectively and G^{hl} is twin anti-Hermitian metric.

We know that the curvature tensor of type (0,4) with respect to Riemannian connection D satisfies the following relations

$$(i) \quad R_{(hj)il} = 0 \quad \text{and} \quad (ii) \quad R_{hj(il)} = 0. \tag{3.9}$$

Now, equation (3.6) can be written as

$$H_{ji} = \frac{1}{2}(R_{hjil} + R_{ljih})\, G^{lh}. \tag{3.10}$$

Interchanging i and j in (3.10), we get

$$H_{ij} = \frac{1}{2}(R_{hijl} + R_{lijh})\, G^{lh}. \tag{3.11}$$

Subtracting (3.11) from (3.10), we have

$$H_{ji} - H_{ij} = \frac{1}{2}(R_{hjil} + R_{ljih} - R_{hijl} - R_{lijh})\, G^{lh} = 0. \tag{3.12}$$

Equation (3.12) implies that

$$H_{ji} = H_{ij}. \tag{3.13}$$

If we take

$$\begin{aligned} &(i) \quad p_{hi} F_i^h = p_{ih} F_j^h, \\ &(ii) \quad p_j F_{hi} = p_i F_{jh} \\ &(iii) \quad p_i^r g_{hj} = p_j^r g_{hj}, \end{aligned} \tag{3.14}$$

then from (1.11) and (3.7) we can say that $\overline{R}_{hjil}$ will be symmetric in first and last indices.

Therefore we can write

$$\overline{H}_{ji} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ljih})\, G^{lh}. \tag{3.15}$$

Interchanging i and j in (3.15), we get

$$\overline{H}_{ij} = \frac{1}{2}(\overline{R}_{hijl} + \overline{R}_{lijh})\, G^{lh}. \tag{3.16}$$

Subtracting (3.16) from (3.15), we get

$$\overline{H}_{ji} - \overline{H}_{ij} = \frac{1}{2}(\overline{R}_{hjil} + \overline{R}_{ljih} - \overline{R}_{hijl} - \overline{R}_{lijh})\, G^{lh} = 0, \tag{3.17}$$

equation (3.17) implies that

$$\overline{H}_{ji} = \overline{H}_{ij}. \tag{3.18}$$

Thus we conclude

Theorem 3.1. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection H_{ij} is symmetric with respect to quarter-symmetric metric connection ∇ if equation (3.14) holds.*

Now equation (3.8) can be written as

$$\begin{aligned}(3.19)\quad & D_h(D_j F_i^h - D_i F_j^h) - \nabla_h(\nabla_j F_i^h - \nabla_i F_j^h) = (S_{jm} F_i^m - H_{ji}) \\ & \qquad - (S_{im} F_j^m - H_{ij}) + (\overline{S}_{im} F_j^m - \overline{H}_{ij}) - (\overline{S}_{jm} F_i^m - \overline{H}_{ji}),\end{aligned}$$

using (1.17) and (2.5) in equation (3.19), we have

$$S_{jm} F_i^m - S_{im} F_j^m = \overline{S}_{jm} F_i^m - \overline{S}_{im} F_j^m. \tag{3.20}$$

Thus we conclude

Theorem 3.2. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection if the Ricci tensor is pure with respect to Riemannian connection then it is also pure with respect quarter-symmetric metric connection if the equation (3.14) holds.*

In 2013, Arif Salimov and S. Turanli [6] defined

$$S^*_{ji} = -H_{jr} F_i^r = -R_{hjrl} G^{lh} F_i^r, \tag{3.21}$$

where S^*_{jr} is *Ricci tensor with respect to Riemannian connection.
Now we are taking

$$\overline{S^*}_{ji} = -\overline{H}_{jr} F_i^r = -\overline{R}_{hjrl} G^{lh} F_i^r, \tag{3.22}$$

where $\overline{S^*}_{ji}$ is *Ricci tensor with respect to quarter-symmetric metric connection.
With the help of equation (1.1), equation (3.21) and (3.22) can be written as

$$S^*_{jr} F_i^r = -H_{ji} \quad \text{and} \quad \overline{S^*}_{jr} F_i^r = -\overline{H}_{ji}, \tag{3.23}$$

using equation (3.23) in (3.8), we have

$$(D_h D_j F_i^h - \nabla_h \nabla_j F_i^h) = (S_{jr} F_i^r + S^*_{jr} F_i^r) - (\overline{S}_{jm} F_i^m + \overline{S}^*_{jm} F_i^m), \tag{3.24}$$

from (3.24), if

$$D_h D_j F_i^h = \nabla_h \nabla_j F_i^h, \tag{3.25}$$

then, we have

$$S_{jr} F_i^r - \overline{S}_{jm} F_i^m = S^*_{jr} F_i^r - \overline{S}^*_{jm} F_i^m, \tag{3.26}$$

which implies $S_{jr} F_i^r = \overline{S}_{jm} F_i^m$, if only if $\overline{S}^*_{jm} F_i^m = S^*_{jr} F_i^r$.
Thus we conclude:

Theorem 3.3. *In a hyperbolic Kaehlerian manifold equipped with a quarter-symmetric metric connection the Ricci tensor with respect to Riemannian connection will be equal to Ricci tensor with respect to quarter-symmetric metrc connection if only if *Ricci tensor with respect to Riemannian connection be equal to *Ricci tensor with respect to quarter symmetric metric connection if equations (3.14) and (3.25) hold.*

4. Conclusions

In this paper we have found that in a hyperbolic Kaehlerian manifold the Ricci tensor is pure with respect to quarter-symmetric metric connection if only if it is pure with respect to Riemannian connection with some conditions.

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