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SOME PROPERTIES OF THE INDICATRIX OF A COMPLEX FINSLER SPACE

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Dedicated to Professor Lajos Tamássy on the occasion of his 90th birthday

Abstract. Following the study of the indicatrix of a real Finsler space, in this paper there are investigated some properties of the complex indicatrix of a complex Finsler space, both in a fixed point and for the indicatrix bundle. In a fixed point $z_0 \in M$, the associated indicatrix $I_{z_0}M$ is a convex hypersurface in the holomorphic tangent space $T'_{z_0}M$ and it can be regarded as a locally Minkowski manifold. Using the submanifold equations, several properties of the indicatrix in a fixed point are obtained in terms of the fundamental function. In the global case, an almost contact structure is introduced on the indicatrix bundle and considering the Gauss-Weingarten equations with respect to the Chern-Finsler connection, a constant value of the mean curvature is determined.

1. Introduction

The study of the indicatrix of a real Finsler space is one of interest ([4, 6, 12, 2, 3, 7], etc.), mainly because it is a compact and strictly convex set surrounding the origin. For example, the indicatrix plays a special role in the definition of the volume of a Finsler space.

In the present paper, based on some ideas from the real case, the indicatrix bundle of a complex Finsler manifold (M, F) is introduced and several of its properties are obtained, both locally and globally.

Firstly, we recall some basic notions about complex Finsler geometry (in Section 1). Then, in Section 2 the notion of complex indicatrix in a fixed point z_0 will be introduced and using the submanifold equations ([10]), the relation

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between the locally coefficients of the second fundamental form and Weingarten operator will be given. Moreover, the properties of the hypersurfaces homothetic to the indicatrix will be obtained. Considering an almost contact metric structure introduced on the indicatrix bundle (as in [4]), we will be able in Section 3 to express the mean curvature of constant value in the global case.

Now, we will make a short overview of the concepts and terminology used in complex Finsler geometry, for more see [1, 8]. Let M be an $n+1$ dimensional complex manifold, and $z := (z^k),\ k = 1, \dots, n+1$, the complex coordinates on a local chart (U, φ). The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits into the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$, i.e. $T_{\mathbb{C}}M = T'M \oplus T''M$. The holomorphic tangent bundle $T'M$ is in its turn a $(2n+2)$-dimensional complex manifold and the local coordinates in a local chart in $u \in T'M$ are $u := (z^k, \eta^k),\ k = 1, \dots, n+1$.

Definition 1. A *complex Finsler space* is a pair (M, F), where $F : T'M \to \mathbb{R}^+,\ F = F(z, \eta)$ is a continuous function that satisfies the following conditions:

i. F is a smooth function on $\widetilde{T'M} := T'M \setminus \{0\}$;
ii. $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
iii. $F(z, \lambda\eta) = |\lambda| F(z, \eta),\ \forall \lambda \in \mathbb{C}$;
iv. the Hermitian matrix $\big(g_{i\bar{j}}(z, \eta)\big)$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$ is the fundamental metric tensor, with $L := F^2$.

By $L := F^2$ we denote the complex Lagrangian associated to the complex Finsler function F.

The fourth condition means that the indicatrix in a fixed point $I_z M = \{\eta \mid g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1\}$ is strongly pseudoconvex, for any $z \in M$.

Moreover, condition iii. says that L is homogeneous with respect to the complex norm $L(z, \lambda\eta) = \lambda\bar{\lambda} L(z, \eta),\ \forall \lambda \in \mathbb{C}$, and by applying Euler's formula we get that:

$$\frac{\partial L}{\partial \eta^k}\eta^k = \frac{\partial L}{\partial \bar{\eta}^k}\bar{\eta}^k = L; \quad \frac{\partial g_{i\bar{j}}}{\partial \eta^k}\eta^k = \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}\bar{\eta}^k = 0 \quad \text{and} \quad L = g_{i\bar{j}}\eta^i \bar{\eta}^j. \tag{1}$$

An immediate consequence of the above homogeneity conditions concerns the following Cartan complex tensors:

$$C_{i\bar{j}k} := \frac{\partial g_{i\bar{j}}}{\partial \eta^k} \quad \text{and} \quad C_{i\bar{j}\bar{k}} := \frac{\partial g_{i\bar{j}}}{\partial \bar{\eta}^k}.$$

They have the following properties:

$$C_{i\bar{j}k} = C_{k\bar{j}i};\ C_{i\bar{j}\bar{k}} = C_{i\bar{k}\bar{j}};\ C_{i\bar{j}k} = \overline{C_{j\bar{i}\bar{k}}} \tag{2}$$

and

$$C_{i\bar{j}k}\eta^k = C_{i\bar{j}\bar{k}}\bar{\eta}^j = C_{i\bar{j}k}\eta^i = C_{i\bar{j}\bar{k}}\bar{\eta}^k = 0 \tag{3}$$

The positivity of $(g_{i\bar{j}})$ from condition *iv.* ensures the existence of the inverse $(g^{\bar{j}i})$, with $g^{\bar{j}i} g_{i\bar{k}} = \delta^{\bar{j}}_{\bar{k}}$.

Roughly speaking, the geometry of a complex Finsler space consists of the study of the geometric objects of the complex manifold $T'M$ endowed with a Hermitian metric structure defined by $g_{i\bar{j}}$. Regarding this, the first step is the study of the sections of the complexified tangent bundle of $T'M$ which splits into the direct sum $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$, where $T''_u(T'M) = \overline{T'_u(T'M)}$. Let be $V(T'M) \subset T'(T'M)$ the vertical bundle, locally spanned by $\left\{\frac{\partial}{\partial \eta^k}\right\}$ and let $V(T''M)$ be its conjugate that contains $(0,1)-$ vector fields.

The idea of complex nonlinear connection, briefly (c.n.c.), is fundamental in "linearization" of this geometry ([8]). A (c.n.c.) is a supplementary complex subbundle to $V(T'M)$ in $T'(T'M)$, i.e. $T'(T'M) = H(T'M) \oplus V(T'M)$. The horizontal distribution $H_u(T'M)$ is locally spanned by $\left\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\right\}$, where $N_k^j(z,\eta)$ are the coefficients of the (c.n.c.). Then, we will call the adapted frame of the (c.n.c.) the pair $\left\{\delta_k := \frac{\delta}{\delta z^k},\ \dot{\partial}_k := \frac{\partial}{\partial \eta^k}\right\}$, which obey the change rules $\delta_k = \frac{\partial z'^j}{\partial z^k}\delta'_j$ and $\dot{\partial}_k = \frac{\partial z'^j}{\partial z^k}\dot{\partial}'_j$. By conjugation everywhere we get an adapted frame $\{\delta_{\bar{k}},\ \dot{\partial}_{\bar{k}}\}$ on $T''_u(T'M)$. The dual adapted bases are $\left\{\mathrm{d}z^k,\ \delta\eta^k := \mathrm{d}\eta^k + N_j^k \mathrm{d}z^j\right\}$, respectively $\{\mathrm{d}\bar{z}^k,\ \delta\bar{\eta}^k\}$, where $\delta\bar{\eta}^k = \mathrm{d}\bar{\eta}^k + N_{\bar{j}}^{\bar{k}}\mathrm{d}\bar{z}^j$.

Let us consider on $T'M$ the Hermitian metric structure G, named the Sasaki type lift of the metric tensor $g_{i\bar{j}}$, as

$$G = g_{i\bar{j}}\mathrm{d}z^i \otimes \mathrm{d}\bar{z}^k + g_{i\bar{j}}\delta\eta^i \otimes \delta\bar{\eta}^j. \tag{4}$$

One main problem of this geometry is to determine a (c.n.c.) related only by the fundamental function of a complex Finsler space (M, L); one almost classical now is the Chern-Finsler (c.n.c.) ([1],[8]), in brief C-F (c.n.c.):

$$N_j^k = g^{\bar{m}k}\frac{\partial g_{l\bar{m}}}{\partial z^j}\eta^l. \tag{5}$$

The next step is to specify the derivation law D on sections of $T_{\mathbb{C}}(T'M)$. A Hermitian connection D, of $(1,0)$-type, which satisfies $D_{JX}Y = JD_XY$, for all horizontal vectors X and J the natural complex structure $J(\delta_k) = \mathrm{i}\delta_k,\ J(\delta_{\bar{k}}) = -\mathrm{i}\delta_{\bar{k}},\ J(\dot{\partial}_k) = \mathrm{i}\dot{\partial}_k,\ J(\dot{\partial}_{\bar{k}}) = -\mathrm{i}\dot{\partial}_{\bar{k}}$, will be the *Chern-Finsler linear connection*, locally given by the next set of coefficients (notations from [8]):

$$L^i_{jk} = g^{\bar{l}i}\delta_k(g_{j\bar{l}}), \quad C^i_{jk} = g^{\bar{l}i}\dot{\partial}_k(g_{j\bar{l}}), \quad L^{\bar{i}}_{\bar{j}k} = 0, \quad C^{\bar{i}}_{\bar{j}k} = 0, \tag{6}$$

where $D_{\delta_k}\delta_j = L^i_{jk}\delta_i,\ D_{\delta_k}\dot{\partial}_j = L^i_{jk}\dot{\partial}_i,\ D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i,\ D_{\dot{\partial}_k}\delta_j = C^i_{jk}\delta_i$. Of course, there is also $\overline{D_XY} = D_{\bar{X}}\bar{Y}$. From the homogeneity conditions (1) it takes: $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$. Moreover, we have $L^i_{jk} = \dot{\partial}_j N^i_k$.

Further we will use the following notation $\bar{\eta}^j =: \eta^{\bar{j}}$ to note a conjugate object.

2. The geometry of complex indicatrix in a fixed point z_0

In this section we limit our research to the indicatrix in a fixed point $z_0 \in M$. Let $T'_{z_0}M$ be the corresponding holomorphic tangent space of M in $z_0 \in M$ and $(g_{i\bar{j}}(z_0,\eta))$ the Hermitian metric from (1). Then, $(T'_{z_0}M, F)$ can be regarded as a locally Minkowski space with the Hermitian metric G defined by:

$$G = g_{j\bar{k}} \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k, \quad \text{with} \quad g_{j\bar{k}} = G\left(\dot{\partial}_j, \dot{\partial}_{\bar{k}}\right),$$

that acts on $V_C(T'M)$.

According to [8], p.12, a linear connection on M extends by linearity to $T_C M$, which is isomorphic to $V_C(T'M)$ via vertical lift, and it is well defined by the next set of coefficients $\overline{\Gamma^i_{jk}} = \Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{jk}} = \Gamma^i_{\bar{j}\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{\bar{j}k}} = \Gamma^i_{j\bar{k}}$, $\overline{\Gamma^{\bar{\imath}}_{j\bar{k}}} = \Gamma^i_{\bar{j}k}$. We require that ∇ be a complex connection with respect to the natural complex structure J, i.e. $\nabla J = 0$. So, it results that ∇ conserves the holomorphic tangent space.

We can choose ∇ to be the Levi-Civita connection, which is a metrical and symmetric connection and we get the next components of the Levi-Civita connection:

$$\begin{aligned}
\Gamma^i_{jk} &= \frac{1}{2} g^{\bar{h}i}\left(\dot{\partial}_k g_{j\bar{h}} + \dot{\partial}_j g_{k\bar{h}}\right) = g^{\bar{h}i} C_{j\bar{h}k} =: C^i_{jk}(\eta),\\
\Gamma^{\bar{\imath}}_{\bar{j}k} &= \frac{1}{2} g^{\bar{\imath}h}\left(\dot{\partial}_k g_{h\bar{j}} - \dot{\partial}_h g_{k\bar{j}}\right) = 0,\\
\Gamma^i_{\bar{j}k} &= \frac{1}{2} g^{\bar{h}i}\left(\dot{\partial}_{\bar{j}} g_{k\bar{h}} - \dot{\partial}_{\bar{h}} g_{k\bar{j}}\right) = 0,\\
\Gamma^{\bar{\imath}}_{jk} &= 0,
\end{aligned}$$

where we used relations (2). Since $\Gamma^i_{\bar{j}k} = \Gamma^{\bar{\imath}}_{jk} = 0$, it takes that the Levi-Civita connection is Hermitian, and has only the following nonzero coefficients:

$$C^i_{jk} := \Gamma^i_{jk} = \overline{\Gamma^{\bar{\imath}}_{\bar{j}\bar{k}}} = g^{\bar{h}i} C_{j\bar{h}k} = g^{\bar{h}i}\dot{\partial}_k g_{j\bar{h}},$$

with $C_{jk} = C_{kj}$ and $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$.

Then, the Riemannian tensor can be written as:

$$R_{i\bar{j}k\bar{h}} = \frac{\partial^2 g_{i\bar{h}}}{\partial \eta^k \partial \bar{\eta}^j} - g^{\bar{p}q}\frac{\partial g_{i\bar{p}}}{\partial \eta^k}\frac{\partial g_{q\bar{h}}}{\partial \bar{\eta}^j}.$$

It fulfills $R_{i\bar{j}k\bar{h}} = R_{k\bar{j}i\bar{h}} = R_{k\bar{h}i\bar{j}}$. Moreover, we have

$$R_{i\bar{j}k\bar{h}} = g_{i\bar{l}}\dot{\partial}_k C^{\bar{l}}_{\bar{j}\bar{h}} = g_{l\bar{h}}\dot{\partial}_{\bar{j}} C^l_{ik} = -g_{l\bar{h}} R^l_{i\bar{j}k} = -g_{i\bar{l}} R^{\bar{l}}_{\bar{j}k\bar{h}}, \tag{7}$$

where $R^i_{j\bar{k}h} = -\dot{\partial}_{\bar{k}} C^i_{jh}$ is the non-zero component of the curvature. Accordingly, from $C^i_{jk}\eta^j = C^i_{jk}\eta^k = 0$, we can write

$$R_{i\bar{j}k\bar{h}}\eta^i = R_{i\bar{j}k\bar{h}}\eta^{\bar{j}} = R_{i\bar{j}k\bar{h}}\eta^k = R_{i\bar{j}k\bar{h}}\eta^{\bar{h}} = 0. \tag{8}$$

The Ricci tensor is $S_{i\bar{j}} = -\sum\limits_k \frac{\partial C^k_{ik}}{\partial \bar{\eta}^j} = \overline{S_{\bar{\imath}j}}$.

The covariant derivative of a tensor $T^h_{i\bar{j}}$ on $T'_{z_0}M$ is:

$$T^h_{i\bar{j}}\,|_k = \frac{\partial T^h_{i\bar{j}}}{\partial \eta^k} + C^h_{mk}T^m_{i\bar{j}} - C^m_{ik}T^h_{m\bar{j}} - C^{\bar{m}}_{\bar{j}k}T^h_{i\bar{m}},$$

and by applying it to the important tensors of the complex Finsler space, by direct calculation we obtain the following relations:

$$F\,|_i = \frac{\partial F}{\partial \eta^i} = l_i := \frac{\eta_i}{2F}; \qquad F\,|_{\bar{\imath}} = \frac{\partial F}{\partial \bar{\eta}^i} = l_{\bar{\imath}} := \frac{\eta_{\bar{\imath}}}{2F};$$
$$l_i\,|_j = -\frac{l_i l_j}{F}; \qquad l_i\,|_{\bar{j}} = l_{\bar{j}}|_i = \frac{g_{i\bar{j}} - 2l_i l_{\bar{j}}}{2F} = \frac{h_{i\bar{j}}}{2F};$$
$$h_{i\bar{j}}\,|_k = \frac{2l_i l_{\bar{j}} l_k - l_i h_{k\bar{j}}}{F}; \qquad h_{i\bar{j}}\,|_{\bar{k}} = \frac{2l_i l_{\bar{j}} l_{\bar{k}} - l_{\bar{j}} h_{i\bar{k}}}{F}.$$

So, for a fixed point z_0 on M, we consider the complex indicatrix in z_0 as:

$$I_{z_0}M = \left\{\eta \in T'_{z_0} \mid F(z_0,\eta) = 1\right\}.$$

The indicatrix $I_{z_0}M$ is an n dimensional immersed complex submanifold in $T'_{z_0}M$. Moreover, it is a locally Minkowski manifold ([11]) and we can take a locally parametrization of this indicatrix as $\eta^i = \eta^i(\theta^\alpha)$; we further consider that the Latin indices $i, j, k, \dots$ run from 1 to $n+1$ and the Greek indices $\alpha, \beta, \gamma, \dots$ run from 1 to n. We shall denote by $B^i_\alpha(\theta) := \frac{\partial \eta^i}{\partial \theta^\alpha}$ the projection factor and by N^i the unit normal vector to $I_{z_0}M$. It can be noticed that $\operatorname{rank}(\frac{\partial \eta^i}{\partial \theta^\alpha}) = n$ and the vector fields $B_\alpha = B^i_\alpha(\theta)\frac{\partial}{\partial \eta^i}$ define a local frame of the holomorphic tangent bundle over $I_{z_0}M$. Then, the induced Hermitian metric tensor relative to $I_{z_0}M$ is given by:

$$g_{\alpha\bar{\beta}} = G\left(\frac{\partial}{\partial\theta^\alpha}, \frac{\partial}{\partial\bar{\theta}^\beta}\right) = G\left(\frac{\partial\eta^i}{\partial\theta^\alpha}\frac{\partial}{\partial\eta^i}, \frac{\partial\bar{\eta}^j}{\partial\bar{\theta}^\beta}\frac{\partial}{\partial\bar{\eta}^j}\right) = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} g_{i\bar{j}}, \tag{9}$$

So, $\operatorname{rank}(g_{\alpha\bar{\beta}}) = n$ and it admits the inverse $\left(g^{\bar{\beta}\alpha}\right)$ such that $g_{\alpha\bar{\beta}}g^{\bar{\beta}\gamma} = \delta^\gamma_\alpha$. Moreover, we can notice that $\dot{\partial}_\alpha := \frac{\partial}{\partial\theta^\alpha} = \frac{\partial\eta^i}{\partial\theta^\alpha}\frac{\partial}{\partial\eta^i} = B^i_\alpha\frac{\partial}{\partial\eta^i} = B^i_\alpha\dot{\partial}_i$.

Since on $I_{z_0}M$ we have $F(z_0,\eta(\theta)) = 1$ and considering that $L = F^2$, if we differentiate the equation $L(z_0,\eta(\theta)) = 1$ with respect to θ^α and we use (1) we get:

$$\frac{\partial L}{\partial\eta^i}\frac{\partial\eta^i}{\partial\theta^\alpha} = 0, \text{ that is equivalent to } g_{i\bar{j}}(z_0,\eta(\theta))\eta^{\bar{j}}(\theta)B^i_\alpha = 0.$$

Now, if we differentiate the same identity with respect to $\bar{\theta}^\alpha$, and using again (1) we obtain $(\dot{\partial}_{\bar{j}}L)\dot{\partial}_{\bar{\alpha}}\bar{\eta}^j = 0$, which is equal to $g_{i\bar{j}}(z_0,\eta(\theta))\eta^i(\theta)B^{\bar{j}}_{\bar{\alpha}} = 0$. Therefore, the vector field $\mathrm{N} = \eta^i(\theta)\dot{\partial}_i$ is a normal and unitary vector field to the indicatrix, because $g_{i\bar{j}}(z_0,\eta(\theta))\eta^i(\theta)\bar{\eta}^j(\theta) = 1$. It represents the vertical Liouville vector field $C = \eta^k\dot{\partial}_k$ restriction to the indicatrix.

We can define $B_i^\alpha = g^{\bar\beta\alpha} B_{\bar\beta}^{\bar j} g_{i\bar j}$ such that the following properties take place:

$$B_\gamma^i B_i^\alpha = \delta_\gamma^\alpha, \quad B_\gamma^i B_j^\gamma = \delta_j^i - \eta^i(\theta)\eta_j(\theta), \quad g^{\bar\alpha\beta} = B_i^\beta B_{\bar j}^{\bar\alpha} g^{\bar j i},$$

and

$$g^{\bar j i} = B_\alpha^i B_{\bar\beta}^{\bar j} g^{\bar\beta\alpha} + \eta^i(\theta)\bar\eta^j(\theta), \tag{10}$$

where $\eta_j(\theta) = g_{j\bar k}\eta^{\bar k}(\theta)$.

Considering $I_{z_0}M$ an n-dimensional holomorphic submanifold immersed in the $n+1$ dimensional complex manifold $T'_{z_0}M$, we can apply the theory of submanifolds and we can denote by $\tilde\nabla$, respectively ∇, their Levi-Civita connections. Then, the Gauss-Weingarten formulae are as follows:

(11) $\nabla_X Y = \tilde\nabla_X Y + h(X,Y), \quad \forall X, Y \in \Gamma(T_C(I_{z_0}M))$

(12) $\nabla_X W = -A_W X + \nabla_X^\perp W, \quad \forall X \in \Gamma(T_C(I_{z_0}M)),\ \forall W \in \Gamma(T_C(I_{z_0}^\perp M))$

with the $\mathcal{F}(T'M)$-bilinear *second fundamental form* of the indicatrix subspace $h : \Gamma(T_C(I_{z_0}M)) \times \Gamma(T_C(I_{z_0}M)) \to \Gamma(T_C(I_{z_0}^\perp M))$ and the $\mathcal{F}(T'M)$-bilinear *shape operator* (or *Weingarten operator*) $A : \Gamma(T_C(I_{z_0}^\perp M)) \times \Gamma(T_C(I_{z_0}M)) \to \Gamma(T_C(I_{z_0}M))$, $A_W X = A(W, X)$. This maps are defined by the following set of coefficients $\overline{h_{\alpha\beta}} = h_{\bar\alpha\bar\beta}$, $\overline{h_{\bar\alpha\beta}} = h_{\alpha\bar\beta}$, $\overline{A_\beta^\alpha} = A_{\bar\beta}^{\bar\alpha}$, $\overline{A_{\bar\beta}^\alpha} = A_\beta^{\bar\alpha}$, regarded as:

$$\begin{array}{ll} h(\dot\partial_\beta, \dot\partial_\alpha) = h_{\alpha\beta}\mathrm{N}, & h(\dot\partial_\beta, \dot\partial_{\bar\alpha}) = h_{\bar\alpha\beta}\bar{\mathrm{N}}, \\ A_{\mathrm{N}}(\dot\partial_\beta) = A_\beta^\alpha \dot\partial_\alpha, & A_{\mathrm{N}}(\dot\partial_{\bar\beta}) = A_{\bar\beta}^\alpha \dot\partial_\alpha. \end{array}$$

The following results are valid in the general context of the complex submanifolds of a Hermitian manifold ([9, 5]). Recall that an $m-$complex submanifold $\widetilde M$ of a complex manifold M is *totally umbilical* iff for every $W \in \Gamma(T'\widetilde M^\perp)$ it exists a real constant λ such that $A_W X = \lambda X$, for all $X \in \Gamma(T'\widetilde M)$. According to [8], a Hermitian structure g on the holomorphic tangent bundle $T'M$ determines a Hermitian metric G on the manifold M, and conversely

$$g(X, Y) = G(X, \bar Y),$$

for any (1,0)-type vector fields. Thus, the *mean curvature* $\mathbb{H}$ is the length of the *mean curvature vector* that is defined with respect to an orthonormal frame $\{E_i\}_{i=\overline{1,m}}$ on $T'\widetilde M$ and is the mean $H = \frac{1}{m}\sum h(E_i, \bar E_i)$. Obvious, it is independent of chosen frame and $\mathbb{H}$ is a real value.

Proposition 1. *Let $\widetilde M$ be an m-dimensional complex submanifold of an n-dimensional complex manifold M, endowed with the Hermitian metric G which is acting on the fibers of $T'M$, such that $G(X, \bar Y) = \overline{G(Y, \bar X)}$. Considering D a linear connection that is metric with respect to G, the following relations take place:*

i) $G(A_W X, \bar Y) = G(W, h(X, \bar Y))$, $\quad G(Y, A_{\bar W} X) = G(h(X, Y), \bar W)$, and their conjugates, for all $X, Y \in \Gamma(T'\widetilde M)$, $W \in \Gamma(T'\widetilde M^\perp)$;

ii) if, in addition, $\widetilde M$ is a totally umbilical submanifold, we can state

a) $A_W X = G(W, H)X, \quad \forall X \in \Gamma(T'\widetilde{M}),\ \forall W \in \Gamma(T'\widetilde{M}^{\perp})$*;*
b) $h(X, \bar{Y}) = G(X, \bar{Y}) \cdot H, \quad \forall X, Y \in \Gamma(T'\widetilde{M})$*,*
where H *represents the mean curvature vector field of* $\widetilde{M}$*.*

The proof of $i)$ is a consequence of $(D_X G)(W, \bar{Y}) = 0, \forall X, Y \in \Gamma(T'\widetilde{M}), W \in \Gamma(T'\widetilde{M}^{\perp})$. We obtain $ii).a)$ by summing the relation $\lambda = G(W, h(E_i, \bar{E}_i))$ and for $ii)$ $b)$ is used $\{W_\alpha\}_{\alpha=\overline{1,n-m}}$ a complementary orthonormal frame in $\Gamma(T'\widetilde{M}^{\perp})$. Then, from $h(X, \bar{Y}) = \sum_{i=1}^{n-m} G(W_i, h(X, \bar{Y}))\bar{W}_i$, by $i)$ and $ii).a)$, we get $G(X, \bar{Y}) \cdot H$.

We note that an equivalent definition of the totally umbilical complex submanifold is $h(X, \bar{Y}) = G(X, \bar{Y})H$, for any $X, Y \in \Gamma(T'\widetilde{M})$, that locally becomes $h_{\bar{\alpha}\beta} = \mathbb{H} g_{\beta\bar{\alpha}}$.

Since the indicatrix $I_{z_0}M$ is an n dimensional immersed complex submanifold in the $(n+1)$-dimensional complex manifold $T'_{z_0}M$, ∇ is the Levi-Civita connection, i.e. it is metrical, and considering $\Gamma(T_C(I^{\perp}_{z_0}M)) = span\{\mathrm{N}, \bar{\mathrm{N}}\}$, we can apply the above Proposition, i), and, between the second fundamental form and shape operator components, we find the next relation:

$$h_{\bar{\alpha}\beta} = A^{\gamma}_{\beta} g_{\gamma\bar{\alpha}}, \quad \text{equivalent to} \quad A^{\alpha}_{\beta} = h_{\bar{\gamma}\beta} g^{\bar{\gamma}\alpha},$$
$$h_{\bar{\alpha}\bar{\beta}} = A^{\gamma}_{\bar{\beta}} g_{\gamma\bar{\alpha}}, \quad \text{equivalent to} \quad A^{\alpha}_{\bar{\beta}} = h_{\bar{\gamma}\bar{\beta}} g^{\bar{\gamma}\alpha}.$$

Using the relations $G(\nabla_X Y, \bar{\mathrm{N}}) = G(h(X, Y), \bar{\mathrm{N}})$, respectively $G(\nabla_X \mathrm{N}, \dot{\partial}_{\bar{\beta}}) = -G(A_{\mathrm{N}} X, \dot{\partial}_{\bar{\beta}})$, we can compute the above components and we obtain:

Proposition 2. *The coefficients of the second fundamental form and Weingarten operator are given by*

$$h_{\alpha\beta} = 0, \qquad h_{\bar{\alpha}\beta} = g_{i\bar{j}} \frac{\partial B^{\bar{j}}_{\bar{\alpha}}}{\partial \theta^{\beta}} \eta^i(\theta)$$
$$A^{\alpha}_{\beta} = -\delta^{\alpha}_{\beta}, \qquad A^{\alpha}_{\bar{\beta}} = 0.$$

Now, looking to the form of the Weingarten operator coefficients from the above Proposition and the definition of a totally umbilical submanifold, we conclude that $I_{z_0}M$ is totally umbilical. Moreover, if we differentiate the orthogonality condition $g_{i\bar{j}} B^{\bar{j}}_{\bar{\alpha}} \eta^i(\theta) = 0$ with respect to θ^{β}, using the relations (1) and (9), we get $g_{i\bar{j}} \eta^i(\theta) \dot{\partial}_{\beta} B^{\bar{j}}_{\bar{\alpha}} = -g_{\beta\bar{\alpha}}$. From Proposition 2, we obtain $h_{\bar{\alpha}\beta} = -g_{\beta\bar{\alpha}}$ and thus, the coefficients of the second fundamental form of $I_{z_0}M$ are Hermitian, i.e. $\overline{h_{\bar{\alpha}\beta}} = h_{\bar{\beta}\alpha}$. Taking into account that $I_{z_0}M$ is a totally umbilical submanifold of $T'_{z_0}M$, $h_{\bar{\alpha}\beta} = -g_{\beta\bar{\alpha}}$ and Proposition 1.ii)b), where G is taken to be the Hermitian structure $G = g_{j\bar{k}} \mathrm{d}\eta^j \otimes \mathrm{d}\bar{\eta}^k$, we can conclude:

Proposition 3. *The indicatrix* $I_{z_0}M$ *of a complex Finsler space is a totally umbilical complex hypersurface with constant mean curvature* $\mathbb{H} = -1$*.*

Expressed in terms of the base vectors fields of $I_{z_0}M$ the Gauss-Weingarten equations (11-12) become:

$$\dot{\partial}_\beta B^i_\alpha + B^k_\beta B^j_\alpha C^i_{jk} = \tilde{C}^\gamma_{\alpha\beta} B^i_\gamma + h_{\alpha\beta}\eta^i(\theta), \quad \dot{\partial}_\beta \eta^i(\theta) = -A^\alpha_\beta B^i_\alpha + \left(C^\perp\right)^i_{\beta k}\eta^k(\theta),$$
$$\dot{\partial}_\beta B^{\bar{\imath}}_{\bar{\alpha}} = \tilde{\Gamma}^{\bar{\gamma}}_{\bar{\alpha}\beta} B^{\bar{\imath}}_{\bar{\gamma}} + h_{\bar{\alpha}\beta}\bar{\eta}^i(\theta), \quad \dot{\partial}_{\bar{\beta}} \eta^i(\theta) = -A^\alpha_{\bar{\beta}} B^i_\alpha + \left(\Gamma^\perp\right)^i_{\bar{\beta}k}\eta^k(\theta).$$

Considering $\dot{\partial}_\beta \eta^i(\theta) = B^i_\beta$, $\dot{\partial}_{\bar{\beta}} \eta^i(\theta) = 0$ and Proposition 2, we find that in the Gauss-Weingarten equalities the coefficients of the orthogonal connection fulfill $\left(C^\perp\right)^i_{\beta k}\eta^k(\theta) = \left(\Gamma^\perp\right)^i_{\bar{\beta}k}\eta^k(\theta) = 0$.

So, given the above properties and since ∇ and $\tilde{\nabla}$ are Levi-Civita connections, we can state:

Proposition 4. *The Gauss-Weingarten formulae of the indicatrix $I_{z_0}M$ are written locally as:*

$$\frac{\partial B^i_\alpha}{\partial \theta^\beta} + B^k_\beta B^j_\alpha C^i_{jk} = \tilde{C}^\gamma_{\alpha\beta} B^i_\gamma, \qquad \frac{\partial B^i_\alpha}{\partial \bar{\theta}^\beta} = -g_{\alpha\bar{\beta}}\eta^i(\theta),$$
$$\frac{\partial \eta^i(\theta)}{\partial \theta^\beta} = B^i_\beta, \qquad \frac{\partial \eta^i(\theta)}{\partial \bar{\theta}^\beta} = 0.$$

Next, we consider

$$\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\bar{\delta}}}{\partial \theta^\gamma \partial \bar{\theta}^\beta} - g^{\bar{\sigma}\lambda}\frac{\partial g_{\alpha\bar{\sigma}}}{\partial \theta^\gamma}\frac{\partial g_{\lambda\bar{\delta}}}{\partial \bar{\theta}^\beta}$$

the Riemannian tensor of $I_{z_0}M$. The Gauss equations ([10]), for the Levi-Civita connections, expressed by means of the metric structures G and $\tilde{G} = g_{\alpha\bar{\beta}} d\theta^\alpha \otimes d\bar{\theta}^\beta$, are as follows:

$$G\left(R(X,Y)Z,U\right)$$
$$= \tilde{G}\left(\tilde{R}(X,Y)Z,U\right) + G\left(H(X,Z),H(Y,U)\right) - G(H(Y,Z),H(X,U)).$$

Using $R(U,Z,X,Y) = G(R(X,Y)Z,U)$ and taking $X = \dot{\partial}_\gamma, Y = \dot{\partial}_{\bar{\delta}}, Z = \dot{\partial}_{\bar{\beta}}, U = \dot{\partial}_\alpha$ we get locally:

$$B^i_\alpha B^{\bar{j}}_{\bar{\beta}} B^k_\gamma B^{\bar{h}}_{\bar{\delta}} R_{i\bar{j}k\bar{h}} = \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} + h_{\bar{\beta}\gamma}h_{\alpha\bar{\delta}} - h_{\bar{\beta}\bar{\delta}}h_{\alpha\gamma}.$$

Considering that $h_{\bar{\beta}\alpha} = h_{\alpha\bar{\beta}} = -g_{\alpha\bar{\beta}}$ and $h_{\alpha\beta} = 0$, we obtain

$$\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}} = B^i_\alpha B^{\bar{j}}_{\bar{\beta}} B^k_\gamma B^{\bar{h}}_{\bar{\delta}} R_{i\bar{j}k\bar{h}} - g_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}$$

Contracting by $g^{\bar{\delta}\alpha}$, considering rank $\left(g_{\alpha\bar{\beta}}\right) = n$ and using (10), we get:

$$\tilde{R}_{\bar{\beta}\gamma} = B^{\bar{j}}_{\bar{\beta}} B^k_\gamma R_{\bar{j}k} - n g_{\gamma\bar{\beta}},$$

where $\tilde{R}_{\bar{\beta}\gamma}$ is the Ricci tensor of $I_{z_0}M$ and $R_{\bar{j}k} = g^{\bar{h}i}R_{i\bar{j}k\bar{h}}$ is v-Ricci tensor. Moreover, contracting by $g^{\bar{\beta}\gamma}$, we have

$$\tilde{R} = R - n^2,$$

where $\tilde{R}$ is the scalar curvature of $I_{z_0}M$ and $R = g^{\bar{j}k}R_{\bar{j}k}$ is v-scalar curvature.

In the following, considering the ideas from the real case [6, 12], we introduce the hypersurfaces homothetic to the complex indicatrix in a fixed point z_0 and we give some properties of them.

Firstly, let $\tilde{M}$ be an n-dimensional hypersurface in $T'_{z_0}M$ represented by the parametric equations $\eta^i = \eta^i(u^\alpha) = \eta^i(u)$. If N^i denotes the unit normal vector to $\tilde{M}$ and $P^i_\alpha = \frac{\partial \eta^i}{\partial u^\alpha}$ the projection factor, the Gauss-Weingarten formulae with respect to the Levi-Civita connections, for the hypersurface $\tilde{M}$, we have the following expression

$$\begin{aligned}
&\frac{\partial P^i_\alpha}{\partial u^\beta} + P^k_\beta P^j_\alpha C^i_{jk} = \tilde{C}^\gamma_{\alpha\beta}P^i_\gamma + h_{\alpha\beta}N^i, && \frac{\partial P^i_\alpha}{\partial \bar{u}^\beta} = h_{\alpha\bar{\beta}}N^i,\\
&\frac{\partial N^i}{\partial u^\beta} + N^j P^k_\beta C^i_{jk} = -A^\alpha_\beta P^i_\alpha, && \frac{\partial N^i}{\partial \bar{u}^\beta} = -A^\alpha_{\bar{\beta}}P^i_\alpha,
\end{aligned}$$

where $h_{\alpha\beta},\ h_{\alpha\bar{\beta}}$ and $A^\alpha_\beta,\ A^\alpha_{\bar{\beta}}$ are the coefficients of the second fundamental form and the shape operator of $\tilde{M}$, respectively, that satisfy $A^\alpha_\beta = h_{\bar{\gamma}\beta}g^{\bar{\gamma}\alpha}$ and $A^\alpha_{\bar{\beta}} = h_{\bar{\gamma}\bar{\beta}}g^{\bar{\gamma}\alpha}$. Moreover, the induced metric tensor is given by $g_{\alpha\bar{\beta}}(u) = P^i_\alpha P^{\bar{j}}_{\bar{\beta}} g_{i\bar{j}}(z_0, \eta(u))$.

A closed (i.e. boundless and compact) hypersurface $\widetilde{M}$ in $T'_{z_0}M$ of equation $F(z_0, \eta) = c$, i.e.

$$g_{i\bar{j}}\eta^i(u)\bar{\eta}^j(u) = c^2, \quad \text{with } c > 0 \text{ a real constant}, \tag{13}$$

will be called *homothetic to the indicatrix*.

On differentiating (13) with respect to $\bar{u}^\alpha$ and using (3), we obtain

$$g_{i\bar{j}}(u)\eta^i(u)P^{\bar{j}}_{\bar{\alpha}} = 0$$

and hence it exists a constant ν such that $\eta^i(u) = \nu N^i$.

We denote by D_α the operator of mixed covariant derivative, that acts on a given T^i_β as follows:

$$D_\alpha T^i_\beta = \frac{\partial T^i_\beta}{\partial \theta^\alpha} + C^i_{jk}T^j_\beta B^k_\alpha - \tilde{C}^\gamma_{\beta\alpha}T^i_\gamma.$$

By a direct computation, using the above Gauss-Weingarten formulae, we get:

$$\begin{aligned}
D_\alpha P^i_\beta &= h_{\beta\alpha}N^i, \quad & D_\alpha N^i &= -h_{\bar{\gamma}\alpha}g^{\bar{\gamma}\beta}P^i_\beta, \quad & D_\alpha \eta^i &= P^i_\alpha,\\
D_{\bar{\alpha}} P^i_\beta &= h_{\beta\bar{\alpha}}N^i, \quad & D_{\bar{\alpha}} N^i &= -h_{\bar{\gamma}\bar{\alpha}}g^{\bar{\gamma}\beta}P^i_\beta, \quad & D_{\bar{\alpha}} \eta^i &= 0,
\end{aligned}$$

and $D_\alpha g_{i\bar{j}} = D_{\bar{\alpha}} g_{i\bar{j}} = 0,\ D_\alpha g_{\beta\bar{\gamma}} = D_{\bar{\alpha}} g_{\beta\bar{\gamma}} = 0$, and their conjugates.

Further, by applying D_β to $g_{i\bar{j}}(u)\eta^i(u)P^{\bar{j}}_{\bar{\alpha}} = 0$, we obtain $h_{\bar{\alpha}\beta} = -\frac{1}{\nu}g_{\beta\bar{\alpha}}$.

Considering Proposition 1.ii)b), we get that a hypersurface homothetic to the indicatrix is a totally umbilical one, with the mean curvature $\mathbb{H} = -\frac{1}{\nu}$. Since $\mathbb{H}$ is a constant real value, we have ν also real, $h_{\alpha\bar{\beta}} = h_{\bar{\beta}\alpha}$, i.e. the second fundamental form coefficients of a hypersurface homothetic to the indicatrix are Hermitian, and $\bar{\eta}^i(u) = \nu \bar{N}^i$. By substituting this into relation (13), we obtain $\nu = \epsilon c$, were $\epsilon = +1$ or $\epsilon = -1$. So, we can conclude:

Proposition 5. *Let $\tilde{M}$, given by the parametric equations $\eta^i = \eta^i(u^\alpha)$, be a hypersurface homothetic to the indicatrix in $\widetilde{T'_{z_0}M}$. Then $\tilde{M}$ is totally umbilical and satisfies the condition:*

$$1 + \epsilon c \mathbb{H} = 0.$$

3. The complex indicatrix bundle

Let (M, F) be a Finsler manifold, $\dim_C M = n+1$, and $\left(\widetilde{T'M}, G\right)$ be the slit holomorphic tangent bundle of M endowed with the Sasaki lift (4), which is a Hermitian metric structure on $\widetilde{T'M} = T'M \backslash \{0\}$. Considering that $\dim_{\mathbb{C}} T'M = 2n+2$, on $T'M$, we take the local coordinates (z^k, η^k), with $k = 1, \dots, n+1$.

We denote by IM the hypersurface of $\widetilde{T'M}$ given by

$$IM = \bigcup_{z \in M} I_z M, \quad I_z M = \{\eta \in T'_z M \mid F(z, \eta) = 1\},$$

which will be called the *indicatrix bundle* of the complex Finsler space (M, F). The above condition can be written, for any $z \in M$, as

$$F(z, \eta) = 1 \quad \text{i.e.} \quad L(z, \eta) = 1 \quad \text{i.e.} \quad g_{i\bar{j}}(z, \eta)\eta^i \bar{\eta}^j = 1.$$

Notice that it takes place the inclusion $IM \overset{i}{\hookrightarrow} T'M$. Locally, we can consider a parametrization of this submanifold as:

$$z^i = z^i(u^\alpha), \quad \eta^i = \eta^i(u^\alpha), \quad \alpha \in \{1, 2, \dots, 2n+1\}.$$

Differentiating $F^2(z, \eta) = 1$ with respect to u^α we obtain: $\frac{\partial F^2}{\partial z^i}\frac{\partial z^i}{\partial u^\alpha} + \frac{\partial F^2}{\partial \eta^i}\frac{\partial \eta^i}{\partial u^\alpha} = 0$. Using $F^2 = L$, we can rewrite:

$$\frac{\partial L}{\partial z^i}\frac{\partial z^i}{\partial u^\alpha} + \frac{\partial L}{\partial \eta^i}\frac{\partial \eta^i}{\partial u^\alpha} = 0.$$

From the homogeneity relations we define: $\eta_i = g_{i\bar{j}}\bar{\eta}^j = \frac{\partial L}{\partial \eta^i}$. Furthermore, on $T'M$ we consider the C-F (c.n.c.) such that $\frac{\delta}{\delta z^i} = \frac{\partial}{\partial z^i} - N_i^k \frac{\partial}{\partial \eta^k}$ and $\frac{\delta L}{\delta z^i} = 0$. Then the above relations can be written as:

$$\left(\frac{\delta L}{\delta z^i} + N_i^k \frac{\partial L}{\partial \eta^k}\right)\frac{\partial z^i}{\partial u^\alpha} + \frac{\partial L}{\partial \eta^i}\frac{\partial \eta^i}{\partial u^\alpha} = 0,$$

that is equivalent to

$$\left(N_i^k \frac{\partial z^i}{\partial u^\alpha} + \frac{\partial \eta^k}{\partial u^\alpha}\right)\eta_k = 0. \tag{14}$$

The natural frame field on IM is represented by

$$\frac{\partial}{\partial u^\alpha} = \frac{\partial z^i}{\partial u^\alpha}\frac{\partial}{\partial z^i} + \frac{\partial \eta^i}{\partial u^\alpha}\frac{\partial}{\partial \eta^i} = \frac{\partial z^i}{\partial u^\alpha}\frac{\delta}{\delta z^i} + \left(N_i^k \frac{\partial z^i}{\partial u^\alpha} + \frac{\partial \eta^k}{\partial u^\alpha}\right)\frac{\partial}{\partial \eta^k}.$$

Then, by (14), we have

$$G\left(\frac{\partial}{\partial u^{\alpha}}, \bar{\eta}^{l}\frac{\partial}{\partial \bar{\eta}^{l}}\right) = \left(N_i^k \frac{\partial z^i}{\partial u^{\alpha}} + \frac{\partial \eta^k}{\partial u^{\alpha}}\right)\bar{\eta}^l g_{k\bar{l}} = 0,$$

where G is the Sasaki lift. Then it follows that the vertical Liouville vector field $N = \eta^l \frac{\partial}{\partial \eta^l}$ is orthogonal to $T'(IM)$, i.e. it is normal to the indicatrix, and $\xi = \mathbb{F}N = \eta^i \delta_i$ is the radial horizontal vector field, unitary and tangent to IM , where $\mathbb{F}$ (which is bold) represents the complex structure defined on $T'M$ as:

$$\mathbb{F}\left(\delta_j\right) = \dot{\partial}_j, \quad \mathbb{F}(\dot{\partial}_j) = -\delta_j, \quad \mathbb{F}\left(\delta_{\bar{j}}\right) = \dot{\partial}_{\bar{j}}, \quad \mathbb{F}(\dot{\partial}_{\bar{j}}) = -\delta_{\bar{j}}.$$

To avoid an eventually confusion with the fundamental function F, we further denote a complex Finsler space as (M, L). So, $(M, \mathbb{F}, G)$ is an almost Hermitian structure on $T'M$ and its integrability implies the integrability of horizontal distribution.

We consider the 1-form θ given by

$$\phi(X) = G(X, \bar{\xi}), \quad \forall X \in \Gamma(T'T'M). \tag{15}$$

Next, we denote by $\tilde{G}$ the induced metric on IM by the Sasaki lift G. Finally, for any vector field X on IM we decompose $\mathbb{F}X$ as:

$$\mathbb{F}X = \varphi X + \phi(X)N, \tag{16}$$

where φX denotes a vector field that is tangent to IM.

Since ξ is a vector field unitary with respect to $\tilde{G}$, from (15) we get that $\phi(\xi) = 1$; moreover, $\phi(N) = 0$. Also, from (15) and (16), we deduce that $\phi \circ \varphi = 0$ and taking $X = \xi$ in (16) we obtain $\varphi\xi = 0$. Further, by applying $\mathbb{F}$ to (16), we obtain $\varphi^2 X = -X + \phi(X)\xi$, i.e.

$$\varphi^2 = -Id + \phi \otimes \xi. \tag{17}$$

Moreover, taking into account that $\mathbb{F}$ is an isometry with respect to $\tilde{G}$ and using (16), we infer that the induced Hermitian metric structure satisfies

$$\tilde{G}(\varphi X, \varphi Y) = \tilde{G}(X, Y) - \phi(X)\phi(Y), \quad \forall X, Y \in \Gamma\left(T_C IM\right).$$

So, we have:

Proposition 6. *Let (M, L) be a complex Finsler manifold. Then $(\varphi, \xi, \phi, \tilde{G})$ is a metric almost contact structure on IM.*

Next, we take a vector field X on M and consider its horizontal and vertical lifts X^h and X^v on $\widetilde{T'M}$, respectively; thus, for $X = X^i \frac{\partial}{\partial z^i}$ we define:

$$X^h = X^i \delta_i \quad \text{and} \quad X^v = X^i \dot{\partial}_i.$$

Then X^h is tangent to IM, while X^v is expressed at the points of IM as follows:

$$X^v = X^t + \phi(X^h)N, \tag{18}$$

where X^t represents the part of X^v tangent to IM and it is called the *tangential lift* of X on IM.

Thus, we can make the notations:

$$HIM = H\widetilde{T'M}_{|IM} \quad \text{and} \quad VIM = V\widetilde{T'M}_{|IM}.$$

Also, we denote by IM^t the bundle of tangential vectors, which are complementary orthogonal to *span*$\{N\}$ in VIM. Then, the tangent bundle of IM admits the orthogonal decomposition:

$$T_C IM = T'IM \oplus T''IM, \text{ with } T'IM = HIM \oplus IM^t \text{ and } T''IM = \overline{T'IM}.$$

Now, by applying $\mathbb{F}$ to (18) and taking into account that $\mathbb{F}(N) = -\xi$, $\phi(X^t) = 0$ and $\mathbb{F}X^v = -X^h$, we deduce that:

$$X^h = -\varphi X^t + \phi(X^h)\xi. \tag{19}$$

Thus, the above decomposition becomes:

$$T'IM = \text{span}\{\xi\} \oplus \varphi\left(IM^t\right) \oplus IM^t.$$

By applying φ to (19) and using (17), we obtain:

$$\varphi X^h = X^t.$$

On $T'M$ we consider the Chern-Finsler (c.n.c.), given by the coefficients (5). So, taking into account the *Chern-Finsler linear connection* on $\widetilde{T'M}$, locally given by the set of coefficients from (6), we get that

$$D_{vX}N = vX, \quad D_{vX}\bar{N} = 0, \quad D_{hX}N = 0 \quad \text{and} \quad D_{hX}\bar{N} = 0,$$

where by h and v we denote the projection morphisms of $T'(T'M)$ on $HT'M$ and $VT'M$, respectively. More precisely, for a vector field X, we consider $hX = X^i\delta_i$ and $vX = X^i\dot{\partial}_i$. So, we deduce that

$$D_{X^t}N = X^t, \quad D_{X^t}\bar{N} = 0, \quad D_{X^h}N = 0 \quad \text{and} \quad D_{X^h}\bar{N} = 0.$$

On the other hand, considering the general framework of the geometry of hypersurfaces, for any $X \in \Gamma(T_C IM)$ we have the Weingarten formula on the indicatrix IM with respect to the induced Chern-Finsler linear connection:

$$D_X N = -A_N X,$$

where $A_N(X) =: A(X)$ is the shape operator of the immersion of IM in $(\widetilde{T'M}, G)$, that satisfies $G(D_X N, \dot{\partial}_{\bar{k}}) = -G(A(X), \dot{\partial}_{\bar{k}})$. Thus, comparing the last two relations, we get:

$$AX^t = -X^t, \quad A\overline{X^t} = 0, \quad AX^h = 0 \quad \text{and} \quad A\overline{X^h} = 0, \tag{20}$$

where $A\overline{X^t} = \overline{A_{\bar{N}}X^t}$.

Let $\tilde{D}$ be the tangent Chern-Finsler connection induced on $(IM, \tilde{G})$. The Gauss formula of the immersed subspace IM is:

$$D_X Y = \tilde{D}_X Y + h(X,Y), \qquad \forall X, Y \in \Gamma(T_{\mathrm{C}} IM),$$

where $h(X,Y) \in \Gamma(T_C IM^{\perp})$ is the normal part of the vector field $D_X Y$. The map $h : \Gamma(T_C IM) \times \Gamma(T_C IM) \to \Gamma(T_C IM^{\perp})$ is $\mathcal{F}(T'M)$-bilinear and it represents the second fundamental form of the indicatrix subspace. Moreover, it takes place: $G(D_X Y, \bar{N}) = G(h(X,Y), \bar{N})$.

Considering that the Chern-Finsler connection is metrical with respect to the Sasaki lift G, we can apply Proposition 1.i), for the immersed subspace IM in $(\widetilde{T'M}, G)$, so between Weingarten operator and the second fundamental tensor it exists the following relation:

$$G\left(A_N X, \bar{Y}\right) = G(N, h(X, \bar{Y})) \quad \text{and} \quad G(Y, A_{\bar{N}} X) = G(h(X,Y), \bar{N}),$$

and their conjugates, for all $X, Y \in \Gamma(T'IM)$.

If we take $X = X^h$, then we obtain $h(X^h, Y) = 0$, equivalent to

$$D_{X^h} Y = \tilde{D}_{X^h} Y. \tag{21}$$

Similar, $D_{\overline{X^h}} Y = \tilde{D}_{\overline{X^h}} Y$, $D_{\overline{X^t}} Y = \tilde{D}_{\overline{X^t}} Y$ and $h(X^t, \overline{X^t}) = -\bar{N}$.

Considering this, we can state the following proposition:

Proposition 7. *The complex indicatrix bundle IM is a hypersurface in* $(\widetilde{T'M}, G)$ *of constant mean curvature:*

$$\mathbb{H} = -\frac{n}{2n+1}.$$

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