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ON THE GROWTH OF SOLUTIONS OF SOME NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we investigate the growth of solutions to the non-homogeneous linear differential equation

$$f^{(k)} + A_{k-1}e^{b_{k-1}z}f^{(k-1)} + \cdots + A_1e^{b_1z}f' + A_0e^{b_0z}f = Fe^{az},$$

where $A_j(z) \not\equiv 0$ $(j = 0, 1, \ldots, k-1)$, $F(z) \not\equiv 0$ are entire functions and $a \neq 0$, $b_j \neq 0$ $(j = 0, 1, \ldots, k-1)$ are complex numbers.

1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory [7, 9, 14]. In addition, we will use $\lambda(f)$ and $\overline{\lambda}(f)$ to denote respectively the exponents of convergence of the zero-sequence and distinct zeros of a meromorphic function f, $\rho(f)$ to denote the order of growth of f. In order to estimate the growth of infinite order solutions, we recall the definition of the hyper-order.

Definition 1.1 ([14])**.** Let f be a meromorphic function. Then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the hyper-order $\rho_2(f)$ of $f(z)$ is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.

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Definition 1.2. [14] Let f be a meromorphic function. Then the hyper-exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z : |z| \leq r\}$. Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z : |z| \leq r\}$.

The complex oscillatory problems of the non-homogeneous linear differential equations are a very important aspect of the complex oscillation theory of differential equations, which has a large number of potential applications. For growth estimates of solutions of a non-homogeneous linear differential equation, in general there exist exceptional solutions that are not easy to discuss, see [9, Chapter 8].

In [11] Wang and Laine investigated the growth of solutions of some second order nonhomogeneous linear differential equation and obtained.

Theorem A ([11]). *Let $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $F(z)$ be entire functions with $\max\{\rho(A_j)\ (j = 0, 1), \rho(F)\} < 1$, and let a, b be complex constants that satisfy $ab \neq 0$ and $a \neq b$. Then every nontrivial solution f of the differential equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = F,$$

is of infinite order.

Remark 1.1. The result of Theorem A was also obtained by the author and El Farissi in [2] by another proof.

Later in [12] Wang and Laine extend the result of Theorem A to higher order non-homogeneous linear differential as follows.

Theorem B ([12]). *Suppose that $A_j(z) = h_j(z)e^{P_j(z)}$ $(j = 0, 1, \ldots, k-1)$ where $P_j(z) = a_{jn}z^n + \cdots + a_{j0}$ are polynomials with degree $n \geq 1$, $h_j(z) \not\equiv 0$ $(j = 0, 1, \ldots, k-1)$ are entire functions of order less than n, and that $H(z) \not\equiv 0$ is an entire function of order less than n. If a_{jn} $(j = 0, 1, \ldots, k-1)$ are distinct complex numbers, then every solution f of the differential equation*

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_1(z)f' + A_0(z)f = H(z)$$

is of infinite order.

Recently, the authors investigated the growth of some nonhomogeneous higher order linear differential equations and obtained the following result.

Theorem C ([3]). *Let $k \geq 2$ be an integer, $I_j \subset \mathbb{N}$ $(j = 0, 1, \ldots, k-1)$ be finite sets such that $I_j \cap I_m = \emptyset$ $(j \neq m)$ and $I = \bigcup_{j=0}^{k-1} I_j$. Suppose that $B_j = \sum_{i \in I_j} A_i e^{P_i(z)}$ $(j = 0, 1, \ldots, k-1)$, where $A_i(z) \not\equiv 0$ $(i \in I)$ are entire functions with $\max\{\rho(A_i), i \in I\} < n$, $P_i(z) = a_{in} z^n + \cdots + a_{i1} z + a_{i0}$ $(i \in I)$ are polynomials with degree $n \geq 1$ and that $F(z) \not\equiv 0$ is an entire function with $\rho(F) < n$. If a_{in} $(i \in I)$ are distinct complex numbers, then every solution f of the differential equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_l f^{(l)} + \cdots + B_1 f' + B_0 f = F$$

satisfies $\rho(f) = +\infty$.

The main purpose of this paper is to investigate the order and the hyper-order of growth to some higher order linear differential equations. In fact we will prove the following results.

Suppose that

$$\begin{aligned} I &= \{0, 1, 2, \ldots, k-1\}, \\ I_1 &= \{i \in I : c_i > 1\} \neq \emptyset, \\ I_2 &= \{i \in I : 0 < c_i < 1\} \neq \emptyset, \\ I_3 &= \{i \in I : c_i < 0\} \neq \emptyset, \\ I_4 &= \{i \in I : c_i = 1\} \neq \emptyset, \end{aligned}$$

where $I_1 \cup I_2 \cup I_3 \cup I_4 = I$ and c_i $(i \in I)$ are real numbers.

Theorem 1.1. *Let $A_j(z) \not\equiv 0$ $(j \in I)$, $F(z) \not\equiv 0$ be entire functions with $\max\{\rho(A_j)(j \in I), \rho(F)\} < 1$, $a \neq 0$ and $b_i \neq 0$ $(i \in I)$ be complex numbers such that $b_i = c_i a$ $(i \in I)$. Suppose that there is one $s \in I_1$ such that $c_s > c_j$ for all $j \in I_1 \setminus \{s\}$, suppose that there is one $l \in I_3$ such that $c_l < c_j$ for all $j \in I_3 \setminus \{l\}$, and suppose that $c_0 \neq 1$ and $c_0 \neq c_j$ for all $j \in I \setminus \{0\}$. Then every solution f of the differential equation*

$$f^{(k)} + A_{k-1} e^{b_{k-1} z} f^{(k-1)} + \cdots + A_1 e^{b_1 z} f' + A_0 e^{b_0 z} f = F e^{az} \tag{1.1}$$

has infinite order and the hyper-order satisfies $\rho_2(f) \leq 1$.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, suppose further that $\varphi(z) \not\equiv 0$ is an entire function with finite order. Then every solution f of (1.1) satisfies*

$$\lambda(f) = \overline{\lambda}(f) = \lambda(f - \varphi) = \overline{\lambda}(f - \varphi) = \rho(f) = \infty$$

and

$$\lambda_2(f) = \overline{\lambda}_2(f) = \lambda_2(f - \varphi) = \overline{\lambda}_2(f - \varphi) = \rho_2(f) \leq 1.$$

2. Auxiliary lemmas

Lemma 2.1 ([13]). *Suppose that $f_1(z), f_2(z), \ldots, f_n(z)$ $(n \geq 2)$ are meromorphic functions and $g_1(z), g_2(z), \ldots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv f_{n+1}$.

(ii) If $1 \leq j \leq n+1$ and $1 \leq k \leq n$, then the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2$, $1 \leq j \leq n+1$ and $1 \leq h < k \leq n$, then the order of f_j is less than the order of $e^{g_h - g_k}$.

Then $f_j(z) \equiv 0$ $(j = 1, 2, \ldots, n+1)$.

Lemma 2.2 ([4]). *Suppose that $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z) \not\equiv 0$ is an entire function with $\rho(A) < n$. Set $g(z) = A(z)e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there is a set $E_1 \subset [0, 2\pi)$ that has linear measure zero, such that for any $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, there is $R > 0$, such that for $|z| = r > R$, we have*

(i) If $\delta(P, \theta) > 0$, then

$$\exp\{(1-\varepsilon)\delta(P,\theta) r^n\} \leq \left|g\left(re^{i\theta}\right)\right| \leq \exp\{(1+\varepsilon)\delta(P,\theta) r^n\}.$$

(ii) If $\delta(P, \theta) < 0$, then

$$\exp\{(1+\varepsilon)\delta(P,\theta) r^n\} \leq \left|g\left(re^{i\theta}\right)\right| \leq \exp\{(1-\varepsilon)\delta(P,\theta) r^n\},$$

where $E_2 = \{\theta \in [0, 2\pi) : \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3 ([6]). *Let f be a transcendental meromorphic function of finite order ρ. Let $\varepsilon > 0$ be a constant, k and j be integers satisfying $k > j \geq 0$. Then the following two statements hold:*

(i) There exists a set $E_3 \subset (1, +\infty)$ which has finite logarithmic measure, such that for all z satisfying $|z| \notin E_3 \cup [0, 1]$, we have

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}. \tag{2.1}$$

(ii) There exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 0$ such that (2.1) holds for all z satisfying $\arg z = \theta$ and $|z| \geq R$.

Lemma 2.4 ([12]). *Let $f(z)$ be an entire function and suppose that*

$$G(z) := \frac{\log^+ \left|f^{(k)}(z)\right|}{|z|^\rho}$$

is unbounded on some ray $\arg z = \theta$ with constant $\rho > 0$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \ldots)$, where $r_n \to +\infty$, such *that* $G(z_n) \to \infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \leq \frac{1}{(k-j)!}(1+o(1))\, r_n^{k-j},\ j=0,1,\ldots,k-1$$

as $n \to +\infty$.

Lemma 2.5 ([12])**.** *Let* $f(z)$ *be an entire function with* $\rho(f)=\rho<+\infty$. *Suppose that there exists a set* $E_5 \subset [0,2\pi)$ *which has linear measure zero, such that* $\log^+\left|f\left(re^{i\theta}\right)\right| \leq Mr^\sigma$ *for any ray* $\arg z=\theta\in[0,2\pi)\setminus E_5$, *where* M *is a positive constant depending on* θ, *while* σ *is a positive constant independent of* θ. *Then* $\rho(f)\leq\sigma$.

Lemma 2.6 ([1, 5])**.** *Let* $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions.*

(i) *If* f *is a meromorphic solution of the equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=F \tag{2.2}$$

with $\rho(f)=+\infty$, *then* f *satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty.$$

(ii) *If* f *is a meromorphic solution of equation* (2.2) *with* $\rho(f)=+\infty$ *and* $\rho_2(f)=\rho$, *then*

$$\overline{\lambda}(f)=\lambda(f)=\rho(f)=+\infty,\ \overline{\lambda}_2(f)=\lambda_2(f)=\rho_2(f)=\rho.$$

By using Wiman–Valiron theory [8], we easily obtain the following result which we omit the proof.

Lemma 2.7. *Let* $B_0(z),\ldots,B_{k-1}(z)$, $H(z)$ *be entire functions of finite order. If* f *is a solution of the equation*

$$f^{(k)}+B_{k-1}(z)f^{(k-1)}+\cdots+B_1(z)f'+B_0(z)f=H,$$

then $\rho_2(f)\leq\max\{\rho(B_0),\ldots,\rho(B_{k-1}),\rho(H)\}$.

3. Proof of Theorem 1.1

Since $b_i=c_ia$ $(i\in I)$, then by (1.1) we get

$$\begin{aligned} e^{-az}f^{(k)}+\sum_{i\in I_1}A_ie^{(c_i-1)az}f^{(i)}+\sum_{i\in I_2}A_ie^{(c_i-1)az}f^{(i)}& \\ +\sum_{i\in I_3}A_ie^{(c_i-1)az}f^{(i)}+\sum_{i\in I_4}A_if^{(i)}&=F. \end{aligned}\tag{3.1}$$

First we prove that every solution f of (1.1) satisfies $\rho(f)\geq1$. We assume that $\rho(f)<1$. It is clear that $f\not\equiv0$. We can rewrite (3.1) in the form

$$f^{(k)}e^{-az}+\sum_{i\in\Gamma}B_ie^{(c_i-1)az}+A_0fe^{(c_0-1)az}=F-\sum_{i\in I_4}A_if^{(i)}, \tag{3.2}$$

where $\Gamma \subseteq I \setminus (I_4 \cup \{0\})$ such that $(c_i - 1)a$ $(i \in \Gamma)$ are distinct numbers and B_i $(i \in \Gamma)$ are entire functions with order less than 1. We can see that $-a$, $(c_0 - 1)\,a$, $(c_i - 1)\,a$ $(i \in \Gamma)$ are distinct numbers. Obviously, $\rho\left(f^{(k)}\right) < 1$ and $\rho\left(A_i f^{(i)}\right) < 1$ $(i \in I)$. Then by (3.2) and the Lemma 2.1, we have $A_0 f \equiv 0$. This is a contradiction. Hence, $\rho(f) \geq 1$. Therefore f is a transcendental solution of equation (1.1).

Now we prove that $\rho(f) = +\infty$. Suppose that $\rho(f) = \rho < +\infty$. Set $\alpha = \max\{\rho(A_j)\,(j = 0, 1, \ldots, k-1), \rho(F)\}$. Then $\alpha < 1$. For any given ε $(0 < 2\varepsilon < 1 - \alpha)$ and for sufficiently large r, we have

$$|F(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.3}$$

$$|A_i(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \; i \in I_4. \tag{3.4}$$

By Lemma 2.2, there exists a set $E \subset [0, 2\pi)$ of linear measure zero, such that whenever $\theta \in [0, 2\pi) \setminus E$, then $\delta(az, \theta) \neq 0$. By Lemma 2.3, there exists a set $E_4 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi) \setminus E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \geq R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq |z|^{2\rho}, \; 0 \leq i < j \leq k. \tag{3.5}$$

For any fixed $\theta \in [0, 2\pi) \setminus (E \cup E_4)$, set

$$\delta_s = \delta\left((c_s - 1)\,az, \theta\right), \; \delta_l = \delta\left((c_l - 1)\,az, \theta\right),$$
$$\delta_1 = \max\left\{\delta\left((c_i - 1)\,az, \theta\right) : i \in I_1 \setminus \{s\}\right\}$$

and

$$\delta_3 = \max\left\{\delta\left((c_i - 1)\,az, \theta\right) : i \in I_3 \setminus \{l\}\right\}.$$

Then $\delta_s \neq 0$, $\delta_l \neq 0$, $\delta_1 \neq 0$ and $\delta_3 \neq 0$. We now discuss two cases separately.

Case 1. $\delta(-az, \theta) > 0$. We know that $\delta\left((c_i - 1)\,az, \theta\right) = (1 - c_i)\,\delta(-az, \theta)$, hence:

If $i \in I_1$, then $\delta\left((c_i - 1)\,az, \theta\right) < 0$.

If $i \in I_2$, then $0 < \delta\left((c_i - 1)\,az, \theta\right) < \delta(-az, \theta)$.

If $i \in I_3 \setminus \{l\}$, then $0 < \delta(-az, \theta) < \delta\left((c_i - 1)\,az, \theta\right) \leq \delta_3 < \delta_l$.

By Lemma 2.2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_l - \delta_3}{\delta_l}, 1 - \alpha\right\}$, we obtain

$$\left|A_l e^{(c_l - 1)az}\right| \geq \exp\left\{(1 - \varepsilon)\,\delta_l r\right\}, \tag{3.6}$$

$$\left|e^{-az}\right| \leq \exp\left\{(1 + \varepsilon)\,\delta(-az, \theta)\,r\right\}, \tag{3.7}$$

$$\left|A_i e^{(c_i - 1)az}\right| \leq \exp\left\{(1 - \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} < 1, \; i \in I_1, \tag{3.8}$$

$$\begin{aligned} \left|A_i e^{(c_i - 1)az}\right| &\leq \exp\left\{(1 + \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} \\ &< \exp\left\{(1 + \varepsilon)\,\delta(-az, \theta)\,r\right\}, \; i \in I_2, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \left|A_i e^{(c_i - 1)az}\right| &\leq \exp\left\{(1 + \varepsilon)\,\delta\left((c_i - 1)\,az, \theta\right) r\right\} \\ &\leq \exp\left\{(1 + \varepsilon)\,\delta_3 r\right\}, \; i \in I_3 \setminus \{l\} \end{aligned} \tag{3.10}$$

for sufficiently large r. We now prove that $\log^+ \left|f^{(l)}(z)\right| / |z|^{\alpha+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+ \left|f^{(l)}(z)\right| / |z|^{\alpha+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+ \left|f^{(l)}(z_m)\right|}{r_m^{\alpha+\varepsilon}} \to +\infty, \tag{3.11}$$

$$\left|\frac{f^{(j)}(z_m)}{f^{(l)}(z_m)}\right| \leq \frac{1}{(l-j)!}(1+o(1))\, r_m^{l-j}, \ (j = 0, 1, \ldots, l-1), \tag{3.12}$$

for m is large enough. From (3.3) and (3.11), we get

$$\left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \to 0, \tag{3.13}$$

for m is large enough. From (3.1), we obtain

$$\begin{aligned} \left|A_l e^{(c_l-1)az}\right| \leq{}& \left|e^{-az}\right| \left|\frac{f^{(k)}}{f^{(l)}}\right| + \sum_{i\in I_1} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| \\ &+ \sum_{i\in I_2} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| + \sum_{i\in I_3\setminus\{l\}} \left|A_i e^{(c_i-1)az}\right| \left|\frac{f^{(i)}}{f^{(l)}}\right| \\ &+ \sum_{i\in I_4} |A_i| \left|\frac{f^{(i)}}{f^{(l)}}\right| + \left|\frac{F}{f^{(l)}}\right|. \end{aligned} \tag{3.14}$$

Substituting (3.4)–(3.10), (3.12) and (3.13) into (3.14), we have

$$\begin{aligned} \exp\{(1-\varepsilon)\delta_l r_m\} &\leq \left|A_l(z_m) e^{(c_l-1)az_m}\right| \\ &\leq \left|e^{-az_m}\right| \left|\frac{f^{(k)}(z_m)}{f^{(l)}(z_m)}\right| + \sum_{i\in I_1} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_2} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_3\setminus\{l\}} \left|A_i(z_m) e^{(c_i-1)az_m}\right| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| \\ &\quad + \sum_{i\in I_4} |A_i(z_m)| \left|\frac{f^{(i)}(z_m)}{f^{(l)}(z_m)}\right| + \left|\frac{F(z_m)}{f^{(l)}(z_m)}\right| \\ &\leq M_0 r_m^{M_1} \exp\{(1+\varepsilon)\delta_3 r_m\} \exp\left\{r_m^{\alpha+\varepsilon}\right\}, \end{aligned} \tag{3.15}$$

where $M_0 > 0$ and $M_1 > 0$ are some constants. By $0 < \varepsilon < \frac{\delta_l - \delta_3}{2\delta_l}$ and (3.15), we can get

$$\exp\left\{\frac{(\delta_l - \delta_3)^2}{2\delta_l} r_m\right\} \leq M_0 r_m^{M_1} \exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<1$. Therefore, $\log^{+}\left|f^{(l)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded and we have

$$\left|f^{(l)}(z)\right| \leq M\exp\left\{r^{\alpha+\varepsilon}\right\}$$

on the ray $\arg z=\theta$. By the same reasoning as in the proof of [10, Lemma 3.1], we immediately conclude that

$$\begin{aligned}|f(z)| &\leq \frac{1}{l!}\left(1+o\left(1\right)\right)r^{l}\left|f^{(l)}(z)\right| \leq \frac{1}{l!}\left(1+o\left(1\right)\right)Mr^{l}\exp\left\{r^{\alpha+\varepsilon}\right\}\\ &\leq M\exp\left\{r^{\alpha+2\varepsilon}\right\}\end{aligned}$$

on the ray $\arg z=\theta$.

Case 2. $\delta\left(-az,\theta\right)<0$. We know that $\delta\left(\left(c_{i}-1\right)az,\theta\right)=\left(1-c_{i}\right)\delta\left(-az,\theta\right)$, hence:

If $i\in I_{1}\setminus\{s\}$, then $0<\delta\left(\left(c_{i}-1\right)az,\theta\right)\leq\delta_{1}<\delta_{s}$.

If $i\in I_{2}\cup I_{3}$, then $\delta\left(\left(c_{i}-1\right)az,\theta\right)<0$.

By Lemma 2.2, for any given ε with $0<2\varepsilon<\min\left\{\frac{\delta_{s}-\delta_{1}}{\delta_{s}},1-\alpha\right\}$, we obtain

$$\left|A_{s}e^{(c_{s}-1)az}\right| \geq \exp\left\{\left(1-\varepsilon\right)\delta_{s}r\right\}, \tag{3.16}$$

$$\left|e^{-az}\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta\left(-az,\theta\right)r\right\}<1, \tag{3.17}$$

$$\begin{aligned}\left|A_{i}e^{(c_{i}-1)az}\right| &\leq \exp\left\{\left(1+\varepsilon\right)\delta\left(\left(c_{i}-1\right)az,\theta\right)r\right\}\\ &\leq \exp\left\{\left(1+\varepsilon\right)\delta_{1}r\right\},\ i\in I_{1}\setminus\{s\},\end{aligned} \tag{3.18}$$

$$\left|A_{i}e^{(c_{i}-1)az}\right| \leq \exp\left\{\left(1-\varepsilon\right)\delta\left(\left(c_{i}-1\right)az,\theta\right)r\right\}<1,\ i\in I_{2}\cup I_{3} \tag{3.19}$$

for sufficiently large r. We now prove that $\log^{+}\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded on the ray $\arg z=\theta$. We assume that $\log^{+}\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is unbounded on the ray $\arg z=\theta$. Then by Lemma 2.4, there is a sequence of points $z_{m}=r_{m}e^{i\theta}$, such that $r_{m}\to+\infty$, and that

$$\frac{\log^{+}\left|f^{(s)}\left(z_{m}\right)\right|}{r_{m}^{\alpha+\varepsilon}}\to+\infty, \tag{3.20}$$

$$\left|\frac{f^{(j)}\left(z_{m}\right)}{f^{(s)}\left(z_{m}\right)}\right| \leq \frac{1}{(s-j)!}\left(1+o\left(1\right)\right)r_{m}^{s-j},\ \left(j=0,1,\ldots,s-1\right), \tag{3.21}$$

for m is large enough. From (3.3) and (3.20) we get

$$\left|\frac{F\left(z_{m}\right)}{f^{(s)}\left(z_{m}\right)}\right|\to 0, \tag{3.22}$$

for m is large enough. From (3.1) we obtain

$$\begin{aligned}&\left|A_{s}e^{(c_{s}-1)az}\right|\\ &\leq \left|e^{-az}\right|\left|\frac{f^{(k)}}{f^{(s)}}\right|+\sum_{i\in I_{1}\setminus\{s\}}\left|A_{i}e^{(c_{i}-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\sum_{i\in I_{2}}\left|A_{i}e^{(c_{i}-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|\end{aligned} \tag{3.23}$$

$$+\sum_{i\in I_3}\left|A_i e^{(c_i-1)az}\right|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\sum_{i\in I_4}|A_i|\left|\frac{f^{(i)}}{f^{(s)}}\right|+\left|\frac{F}{f^{(s)}}\right|.$$

Substituting (3.4), (3.5), (3.16)–(3.19), (3.21) and (3.22) into (3.23), we have

$$\exp\left\{(1-\varepsilon)\,\delta_s r_m\right\} \le \left|A_s\left(z_m\right) e^{(c_s-1)az_m}\right| \tag{3.24}$$

$$\le \left|e^{-az_m}\right|\left|\frac{f^{(k)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|+\sum_{i\in I_1\setminus\{s\}}\left|A_i\left(z_m\right)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|$$

$$+\sum_{i\in I_2}\left|A_i\left(z_m\right)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|$$

$$+\sum_{i\in I_3}\left|A_i\left(z_m\right)e^{(c_i-1)az_m}\right|\left|\frac{f^{(i)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|$$

$$+\sum_{i\in I_4}\left|A_i\left(z_m\right)\right|\left|\frac{f^{(i)}\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|+\left|\frac{F\left(z_m\right)}{f^{(s)}\left(z_m\right)}\right|$$

$$\le M_2 r_m^{M_3}\exp\left\{(1+\varepsilon)\,\delta_1 r_m\right\}\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

where $M_2>0$ and $M_3>0$ are some constants. By $0<\varepsilon<\frac{\delta_s-\delta_1}{2\delta_s}$ and (3.24), we get

$$\exp\left\{\frac{\left(\delta_s-\delta_1\right)^2}{2\delta_s}r_m\right\}\le M_2 r_m^{M_3}\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<1$. Therefore, $\log^+\left|f^{(s)}(z)\right|/|z|^{\alpha+\varepsilon}$ is bounded and we have

$$\left|f^{(s)}(z)\right|\le M\exp\left\{r^{\alpha+\varepsilon}\right\}$$

on the ray $\arg z=\theta$. This implies, as in Case 1, that

$$|f(z)|\le M\exp\left\{r^{\alpha+2\varepsilon}\right\}. \tag{3.25}$$

Therefore, for any given $\theta\in[0,2\pi)\setminus(E\cup E_4)$, we have got (3.25) on the ray $\arg z=\theta$, provided that r is large enough. Then by Lemma 2.5, we have $\rho(f)\le\alpha+2\varepsilon<1$, which is a contradiction. Hence every transcendental solution f of (1.1) must be of infinite order. Since

$$\max\left\{\rho\left(A_j e^{b_j z}\right)\ (j=0,\ldots,k-1),\ \rho\left(Fe^{az}\right)\right\}=1,$$

then by Lemma 2.7 we have $\rho_2(f)\le 1$.

4. Proof of Theorem 1.2

Suppose that f is a solution of equation (1.1). Then, by Theorem 1.1 we have $\rho(f)=+\infty$ and $\rho_2(f)\le 1$. Set $g(z)=f(z)-\varphi(z)$. Then $g(z)$ is an entire function with $\rho(g)=\rho(f)=+\infty$ and $\rho_2(g)=\rho_2(f)\le 1$. Substituting $f=g+\varphi$ into (1.1) , we have

$$g^{(k)}+A_{k-1}e^{b_{k-1}z}g^{(k-1)}+\cdots+A_1e^{b_1z}g'+A_0e^{b_0z}g=D, \tag{4.1}$$

where

$$D = Fe^{az} - \left[\varphi^{(k)} + A_{k-1}e^{b_{k-1}z}\varphi^{(k-1)} + \cdots + A_1e^{b_1z}\varphi' + A_0e^{b_0z}\varphi\right].$$

We prove that $D \not\equiv 0$. In fact, if $D \equiv 0$, then

$$\varphi^{(k)} + A_{k-1}e^{b_{k-1}z}\varphi^{(k-1)} + \cdots + A_1e^{b_1z}\varphi' + A_0e^{b_0z}\varphi = Fe^{az}.$$

Hence $\rho(\varphi) = +\infty$, which is a contradiction. Therefore $D \not\equiv 0$. We know that the functions $A_je^{b_jz}$ $(j = 0, 1, \ldots, k-1)$, D are of finite order. By Lemma 2.6 and (4.1) we have

$$\overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f-\varphi) = \rho(f) = \infty,$$

$$\overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f-\varphi) = \rho_2(f) \leq 1.$$

Then, by f is infinite order solution of equation (1.1) and Lemma 2.6 we obtain

$$\lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f-\varphi) = \lambda(f-\varphi) = \rho(f) = \infty$$

$$\lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f-\varphi) = \lambda_2(f-\varphi) = \rho_2(f) \leq 1.$$

which completes the proof.

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