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THE CONNECTED VERTEX MONOPHONIC NUMBER OF A GRAPH

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Abstract. For a connected graph G of order $p \geq 2$ and a vertex x of G, a set $S \subseteq V(G)$ is an *x-monophonic set* of G if each vertex $v \in V(G)$ lies on an $x - y$ monophonic path for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$. A *connected x-monophonic set* of G is an x-monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-monophonic set of G is defined as the *connected x-monophonic number* of G and is denoted by $\mathrm{cm}_x(G)$. We determine bounds for it and find the same for some special classes of graphs. If p, a and b are positive integers such that $2 \leq a \leq b \leq p-1$, then there exists a connected graph G of order p, $m_x(G) = a$ and $\mathrm{cm}_x(G) = b$ for some vertex x in G. Also, if p, d_m and n are positive integers such that $2 \leq d_m \leq p-2$ and $1 \leq n \leq p$, then there exists a connected graph G of order p, monophonic diameter d_m and $\mathrm{cm}_x(G) = n$ for some vertex x in G.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges of order at least 2. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology we refer to [1, 4]. For vertices x and y in a connected graph G, the *distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an *$x - y$ geodesic*. It is known that d is a metric on the vertex set V of G. The *neighbourhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. The *closed neighbourhood* of a vertex v is the set $N[v] = N(v) \bigcup \{v\}$. A vertex v is a *simplicial vertex* if the subgraph induced by its neighbors is complete. The *closed interval* $I[x, y]$ consists of all vertices lying on some $x - y$ geodesic of G, while for $S \subseteq V$, $I[S] = \bigcup\limits_{x,y \in S} I[x, y]$. A set S of vertices is a *geodetic set* if $I[S] = V$, and the minimum cardinality of a

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geodetic set is the *geodetic number* $g(G)$. A geodetic set of cardinality $g(G)$ is called a *g-set*. The geodetic number of a graph was introduced in [1, 5] and further studied in [2, 3].

The concept of vertex geodomination number was introduced in [6] and further studied in [8]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-geodominating set* of G if each vertex v of G lies on an $x-y$ geodesic in G for some element y in S. The minimum cardinality of an *x-geodominating set* of G is defined as the *x-geodomination number* of G and is denoted by $g_x(G)$. An x-geodominating set of cardinality $g_x(G)$ is called a g_x-set.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. The *closed interval* $I_m[x, y]$ consists of all vertices lying on some $x-y$ monophonic path of G. For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophoic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $\text{rad}_m(G)$ of G is $\text{rad}_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $\text{diam}_m(G)$ of G is $\text{diam}_m(G) = \max\{e_m(v) : v \in V(G)\}$. For any vertex x in G, a vertex y in G is said to be an *x-monophonic superior vertex* if for any vertex z with $d_m(x, y) < d_m(x, z)$, z lies on an $x-y$ monophonic path. The monophonic distance was introduced and studied in [7].

The concept of vertex monophonic number was introduced in [9]. Let x be a vertex of a connected graph G. A set S of vertices of G is an *x-monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path for some element y in S. The minimum cardinality of an x-monophonic set of G is defined as the *x-monophonic number* of G, denoted by $m_x(G)$.

The following theorems will be used in the sequel.

Theorem 1.1 ([4])**.** *Let v be a vertex of a connected graph G. The following statements are equivalent:*

(i) *v is a cut vertex of G.*
(ii) *There exists u and w distinct from v such that v is on every $u-w$ path.*
(iii) *There exists a partition of the set of vertices $V - \{v\}$ into subsets U and W such that for any vertices $u \in U$ and $w \in W$, the vertex v is on every $u - w$ path*

Theorem 1.2 ([9])**.** *For a vertex x in a graph G, $m_x(G) = 1$ if and only if there exists an x-monophonic superior vertex y in G such that every vertex of G is on an $x-y$ monophonic path.*

Theorem 1.3 ([9])**.** *Let x be a vertex of a connected graph G.*

(i) *Every simplicial vertex of G other than the vertex x(whether x is simplicial vertex or not) belongs to every m_x-set.*
(ii) *No cut vertex of G belongs to any m_x-set.*

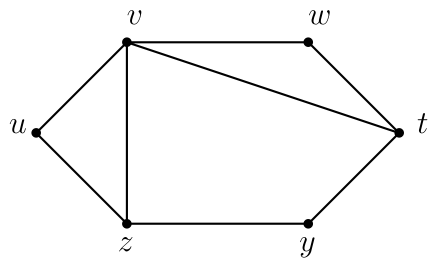

FIGURE 2.1. G

Theorem 1.4 ([9]). *Let T be a tree with t end-vertices. Then $m_x(T) = t - 1$ or t according as x is an end-vertex or not.*

Throughout this paper G denotes a connected graph with at least two vertices.

2. Connected Vertex Monophonic Number

Definition 2.1. Let x be any vertex of a connected graph G. A connected x-monophoninic set of G is an x-monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected x-monophonic set of G is the connected x-monophonic number of G and is denoted by $\mathrm{cm}_x(G)$. A connected x-monophonic set of cardinality $\mathrm{cm}_x(G)$ is called a cm_x-set of G.

Example 2.2. For the graph G given in Figure 2.1, the minimum vertex monophonic sets, the vertex monophonic numbers, the minimum connected vertex monophonic sets and the connected vertex monophonic numbers are given in Table 2.1.

We observe that in the case of connected x-monophonic sets, there can be more than one minimum connected x-monophonic set. For the vertex v of the graph G in Figure 2.1, $\{u, z, y, t, w\}, \{u, v, w, t, y\}$ and $\{y, z, u, v, w\}$ are three distinct cm_v-sets of G. It is observed in [9] that x is not an element of any m_x-set of G, whereas x may belong to a cm_x-set of G. For the graph G given in Figure 2.1, the vertex v is an element of a cm_v-set.

Vertex x	m_x-sets	$m_x(G)$	cm_x-sets	$\mathrm{cm}_x(G)$
u	$\{w\}$	1	$\{w\}$	1
v	$\{u, w, y\}$	3	$\{u, z, y, t, w\}, \{u, v, w, t, y\}, \{u, v, w, z, y\}$	5
w	$\{u\}$	1	$\{u\}$	1
z	$\{u, w\}$	2	$\{u, v, w\}$	3
y	$\{u, w\}$	2	$\{u, v, w\}$	3
t	$\{u, w\}$	2	$\{u, v, w\}$	3

Table 2.1

In the following theorem we establish the relationship between the vertex monophonic number and a connected vertex monophonic number of a graph G.

Theorem 2.3. *For any vertex x in G, $m_x(G) \leq \mathrm{cm}_x(G)$.*

Proof. Since every connected x-monophonic set is also an x-monophonic set, it follows that $m_x(G) \leq \mathrm{cm}_x(G)$. □

Theorem 2.4. *If $y \neq x$ be a simplicial vertex, then y belongs to every connected x-monophonic set of G.*

Proof. Let S_x be an x-monophonic set of G. Suppose that y does not belong to S_x. Then y is an internal vertex of an $x - u$ monophonic path, say P, for some $u \in S_x$. Let v and w be the neighbours of y on P. Then v and w are not adjacent and so y is not a simplicial vertex, which is a contradiction. Thus y belongs to every x-monophonic set of G. Since every connected x-monophonic set is an x-monophonic set, y belongs to every connected x-monophonic set of G. □

Theorem 2.5.
(i) *For the complete graph K_p, $\mathrm{cm}_x(K_p) = p - 1$ for any vertex x in K_p.*
(ii) *For any vertex x in a cycle $C_p (p \geq 4)$, $\mathrm{cm}_x(C_p) = 1$*
(iii) *For the wheel $W_p = K_1 + C_{p-1} (p \geq 5)$, $\mathrm{cm}_x(W_p) = p - 1$ or 1 according as x is K_1 or x is in C_{p-1}.*

Proof. (i) For any vertex x in K_p, let $S = V(K_p) - \{x\}$. Since every vertex of K_p is a simplicial vertex, it follows from Theorem 2.4 that $\mathrm{cm}_x(K_p) \geq |S| = p - 1$. It is clear that S is a connected x-monophonic set of G and so $\mathrm{cm}_x(K_p) = p - 1$.

(ii) Let C_p be a cycle. For any vertex x in C_p, let y be a non-adjacent vertex of x. Clearly every vertex of C_p lies on an $x - y$ monophonic path and so $\{y\}$ is a connected x-monophonic set of C_p so that $\mathrm{cm}_x(C_p) = 1$.

(iii) Let x be the vertex of K_1. Clearly $S = V(C_{p-1})$ is the minimum x-monophonic set of W_p. Since the induced subgraph $G[S]$ is connected, $\mathrm{cm}_x(W_p) = p - 1$.

Let $C_{p-1} : u_1, u_2, \ldots, u_{p-1}, u_1$ be the cycle in W_p. Let x be any vertex in C_{p-1}. Let y be a non-adjacent vertex of x in W_p. Then any vertex v in W_p lies on an $x - y$ monophonic path and so $\{y\}$ is a connected x-monophonic set of W_p. Thus $\mathrm{cm}_x(W_p) = 1$ □

Theorem 2.6. *Let $K_{m,n}(2 \leq m \leq n)$ be the complete bipartite graph with bipartition (V_1, V_2). Then*

(i) $\mathrm{cm}_x(K_{2,2}) = 1$ *for any vertex x*

(ii) $\mathrm{cm}_x(K_{2,n}) = \begin{cases} 1, & \text{if } x \in V_1 \\ n, & \text{if } x \in V_2 \text{ for } n \geq 3 \end{cases}$

(iii) $\mathrm{cm}_x(K_{m,n}) = \begin{cases} m, & \text{if } x \in V_1 \\ n, & \text{if } x \in V_2 \text{ for } m, n \geq 3 \end{cases}$

Proof. (i) By Theorem 2.5(ii), $\mathrm{cm}_x(K_{2,2}) = 1$ for any vertex x in $K_{2,2}$.

(ii) Let $x \in V_1$ be any vertex. Let y be the other vertex of V_1. Then any vertex v of V_2 lies on an $x - y$ monophonic path x,v,y and so $\{y\}$ is a connected x-monophonic set of $K_{2,n}$. Thus $\mathrm{cm}_x(K_{2,n}) = 1$.

Let $x \in V_2$ be any vertex. Since every element of V_1 is adjacent to x, no element of V_2 is an internal vertex of any monophonic path starting from x. Thus every x-monophonic set of G contains $S = V_2 - \{x\}$. Also, any vertex v of V_1 lies on an $x - u$ monophonic path x,v,u where $u \in S$ so that S is an x-monophonic set of $K_{2,n}$. Since $n \geq 3$, the induced subgraph $G[S]$ is disconnected so that $\mathrm{cm}_x(K_{2,n}) > n - 1$. Now, the induced subgraph $G[S \bigcup \{w\}]$ is connected for any vertex w in V_1 and so $\mathrm{cm}_x(K_{2,n}) = n$.

(iii) The proof is similar to the second part of the proof of (ii). □

Theorem 2.7. (i) *If T is any tree of order p, then $\mathrm{cm}_x(T) = p$ for any cut-vertex x of T.*

(ii) *If T is any tree of order p which is not a path, then for an end vertex x, $\mathrm{cm}_x(T) = p - d_m(x, y)$, where y is the vertex of T with $\deg(y) \geq 3$ such that $d_m(x, y)$ is minimum.*

(iii) *If T is a path, then $\mathrm{cm}_x(T) = 1$ for any end vertex x of T.*

Proof. (i) Let x be a cut vertex of T and let S be any connected x-monophonic set of T. By Theorem 2.4, every connected x-monophonic set of T contains all simplicial vertices. If $S \neq V(T)$, there exists a cut vertex v of T such that $v \notin S$. Let u and w be two end vertices belonging to different components of $T - \{v\}$. Since v lies on the unique path (monophonic path) joining u and w, it follows that the subgraph $G[S]$ induced by S is disconnected, which is a contradiction. Hence $\mathrm{cm}_x(T) = p$.

(ii) Let T be a tree which is not a path and x an end vertex of T. Let $S = (V(T) - I_m[x, y]) \bigcup \{y\}$. Clearly, S is a connected x-monophonic set of T and so $\mathrm{cm}_x(T) \leq |S| = p - d_m(x, y)$. We claim that $\mathrm{cm}_x(T) = p - d_m(x, y)$. Otherwise, there is a connected x-monophonic set M of T with $|M| < p - d_m(x, y)$. By Theorem 2.4, every connected x-monophonic set of T contains all simplicial vertices except possibly x and hence there exists a cut-vertex v of T such that

$v \in S$ and $v \notin M$. Let $B_1, B_2, \ldots, B_m (m \geq 3)$ be the components of $T - \{y\}$. Assume that x belongs to B_1.

Case 1. Suppose that $v = y$. Let $z \in B_2$ and $w \in B_3$ be two end vertices of T. By Theorem 1.1, v lies on the unique $z - w$ monophonic path. Since z and w belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

Case 2. Suppose that $v \neq y$. Let $v \in B_i (i \neq 1)$. Now, choose an end vertex $u \in B_i$ such that v lies on the $y - u$ monophonic path. Let $a \in B_j (j \neq i, 1)$ be an end vertex of T. By Theorem 1.1, y lies on the $u - a$ monophonic path. Hence it follows that v lies on the $u - a$ monophonic path. Since u and a belong to M and $v \notin M$, $G[M]$ is not connected, which is a contradiction.

(iii) Let T be a path. For an end vertex x in T, let y be the other end vertex of T. Clearly every vertex of T lies on the $x - y$ monophonic path and so $\{y\}$ is a connected x-monophonic set of T so that $\mathrm{cm}_x(T) = 1$. $\square$

Corollary 2.8. *For any tree T, $\mathrm{cm}_x(T) = p$ if and only if x is a cut vertex of T.*

Proof. This follows from Theorem 2.7. $\square$

Theorem 2.9. *For any vertex x in a connected graph G, $1 \leq \mathrm{cm}_x(G) \leq p$.*

Proof. Since $V(G)$ induces a connected x-monophonic set of G, it follows that $\mathrm{cm}_x(G) \leq p$. Also it is clear that $\mathrm{cm}_x(G) \geq 1$ and so $1 \leq \mathrm{cm}_x(G) \leq p$. $\square$

Remark 2.10. The bounds for $\mathrm{cm}_x(G)$ in Theorem 2.9 are sharp. For the cycle $C_p (p \geq 4)$, $\mathrm{cm}_x(C_p) = 1$ for any vertex x. Also, for any non-trivial path P_n, $\mathrm{cm}_x(P_n) = 1$ for any end vertex x. For any path $P_n (n \geq 3)$, $\mathrm{cm}_x(P_n) = n$ for any cut vertex x.

Theorem 2.11. *Let x be any vertex of a connected graph G. Then the following are equivalent:*

(i) $\mathrm{cm}_x(G) = 1$.
(ii) $m_x(G) = 1$.
(iii) *There exists an x-monophonic superior vertex y in G such that every vertex of G is on an $x - y$ monophonic path.*

Proof. *(i)* $\Rightarrow$ *(ii)* Let $\mathrm{cm}_x(G) = 1$. By Theorem 2.3, $m_x(G) \leq \mathrm{cm}_x(G) = 1$ and so $m_x(G) = 1$.

(ii) $\Rightarrow$ *(iii)* This follows from Theorem 1.2.

(iii) $\Rightarrow$ *(i)* Let y be an x-monophonic superior vertex of x in G such that every vertex of G is on an $x - y$ monophonic path. Then $\{y\}$ is a connected x-monophonic set of G so that $\mathrm{cm}_x(G) = 1$. $\square$

We proved (Theorem 2.3) that $m_x(G) \leq \mathrm{cm}_x(G)$ for any vertex x in G. The following theorem gives a realization for these parameters when $2 \leq a \leq b \leq p - 1$.

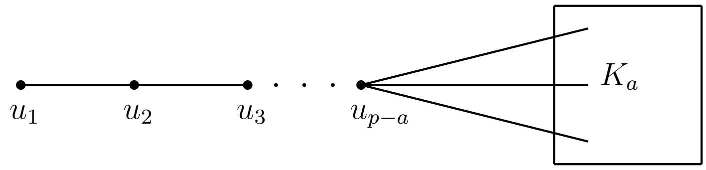

FIGURE 2.2. G

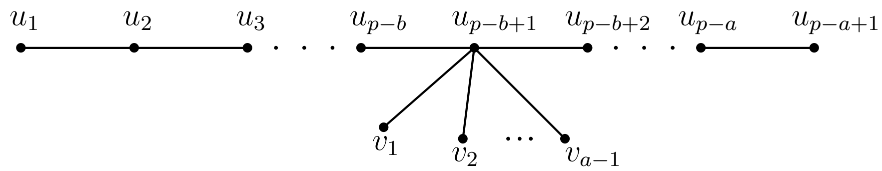

FIGURE 2.3. G

Theorem 2.12. *If p, a and b are positive integers such that $2 \leq a \leq b \leq p-1$, then there exists a connected graph G of order p, $m_x(G) = a$ and $\mathrm{cm}_x(G) = b$ for some vertex x in G.*

Proof. We prove this theorem by considering two cases.

Case 1. $2 \leq a = b \leq p-1$. Let $P_{p-a} : u_1, u_2, \ldots, u_{p-a}$ be a path of order $p-a$ and K_a be the complete graph of order a. Let G be the graph obtained by joining u_{p-a} to every vertex of K_a and it is shown in Figure 2.2.

Then G is of order p and it has $a+1$ simplicial vertices $\{u_1\} \bigcup V(K_a)$. By Theorem 1.3(i), every m_x-set of G contains $V(K_a)$ for $x = u_1$ and hence $m_x(G) \geq a$. Now, every vertex $u_i (1 \leq i \leq p-a)$ lies on the $x-v$ monophonic path for some $v \in V(K_a)$, it follows that $V(K_a)$ is an x-monophonic set of G and so $m_x(G) = a$. Also, since K_a is connected, $\mathrm{cm}_x(G) = a$.

Case 2. $2 \leq a < b \leq p-1$. Let $P_{p-a+1} : u_1, u_2, \ldots, u_{p-a+1}$ be a path of order $p-a+1$. Add $a-1$ new vertices $v_1, v_2, \ldots, v_{a-1}$ to P_{p-a+1} and join these to u_{p-b+1}, there by producing the tree G of Figure 2.3. Then G is of order p with $a+1$ end vertices. For the vertex $x = u_1$, $m_x(G) = a$ by Theorem 1.4 and $\mathrm{cm}_x(G) = b$ by Theorem 2.7 (ii). $\square$

In the following, we construct a graph of prescribed order, monophonic diameter and connected vertex monophonic number under some conditions.

Theorem 2.13. *If p, d_m and n are positive integers such that $2 \leq d_m \leq p-2$ and $1 \leq n \leq p$, then there exists a connected graph G of order p, monophonic diameter d_m and $\mathrm{cm}_x(G) = n$ for some vertex x in G.*

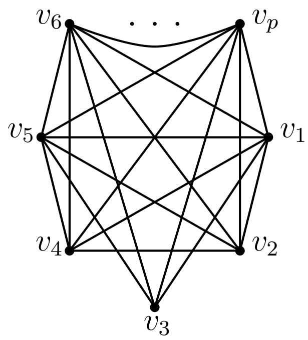

FIGURE 2.4. G

Proof. We prove this theorem by considering two cases.

Case 1. Let $d_m = 2$. If $n = p - 1$ or p, then take $G = K_{1,p-1}$. By Theorem 2.7, $\mathrm{cm}_x(G) = p-1$ or p according as x is an end vertex or the cut vertex. Now we consider three subcases. First, let $n = 1$. Let G be the complete bipartite graph $K_{2,p-2}$ with partite sets $X = \{u_1, u_2\}$ and $Y = \{w_1, w_2, \ldots, w_{p-2}\}$. Then G has order p and monophonic diameter $d_m = 2$. For the vertex $x = u_1$, $\mathrm{cm}_x(G) = 1$ by Theorem 2.6(ii). Let $n = 2$. Let $\{v_1, v_2, \ldots, v_p\}$ be the vertex set of the complete graph K_p. The graph G is obtained by removing the edges v_2v_3 and v_3v_4 from the complete graph K_p. The graph G has order p and monophonic diameter $d_m = 2$ and is shown in Figure 2.4. Let $S = \{v_2, v_3, v_4\}$ be the set of all simplicial vertices of G. Then by Theorem 2.4, every connected x-monophonic set of G contains $S - \{v_3\}$ for the vertex $x = v_3$. It is clear that S is a connected x-monophonic set of G and so $\mathrm{cm}_x(G) = 2$.

Now, let $3 \le n \le p - 2$. Let $K_{1,n}$ be a star with end vertices $u_1, u_2, \ldots, u_n$ and cut vertex y. Let G be the graph obtained from $K_{1,n}$ by adding $p - n - 1$ new vertices $w_1, w_2, \ldots, w_{p-n-1}$ and joining each $w_i (1 \le i \le p-n-1)$ to both u_1, u_2 and y. The graph G has order p and monophonic diameter $d_m = 2$ and is shown in Figure 2.5.

Let $S = \{u_3, u_4, \ldots, u_n\}$ be the set of all simplicial vertices of G. Then by Theorem 2.4, every connected x-monophonic set of G contains S for the vertex $x = u_1$. It is clear that S and $S \bigcup \{z\}$, where $z \in V(G) - S$, are not connected x-monophonic sets of G and so $\mathrm{cm}_x(G) > n - 1$. Clearly $S' = S \bigcup \{u_2, y\}$ is a minimum connected x-monophonic set of G and so $\mathrm{cm}_x(G) = n$.

Case 2. Let $3 \le d_m \le p - 2$. Let $P_{d_m+1} : u_0, u_1, \ldots, u_{d_m}$ be a path of length d_m.

Subcase 1. Let $n = 1$. Add $p - d_m - 1$ new vertices $w_1, w_2, \ldots, w_{p-d_m-1}$ to P_{d_m+1} and join these to both u_0 and u_2, there by producing the graph G of Figure 2.6. Then G has order p and monophonic diameter d_m. For the vertex

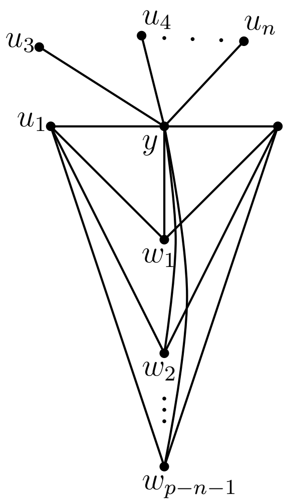

FIGURE 2.5. G

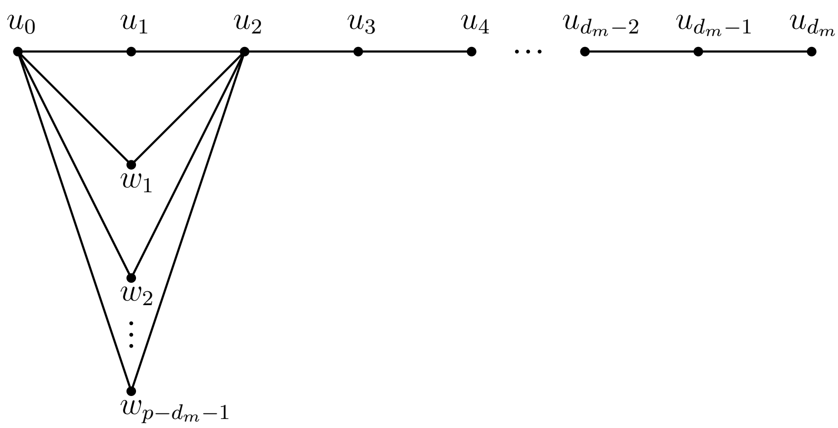

FIGURE 2.6. G

$x = u_0$, clearly $\{u_{d_m}\}$ is the minimum connected x-monophonic set of G so that $\mathrm{cm}_x(G) = 1$.

Subcase 2. Let $n = 2$. Add $p - d_m - 1$ new vertices $w_1, w_2, \ldots, w_{p-d_m-2}, v$ to P_{d_m+1} and join each $w_i (1 \leq i \leq p - d_m - 2)$ to both u_0 and u_2 and join v

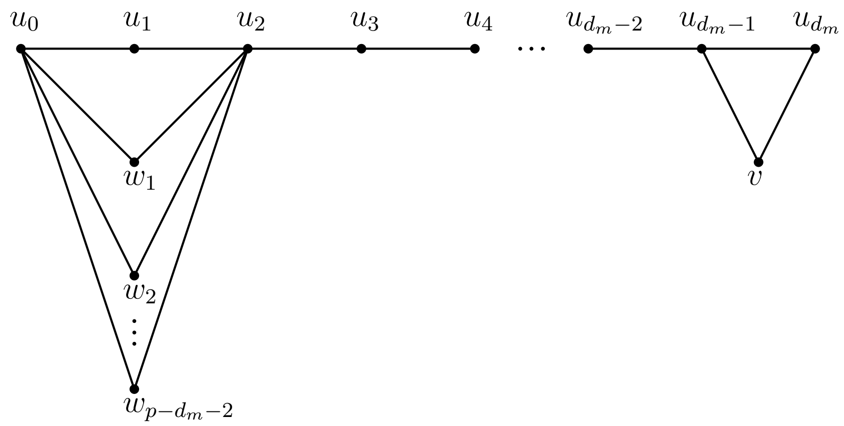

FIGURE 2.7. G

to both u_{d_m-1} and u_{d_m}, there by producing the graph G of Figure 2.7. Then G has order p and monophonic diameter d_m. For the vertex $x = u_0$, clearly $\{u_{d_m}, v\}$ is the cm_x-set so that $\mathrm{cm}_x(G) = 2$.

Subcase 3. Let $3 \leq n \leq p-1$. We consider two cases. If $n \leq p-d_m$, then add $p-d_m-1$ new vertices $w_1, w_2, \ldots, w_{p-d_m-n+1}, v_1, v_2, \ldots, v_{n-2}$ to P_{d_m+1} and join each $w_i (1 \leq i \leq p-d_m-n+1)$ to both u_0 and u_2 and join each $v_j (1 \leq j \leq n-2)$ to u_{d_m-1}, there by producing the graph G of Figure 2.8. Then G has order p and monophonic diameter d_m. Clearly $S = \{u_{d_m}, v_1, v_2, \ldots, v_{n-2}\}$ is the set of all simplicial vertices of G. Let $x = u_0$. By Theorem 2.4, $\mathrm{cm}_x(G) \geq |S| = n-1$. Since the induced subgraph $G[S]$ is not connected, $\mathrm{cm}_x(G) > |S| = n-1$. Let $S' = S \bigcup \{u_{d_m-1}\}$. Then S' is an x-monophonic set of G and $G[S']$ is connected so that $\mathrm{cm}_x(G) = |S'| = n$.

If $n > p - d_m$, then add $p - d_m - 1$ new vertices $v_1, v_2, \ldots, v_{p-d_m-1}$ to P_{d_m+1} and join each $v_i (1 \leq i \leq p - d_m - 1)$ to u_{p-n}, there by producing the graph G of Figure 2.9. Then G has order p and monophonic diameter d_m. Since G is a tree, by Theorem 2.7 (ii), $\mathrm{cm}_x(G) = p - (p - n) = n$ for the vertex $x = u_0$.

Subcase 4. Let $n = p$. Let G be any tree of order p and monophonic diameter d_m. Then for any cut vertex x in G, $\mathrm{cm}_x(G) = p$, by Theorem 2.7(i). $\square$

For every connected graph G, $\mathrm{rad}_m(G) \leq \mathrm{diam}_m(G)$. It is shown in [7] that every two positive integers a and b with $a \leq b$ are realizable as the monophonic radius and monophonic diameter, respectively, of some connected graph. This theorem can be extended so that the connected vertex monophonic number can also be prescribed.

Theorem 2.14. *For positive integers a, b and $n \geq 3$ with $a \leq b$, there exists a connected graph G with $\mathrm{rad}_m(G) = a$, $\mathrm{diam}_m(G) = b$ and $\mathrm{cm}_x(G) = n$ for some vertex x in G.*

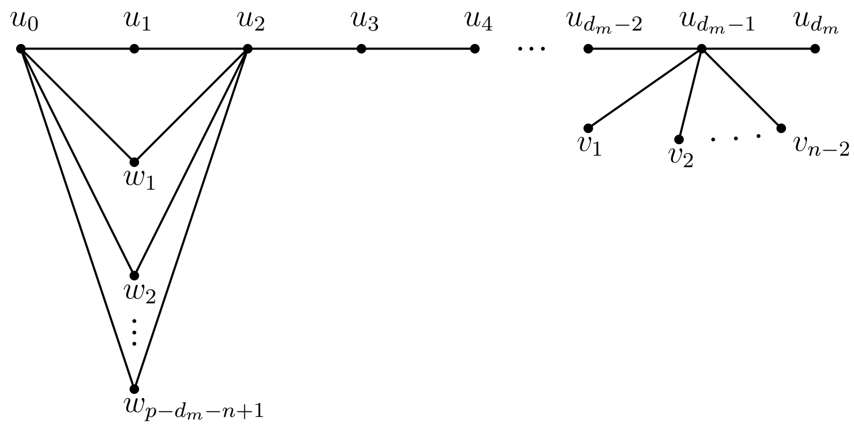

FIGURE 2.8. G

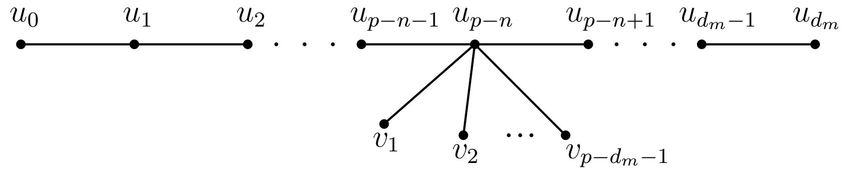

FIGURE 2.9. G

Proof. We prove this theorem by considering four cases.

Case 1. $a = b = 1$. Let $G = K_{n+1}$. Then by Theorem 2.5 (i), $\text{cm}_x(G) = n$ for any vertex x in G.

Case 2. $a = b \geq 2$. Let $C_{a+2} : v_1, v_2, \ldots, v_{a+2}, v_1$ be a cycle of order $a + 2$. Let G be the graph obtained from C_{a+2} by adding $n - 2$ new vertices $u_1, u_2, \ldots, u_{n-2}$ and joining each vertex $u_i (1 \leq i \leq n - 2)$ to both v_1 and v_3. The graph G is shown in Figure 2.10.

It is easily verified that the monophonic eccentricity of each vertex of G is a and so $\text{rad}_m(G) = \text{diam}_m(G) = a$. Also, for the vertex $x = v_2$, it is clear that $S = \{v_{a+2}, u_1, u_2, \ldots, u_{n-2}\}$ is a minimum x-monophonic set of G and so $m_x(G) = n - 1$. Since the induced subgraph $G[S]$ is disconnected and no $n - 1$ point subset of $V(G)$ is a connected x-monophonic set of G, we have $\text{cm}_x(G) > n - 1$. Let $S' = S \bigcup \{v_1\}$. Then S' is a connected x-monophonic set of G so that $\text{cm}_x(G) = n$.

Case 3. $1 \leq a < b$. Let $C_{b+1} : v_1, v_2, \ldots, v_{b+1}, v_1$ be a cycle of order $b+1$ and K_{n+2} be the complete graph of order $n+2$. Let G be the graph obtained from the cycle C_{b+1} and the complete graph K_{n+2} by identifying the edge v_1v_{b+1} of C_{b+1} with an edge of K_{n+2} and joining each vertex $v_i (a + 2 \leq i \leq b)$ to

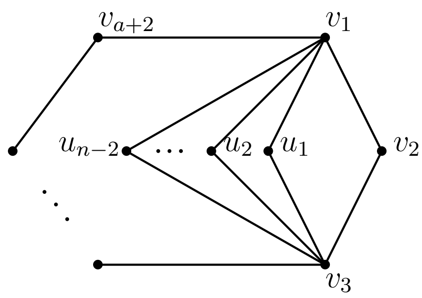

FIGURE 2.10. G

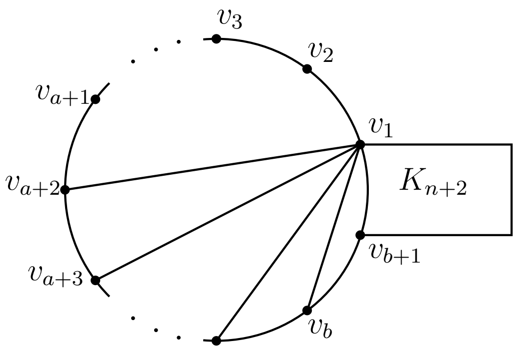

FIGURE 2.11. G

the vertex v_1. The graph G is shown in Figure 2.11. It is easily verified that $e_m(v_1) = a$, $e_m(v_2) = b$ and $a \leq e_m(x) \leq b$ for any vertex x in G. Then $\text{rad}_m(G) = a$ and $\text{diam}_m(G) = b$.

Subcase 1. Let $a = 1$. Then $S = (V(K_{n+2}) - \{v_1, v_{b+1}\}) \bigcup \{v_2\})$ is the set of all simplicial vertices of G. Then by Theorem 2.4, every connected x-monophonic set of G contains $S - \{v_2\}$ for the vertex $x = v_2$. Also, since $S - \{v_2\}$ is a connected x-monophonic set of G, $\text{cm}_x(G) = |S - \{v_2\}\,| = n$.

Subcase 2. Let $a \geq 2$. Then $S = V(K_{n+2}) - \{v_1, v_{b+1}\}$ is the set of all simplicial vertices of G. Then by Theorem 2.4, every connected x-monophonic set of G contains S for the vertex $x = v_2$. Also, since S is a connected x-monophonic set of G, $\text{cm}_x(G) = |S| = n$. $\square$

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