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COMPLETE INTERPOLATION VS. RIESZ BASES OF REPRODUCING KERNELS

SIMON COWELL AND PHILIPPE POULIN

Abstract. In the study of Hilbert spaces of analytic functions, it is noticed that complete interpolating sequences and Riesz bases of reproducing kernels are dual notions. In this work we make this duality explicit by identifying sequences of complex numbers with linear operators.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space of entire functions such that the evaluation at w, $\mathcal{H} \to \mathbb{C}$, $f \mapsto f(w)$, is a continuous linear functional for every $w \in \mathbb{C}$. By Riesz' lemma, $\mathcal{H}$ then admits *reproducing kernels*, that is, functions $k_w \in \mathcal{H}$ satisfying[1]

$$\langle k_w, f\rangle = f(w).$$

Let $\sigma = \{\sigma_n\}$ be a sequence of complex numbers and let

$$\mathcal{D}_\sigma = \left\{ \{d_n\} \ ; \ \sum |d_n|^2/\|k_{\sigma_n}\|^2 < \infty \right\}.$$

We say that σ is *complete interpolating* [7] if for all $\{d_n\} \in \mathcal{D}_\sigma$ there exists a unique $f \in \mathcal{H}$ such that

$$f(\sigma_n) = d_n.$$

In the seminal case where $\mathcal{H}$ is the Paley–Wiener space L^2_π (see Section 2 for the definition), it is noticed [7, 3, 5] that σ is complete interpolating if and only if $\{k_{\sigma_n}/\|k_{\sigma_n}\|\}$ is a Riesz basis; in other words, iff $\{k_{\sigma_n}/\|k_{\sigma_n}\|\}$ is the image of an orthonormal basis under a bounded invertible linear operator with bounded inverse.

The aim of this paper is to make the duality between these two notions explicit. It starts from the observation that, assuming the existence of a complete interpolating sequence for $\mathcal{H}$, each sequence of complex numbers may be

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[1] We use the convention that the Hermitian product is conjugate-linear in its first component.

identified with a linear operator. The aforementioned duality is then seen in invertibility conditions on this operator and its adjoint. Special attention is given to de Branges' spaces.

2. Historical account

In Fourier analysis, it is well known that the system of exponentials

$$\{\mathrm{e}^{inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$$

is an orthonormal basis of $L^2[-\pi,\pi]$. Paley and Wiener [6] investigated the following stability problem: how much may the nodes $n\in\mathbb{Z}$ be perturbed so the resulting exponential system remains a Riesz basis?

An equivalent formulation may be obtained by applying an inverse Fourier transform. By the Paley–Wiener theorem and Plancherel's formula, $\mathcal{F}^{-1}$, defined by

$$\mathcal{F}^{-1}[\varphi](z)=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}zt}\varphi(t)\,\mathrm{d}t,$$

is an isometry from $L^2[-\pi,\pi]$ to L^2_π, where

$$L^2_\pi=\{f \text{ entire };\ \|f\|_2<\infty,\ f \text{ of exponential type } \le\pi\}$$

(equipped with the usual L^2 norm on $\mathbb{R}$).

Observe that L^2_π admits reproducing kernels. Indeed, for arbitrary $\mathcal{F}^{-1}[\varphi]\in L^2_\pi$ and $w\in\mathbb{C}$, and for x varying in $[-\pi,\pi]$,

$$\begin{aligned}\frac{1}{\sqrt{2\pi}}\langle\mathcal{F}^{-1}[\mathrm{e}^{-\mathrm{i}\bar{w}x}],\mathcal{F}^{-1}[\varphi]\rangle_2&=\frac{1}{\sqrt{2\pi}}\langle\mathrm{e}^{-\mathrm{i}\bar{w}x},\varphi\rangle_{L^2[-\pi,\pi]}\\&=\frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\mathrm{e}^{\mathrm{i}wx}\varphi(x)\,\mathrm{d}x\\&=\mathcal{F}^{-1}[\varphi](w),\end{aligned}$$

yielding

$$k_w(z)=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[\mathrm{e}^{-\mathrm{i}\bar{w}x}]=\frac{\sin(\pi(z-\bar{w}))}{\pi(z-\bar{w})}. \tag{1}$$

Since $\mathcal{F}^{-1}$ is an isometry, stability of the Riesz basis property of the exponential system $\{\mathrm{e}^{inx}\}_{n\in\mathbb{Z}}$ in $L^2[-\pi,\pi]$ for small, complex perturbations of n is equivalent to stability of the Riesz basis property of the reproducing kernels $\{k_n\}_{n\in\mathbb{Z}}$ in L^2_π.

As already mentioned, $\{\mathrm{e}^{inx}/\sqrt{2\pi}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\pi,\pi]$, and hence $\{k_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis of L^2_π. In particular, $\mathbb{Z}$ is complete interpolating for L^2_π:

$$f(n)=d_n\Leftrightarrow f(z)=\sum_{n\in\mathbb{Z}}d_n\frac{\sin(\pi(z-n))}{\pi(z-n)}. \tag{2}$$

The stability problem for $\{k_n\}_{n\in\mathbb{Z}}$ may thus be translated to a stability problem for complete interpolating sequences (see Section 3).

Here are some highlights [4] in the study of the stability problem from its origin to its complete solution. Paley and Wiener first proved that $\{e^{i\sigma_n x}\}_{n\in\mathbb{Z}}$ is a Riesz basis if $|\sigma_n - n| < d$ for certain $d > 0$, namely for all $d < 1/\pi^2$. Then, Ingham showed that $d = 1/4$ is not admissible. In 1964, Kadets showed that $1/4$ is indeed the lowest upper bound of the valid d.

Later, Pavlov *et al.* [3] obtained a geometric characterization of all complex sequences $\{\sigma_n\}$ such that $\{k_{\sigma_n}\}$ is a Riesz basis. The result was then revisited by Seip and Lyubarskii [5].

3. Duality

In the sequel we consider a Hilbert space $\mathcal{H}$ of entire functions with non-vanishing reproducing kernels at every point. We denote by $\tilde{k}_w = k_w/\|k_w\|$ the normalized reproducing kernel at $w \in \mathbb{C}$. We assume that $\mathcal{H}$ admits Riesz bases of normalized reproducing kernels, among which we distinguish an arbitrary one, $\{\tilde{k}_{\lambda_n}\}$. We let T be the linear operator, invertible in $\mathcal{B}(\mathcal{H})$, such that $T\tilde{k}_{\lambda_n} = e_n$, where $\{e_n\}$ is a certain orthonormal basis in $\mathcal{H}$.

Doing so, the sequence $\lambda = \{\lambda_n\}$ is complete interpolating: for any $\{d_n\} \in \mathcal{D}_\lambda$, letting $g = \sum(d_n/\|k_{\lambda_n}\|)e_n$, T^*g solves the interpolation problem $f(\lambda_n) = d_n$, since

$$T^*g(\lambda_n) = \langle k_{\lambda_n}, T^*g\rangle = \langle Tk_{\lambda_n}, g\rangle = \langle \|k_{\lambda_n}\|e_n, g\rangle = d_n.$$

Moreover the solution is unique: $f(\lambda_n) = 0$ for all n is equivalent to

$$\langle T^{-1}Tk_{\lambda_n}, f\rangle = \|k_{\lambda_n}\|\langle e_n, (T^{-1})^*f\rangle = 0 \quad \text{for all } n,$$

and hence implies that $f = 0$.

Observe in addition that for all $f \in \mathcal{H}$, $\{f(\lambda_n)\} \in \mathcal{D}_\lambda$, since

$$\sum |f(\lambda_n)|^2/\|k_{\lambda_n}\|^2 = \sum \left|\langle T^{-1}T\tilde{k}_{\lambda_n}, f\rangle\right|^2 = \sum |\langle e_n, (T^{-1})^*f\rangle|^2 < \infty.$$

The presence of the complete interpolating sequence $\lambda = \{\lambda_n\}$ allows us to associate with any sequence $\sigma = \{\sigma_n\}$ a linear operator Λ_σ: The domain of Λ_σ consists of the functions whose restriction to σ is in $\mathcal{D}_\sigma$, namely,

$$\operatorname{dom}\Lambda_\sigma = \{f \in \mathcal{H}\ ;\ \sum |f(\sigma_n)|^2/\|k_{\sigma_n}\|^2 < \infty\}.$$

$\Lambda_\sigma f$ is defined as the unique solution to the interpolation problem

$$\Lambda_\sigma f(\lambda_n) = (\|k_{\lambda_n}\|/\|k_{\sigma_n}\|)f(\sigma_n).$$

Observe that

$$\Lambda_\sigma f = T^*T\sum(f(\sigma_j)/\|k_{\sigma_j}\|)\,\tilde{k}_{\lambda_j}, \tag{3}$$

since the scalar product of k_{λ_n} times this last expression gives

$$\begin{aligned}\sum_j(f(\sigma_j)/\|k_{\sigma_j}\|)\langle Tk_{\lambda_n}, T\tilde{k}_{\lambda_j}\rangle &= \sum_j(f(\sigma_j)\|k_{\lambda_n}\|/\|k_{\sigma_j}\|)\langle e_n, e_j\rangle \\ &= (\|k_{\lambda_n}\|/\|k_{\sigma_n}\|)f(\sigma_n).\end{aligned}$$

In particular, $\Lambda_\sigma \in \mathcal{B}(\mathcal{H})$ if and only if

$$\sup_{\|f\|=1} \sum \frac{|f(\sigma_n)|^2}{\|k_{\sigma_n}\|^2} < \infty.$$

Complete interpolating sequences are then easily characterized:

Proposition 1. *The sequence σ is complete interpolating iff $\Lambda_\sigma \colon \operatorname{dom} \Lambda_\sigma \to \mathcal{H}$ is bijective.*

Proof. Suppose that σ is complete interpolating. Let us denote by $\Sigma_\lambda f$ the unique solution to the interpolation problem $\Sigma_\lambda f(\sigma_n) = (\|k_{\sigma_n}\|/\|k_{\lambda_n}\|)f(\lambda_n)$. Since
$\{f(\lambda_n)\} \in \mathcal{D}_\lambda$, Σ_λ maps the whole $\mathcal{H}$ to $\operatorname{dom} \Lambda_\sigma$. Moreover, for all $f \in \operatorname{dom} \Lambda_\sigma$, $\Sigma_\lambda \Lambda_\sigma f(\sigma_n) = f(\sigma_n)$, while for all $f \in \mathcal{H}$, $\Lambda_\sigma \Sigma_\lambda f(\lambda_n) = f(\lambda_n)$. It follows that $\Lambda_\sigma \colon \operatorname{dom} \Lambda_\sigma \to \mathcal{H}$ is a bijection.

Conversely, suppose that Λ_σ has an inverse $\Lambda_\sigma^{-1} \colon \mathcal{H} \to \operatorname{dom} \Lambda_\sigma$ and let $\{d_n\} \in \mathcal{D}_\sigma$. Since λ is complete interpolating, there exists a unique $g \in \mathcal{H}$ such that $g(\lambda_n) = (\|k_{\lambda_n}\|/\|k_{\sigma_n}\|)d_n$. Therefore, $\Lambda_\sigma^{-1} g$ is the unique solution to the interpolation problem $f(\sigma_n) = d_n$, because

$$d_n = (\|k_{\sigma_n}\|/\|k_{\lambda_n}\|)\Lambda_\sigma \Lambda_\sigma^{-1} g(\lambda_n) = (\Lambda_\sigma^{-1} g)(\sigma_n).$$

Consequently, σ is complete interpolating. $\square$

Using duality, Riesz bases of normalized reproducing kernels are also characterized by an invertibility condition, as shown below.

Lemma 1. *If $\operatorname{dom} \Lambda_\sigma = \mathcal{H}$, then Λ_σ is bounded.*

Proof. Let $S_N \colon \mathcal{H} \to \ell^2$ be the linear operator defined by $S_N f = \{\langle \tilde{k}_{\sigma_n}, f\rangle\}_{n=1}^N$. Observe that $\|S_N f\|_{\ell^2} \le \sqrt{N}\|f\|$, so each S_N is bounded. By the hypothesis, $\sum |f(\sigma_n)|^2/\|k_{\sigma_n}\|^2 < \infty$ for all $f \in \mathcal{H}$. In particular, the operator $S \colon \mathcal{H} \to \ell^2$, $Sf = \{\langle \tilde{k}_{\sigma_n}, f\rangle\}_{n=1}^\infty$ exists. By the Banach–Steinhaus theorem [1] it must be continuous, so the relation (3) implies

$$\|\Lambda_\sigma\| \le \|T^*\| \sup_{\|f\|=1} \sqrt{\sum \frac{|f(\sigma_n)|^2}{\|k_{\sigma_n}\|^2}} < \infty. \qquad \square$$

Proposition 2. *The system of normalized reproducing kernels $\{\tilde{k}_{\sigma_n}\}$ is a Riesz basis if and only if $\operatorname{dom} \Lambda_\sigma = \mathcal{H}$ and $\Lambda_\sigma \colon \mathcal{H} \to \mathcal{H}$ is bijective.*

Proof. Assume that $\{\tilde{k}_{\sigma_n}\}$ is a Riesz basis. Then $\operatorname{dom} \Lambda_\sigma = \mathcal{H}$ and hence, by the lemma, $\Lambda_\sigma \in \mathcal{B}(\mathcal{H})$. Observe that $\langle \tilde{k}_{\lambda_n}, \Lambda_\sigma f\rangle = \langle \tilde{k}_{\sigma_n}, f\rangle$ for all $f \in \mathcal{H}$, yielding $\Lambda_\sigma^* \tilde{k}_{\lambda_n} = \tilde{k}_{\sigma_n}$. Since $\{\tilde{k}_{\lambda_n}\}$ and $\{\tilde{k}_{\sigma_n}\}$ are Riesz bases, it follows that Λ_σ^*, and hence Λ_σ, are invertible in $\mathcal{B}(\mathcal{H})$.

Conversely, assume that Λ_σ is a bijection from $\mathcal{H}$ to $\mathcal{H}$. By the lemma, Λ_σ is bounded. The inverse mapping theorem implies that its inverse is also bounded. In particular, Λ_σ^* is invertible in $\mathcal{B}(\mathcal{H})$. Since $\Lambda_\sigma^* \tilde{k}_{\lambda_n} = \tilde{k}_{\sigma_n}$ and $\{\tilde{k}_{\lambda_n}\}$ is a Riesz basis, $\{\tilde{k}_{\sigma_n}\}$ is also a Riesz basis. $\square$

A complete interpolating sequence σ such that $\{f(\sigma_n)\} \in \mathcal{D}_\sigma$ for all $f \in \mathcal{H}$ is said to be *universal complete interpolating* [7]. From the Propositions 1 and 2, we have recovered this classical result: under the assumption that $\mathcal{H}$ admits a Riesz basis of normalized reproducing kernels, $\{\sigma_n\}$ *is a universal complete interpolating sequence if and only if* $\{\tilde{k}_{\sigma_n}\}$ *is a Riesz basis.* The classical proof [7] is however a simpler alternative, based on the observation that $\mathcal{H} \to \ell^2$, $f \mapsto \{\langle \tilde{k}_{\sigma_n}, f\rangle\}$ and $\ell^2 \to \mathcal{H}$, $\{c_n\} \mapsto \sum c_n \tilde{k}_{\sigma_n}$ are adjoint.
Remark. What is the analogue for non normalized Riesz bases of reproducing kernels? If $\{k_{\lambda_n}\}$ is a Riesz basis, then necessarily the norms $\|k_{\lambda_n}\|$ are comparable with 1, and hence the associated interpolation data space is always ℓ^2. In these circumstances, the normalisation of $\{k_{\lambda_n}\}$ preserves the Riesz basis property. Our results persist trivially. In fact, a simpler definition of Λ_σ may be used, by removing the normalisation factors.

4. Scope of applications

Our results apply to every space of entire functions $\mathcal{H}$ containing a Riesz basis of normalized reproducing kernels, in particular, to any de Branges space. Recall that $\mathcal{H}$ is a *de Branges space* if it satisfies the following axioms [2]:

(1) For all $w \in \mathbb{C}$ the linear functional $\mathcal{H} \to \mathbb{C}$, $f \mapsto f(w)$ is continuous;
(2) If $f \in \mathcal{H}$, then $f^*(z) = \overline{f(\bar{z})}$ is also in $\mathcal{H}$ with the same norm;
(3) If $f \in \mathcal{H}$ and $f(w) = 0$, then $f(z)(z-\bar{w})/(z-w)$ is also in $\mathcal{H}$ with the same norm.

By the first axiom, such a space admits reproducing kernels.

There is an intimate connection between de Branges spaces and the so-called *Hermite–Biehler functions*, that is, the entire functions satisfying $|E(\bar{z})| < |E(z)|$ for $\Im z > 0$. Indeed, a celebrated theorem of de Branges establishes that each de Branges space is isomorphically equal to a space of the form

$$\mathcal{H}(E) = \{f \text{ entire } ; \ f/E,\ f^*/E \in H^2(\mathbb{C}^+)\}$$

equipped with the norm $\|f\| = \|f/E\|_2$, where E is in the Hermite–Biehler class. Here, $H^2(\mathbb{C}^+)$ denotes the usual Hardy space,

$$H^2(\mathbb{C}^+) = \{f \text{ analytic in } \mathbb{C}^+ \ ; \ \sup_{y>0} \int_{-\infty}^{\infty} |f(t+\mathrm{i}y)|^2 \,\mathrm{d}t < \infty\}.$$

Without loss of generality we assume that E does not vanish on the real axis.[2] In particular, E has a polar decomposition on the real line, $E(x) = |E(x)|\mathrm{e}^{-\mathrm{i}\varphi(x)}$, where $\varphi(x)$ is the so-called *phase.*

Theorem 1 (de Branges)**.** *Given an Hermite–Biehler function E of phase φ, let $\{\lambda_n\}$ be the solution set of* $\sin(\varphi(x) - \alpha) = 0$, *where* $0 \le \alpha < 2\pi$. *Except*

[2]If it does, then all elements in $\mathcal{H}(E)$ inherit the real zeroes of E with multiplicity. Then, removal of these common zeroes is an isometry from $\mathcal{H}(E)$ to a de Branges space with an Hermite–Biehler function of the right kind.

for at most one exceptional α, the system of normalized reproducing kernels $\{\tilde{k}_{\lambda_n}\}$ is an orthonormal basis of $\mathcal{H}(E)$.

The existence of an orthonormal basis $\{\tilde{k}_{\lambda_n}\}$ of normalized reproducing kernels is thus granted. The associated interpolation data space is

$$\mathcal{D}_\lambda = \{\{d_n\}\ ;\ \sum |d_n|^2/\|k_{\lambda_n}\|^2 < \infty\},$$

so $\lambda = \{\lambda_n\}$ is complete interpolating:

$$f(\lambda_n) = d_n \Leftrightarrow f = \sum \frac{d_n}{\|k_{\lambda_n}\|}\tilde{k}_{\lambda_n}. \tag{4}$$

An explicit formula for the reproducing kernel is also available [2]:

$$k_w(z) = \frac{E^*(z)\overline{E^*(w)} - E(z)\overline{E(w)}}{2\pi\mathrm{i}(z-\bar{w})}.$$

In particular, for $\lambda_n \in \lambda$, using the fact that $\lambda_n \in \mathbb{R}$,

$$k_{\lambda_n}(z) = \frac{E^*(z)E(\lambda_n) - E(z)\overline{E(\lambda_n)}}{2\pi\mathrm{i}(z-\lambda_n)},$$

$$\|k_{\lambda_n}\|^2 = k_{\lambda_n}(\lambda_n) = (1/\pi)\varphi'(\lambda_n)|E(\lambda_n)|^2,$$

and hence

$$\tilde{k}_{\lambda_n}(z) = \frac{\mathrm{e}^{-\mathrm{i}\varphi(\lambda_n)}E^*(z) - \mathrm{e}^{\mathrm{i}\varphi(\lambda_n)}E(z)}{\mathrm{i}\sqrt{\pi\varphi'(\lambda_n)}(z-\lambda_n)}.$$

In the seminal case where $\mathcal{H}(E) = L^2_\pi$, one may let $E(z) = \mathrm{e}^{-\mathrm{i}\pi z}$, so $\varphi(t) = \pi t$. The value $\alpha = 0$ in de Branges' theorem is valid, so $\lambda = \mathbb{Z}$ are the nodes of the Riesz basis of reproducing kernels $\{k_n\}_{n\in\mathbb{Z}}$. Notice that the k_n are already normalized, so the corresponding space of interpolation data is ℓ^2. Indeed, $k_w(z)$ is given by (1), so the expansion (4) corresponds to (2). In this classical case, a theorem of Plancherel and Pólya ensures that all complete interpolating sequences are universally complete interpolating. There, the correspondence between complete interpolating sequences and Riesz bases of normalized reproducing kernels is celebrated.

In the more general case where $\mathcal{H}$ is an arbitrary de Branges space, the presence of an orthonormal basis $\{\tilde{k}_{\lambda_n}\}$ ensures that our propositions hold.

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Simon R. Cowell
Department of Mathematical Sciences,
United Arab Emirates University,
Al Ain, Abu Dhabi, UAE, P.O. Box 15551
E-mail address: scowell@uaeu.ac.ae

Philippe Poulin
Department of Mathematical Sciences,
United Arab Emirates University,
Al Ain, Abu Dhabi, UAE, P.O. Box 15551
E-mail address: philippepoulin@uaeu.ac.ae