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SOME NEW LACUNARY STRONG CONVERGENT VECTOR-VALUED MULTIPLIER DIFFERENCE SEQUENCE SPACES DEFINED BY A MUSIELAK-ORLICZ FUNCTION

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Abstract. In the present paper we introduce some vector-valued multiplier difference sequence spaces defined by a Musielak-Orlicz function, concepts of lacunary convergence and strong (A, u)-convergence, where $A = (a_{ik})$ is an infinite matrix of complex numbers and $u = (u_i)$ be any sequence of strictly positive real numbers. We also make an effort to study some topological properties and some inclusion relations between these spaces.

1. Introduction and Preliminaries

The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [4] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$. Let w be the space of all real or complex sequences $x = (x_k)$. Let m, n be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+n})$ and $\Delta_n^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+nv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [9].

An Orlicz function $M\colon [0,\infty) \to [0,\infty)$ is convex and continuous such that $M(0) = 0$, $M(x) > 0$ for $x > 0$. Lindenstrauss and Tzafriri [11] used the idea

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of Orlicz function to define the following sequence space,

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

It is shown in [11] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. An Orlicz function M satisfies Δ_2-condition if and only if for any constant $L > 1$ there exists a constant $K(L)$ such that $M(Lu) \leq K(L)M(u)$ for all values of $u \geq 0$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function (see [13, 16]). A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) : u \geq 0\}, \quad k = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function $\mathcal{M}$. For a given Musielak-Orlicz function $\mathcal{M}$, the Musielak-Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\},$$

$$h_{\mathcal{M}} = \{x \in w : I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), \quad x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$||x|| = \inf\left\{ k > 0 : I_{\mathcal{M}}\left(\frac{x}{k}\right) \leq 1 \right\}$$

or equipped with the Orlicz norm

$$||x||^0 = \inf\left\{ \frac{1}{k}\left(1 + I_{\mathcal{M}}(kx)\right) : k > 0 \right\}.$$

A Musielak-Orlicz function (M_k) is said to satisfy Δ_2-condition if there exist constants $a, K > 0$ and a sequence $c = (c_k)_{k=1}^{\infty} \in \ell_+^1$ (the positive cone of ℓ^1) such that the inequality

$$M_k(2u) \leq KM_k(u) + c_k$$

holds for all $k \in \mathbb{N}$ and $u \in R_+$, whenever $M_k(u) \leq a$.

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;
(2) $p(-x) = p(x)$, for all $x \in X$;
(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$;
(4) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [25, Theorem 10.4.2]). For more details about sequence spaces (see [14, 17, 18, 20, 19]) and references therein.

The space of lacunary strong convergence have been introduced by Freedman et al. [6]. A sequence of positive integers $\theta = (k_r)$ is called *lacunary* if $k_0 = 0, 0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$, as $r \to \infty$. The intervals determined by θ are denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. the space of lacunary strongly convergent sequences N_θ is defined by Freedman et al. [6] as follows:

$$N_\theta = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} |x_i - s| = 0, \text{ for some } s \right\}.$$

The space $|\sigma_1|$ of strongly Cesaro summable sequences is

$$|\sigma_1| = \left\{ x = (x_k) : \text{ there exists } L \text{ such that } \frac{1}{n} \sum_{i=1}^{n} |x_i - L| \to 0, \text{ as } n \to \infty \right\}.$$

In case, when $\theta = (2^r), N_\theta = |\sigma_1|$. Recently, Bilgin [1] in his paper generalized the concept of lacunary convergence and introduced the space $N_\theta(A, f)$, as

$$N_\theta(A, f) = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} f(|A_i(x) - s|) = 0, \text{ for some } s \right\},$$

where f is a modulus function and $A = (A_i(x))$, $A_i x = \sum_{k=1}^{\infty} a_{ik} x_k$ converges for each i. Later Bilgin [2] generalized lacunary strongly A-convergent sequences with respect to a sequence of modulus function $F = (f_i)$ as follows:

$$N_\theta(A, F) = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{i \in I_r} f_i(|A_i(x) - s|) = 0, \text{ for some } s \right\}.$$

By $A_i(\Delta_n^m x_k)$ we mean

$$A_i(\Delta_n^m x_k) = \sum_{k=1}^{\infty} a_{ik} (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1}).$$

The main aim of this paper is to introduce the concept of lacunary strongly (A,u)-convergence for difference sequences with the elements chosen from a Banach space $(E,\|.\|)$ over the complex field $\mathbb{C}$, with respect to a Musielak-Orlicz function $\mathcal{M}=(M_i)$. We have studied some topological properties and also make an effort to study some inclusion relations between below defined sequence spaces.

Let $A=(a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M}=(M_i)$ be a Musielak-Orlicz function and $u=(u_i)$ be any sequence of strictly positive real numbers. In the present paper we define the following sequence spaces:

$$\Delta_n^m N_\theta(E,A,u,\mathcal{M}) = \Big\{x=(x_k)\in E : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i\left[M_i\left(\frac{\|A_i(\Delta_n^m x_k)-s_ie_i\|}{\rho^{(i)}}\right)\right]=0 \text{ for some } s=(s_1,s_2,\dots)\in E, e_i\in\mathbb{C} \text{ and } \rho^{(i)}>0\Big\}$$

and

$$\Delta_n^m N_\theta^0(E,A,u,\mathcal{M}) = \Big\{x=(x_k)\in E : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i\left[M_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\right]=0 \text{ for some } \rho^{(i)}>0\Big\}$$

If we take $\mathcal{M}(x)=x$, we have

$$\Delta_n^m N_\theta(E,A,u) = \Big\{x=(x_k)\in E : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i\left[\frac{\|A_i(\Delta_n^m x_k)-s_ie_i\|}{\rho^{(i)}}\right]=0 \text{ for some } s=(s_1,s_2,\dots)\in E, e_i\in\mathbb{C} \text{ and } \rho^{(i)}>0\Big\}.$$

and

$$\Delta_n^m N_\theta^0(E,A,u) = \Big\{x=(x_k)\in E : \lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i\left[\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right]=0 \text{ for some } \rho^{(i)}>0\Big\},$$

where

$$e_i=\begin{cases}1, & \text{at the } i^{\text{th}} \text{ place},\\ 0, & \text{otherwise.}\end{cases}$$

2. Some topological properties of the spaces $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$

Theorem 2.1. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function and $u = (u_i)$ be any sequence of strictly positive real numbers. Then $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Suppose that $x = (x_k), y = (y_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $(x_k) \xrightarrow{\Delta_n^m} s, (y_k) \xrightarrow{\Delta_n^m} t$, then for some $s = (s_1, s_2, \ldots), t = (t_1, t_2, \ldots) \in E$, $\quad e_i \in \mathbb{C}$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho_1^{(i)}} \right) \right] = 0, \text{ for some } \rho_1^{(i)} > 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m y_k) - t_i e_i\|}{\rho_2^{(i)}} \right) \right] = 0, \text{ for some } \rho_2^{(i)} > 0.$$

Let $\alpha, \beta \in \mathbb{C}$. Without loss of generality we may assume that there exists $P_1 > 1, P_2 > 1$ such that $|\alpha| \le P_1$ and $|\beta| \le P_2$. Let $\rho^{(i)} = \max(2\rho_1^{(i)}, 2\rho_2^{(i)})$. Then

$$\begin{aligned}
&\lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\alpha \Delta_n^m x_k + \beta \Delta_n^m y_k) - (\alpha s_i e_i + \beta t_i e_i\|)}{\rho^{(i)}} \right) \right] \\
&\quad \le \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|\alpha A_i(\Delta_n^m x_k) - \alpha s_i e_i\| + \|\beta A_i(\Delta_n^m y_k) - \beta t_i e_i\|}{\rho^{(i)}} \right) \right] \\
&\quad = \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{|\alpha| \|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho_1^{(i)}} \right) \right] \\
&\qquad + \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{|\beta| \|A_i(\Delta_n^m y_k - t_i e_i)\|}{\rho_2^{(i)}} \right) \right] \\
&\quad \le \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{P_1 \|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho_1^{(i)}} \right) \right] \\
&\qquad + \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{P_2 \|A_i(\Delta_n^m y_k) - t_i e_i\|}{\rho_2^{(i)}} \right) \right]
\end{aligned}$$

$$\leq K_1 \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho_1^{(i)}} \right) \right]$$

$$+ K_2 \lim_{r\to\infty} \frac{1}{h_r} \sum_{i\in I_r} \frac{1}{2} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m y_k) - t_i e_i\|}{\rho_2^{(i)}} \right) \right]$$

$$\to 0 \text{ as } r \to \infty.$$

Therefore, $(\alpha x_k + \beta y_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

This proves that $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$ is a linear space. Similarly we can prove that $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$ is a linear space. □

Theorem 2.2. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function and $u = (u_i)$ be any sequence of strictly positive real numbers. Then $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$ are normal spaces, when E is normal.*

Proof. Let $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $(x_k) \xrightarrow{\Delta_n^m} s$, where $s = (s_1, s_2, \ldots) \in E$, $e_i \in \mathbb{C}$. Let $\|y_k\| \leq \|x_k\|$. Then

$$\|A_i(\Delta_n^m y_k) - s_i e_i\| \leq \|A_i(\Delta_n^m x_k) - s_i e_i\|.$$

Since, $\mathcal{M} = (M_i)$ is an increasing,

$$\frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m y_k) - s_i e_i\|}{\rho^{(i)}} \right) \right] \leq \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}} \right) \right]$$

Consequently, $y = (y_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$. This completes the proof of the theorem. Similarly, we can prove that $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$ is normal space. □

Theorem 2.3. *The spaces $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $\Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$ are paranormed spaces, with respect to the paranorm*

$$\|x\|_{\Delta_n^m} = \inf \Bigg\{ \rho^{(i)} > 0 : u_i M_i \left(\frac{\|a_{i0} x_1\|}{\rho^{(i)}} \right) + $$
$$+ \sup_{r\geq 1} \frac{1}{h_r} \sum_{i\in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}} \right) \right] \leq 1, \ \ \rho^{(i)} \geq 0 \Bigg\}.$$

It is easy to prove, so we omit the details.

3. Relation between the spaces $\Delta_n^m N_\theta(E, A, u)$ and $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$

The main purpose of this section is to study some interesting relations between the spaces $\Delta_n^m N_\theta(E, A, u)$ and $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

Theorem 3.1. *Let $A = (a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function satisfying Δ_2-condition and $u = (u_i)$ be any sequence of strictly positive real numbers. If $x = (x_k)$ is Δ_n^m-lacunary strong*

(A,u)-convergent to s, with respect to $\mathcal{M}$ and $(E,\|.\|)$ is a normal Banach space, then $\Delta_n^m N_\theta(E,A,u) \subset \Delta_n^m N_\theta(E,A,u,\mathcal{M})$.

Proof. Let $x=(x_k)\in \Delta_n^m N_\theta(E,A,u)$ and $(x_k)\xrightarrow{\Delta_n^m} s$, where $s=(s_1,s_2,\ldots)\in E, e_i\in\mathbb{C}$. Then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}u_i\left(\frac{\|A_i(\Delta_n^m x_k)-s_ie_i\|}{\rho}\right)=0 \text{ for some } \rho>0.$$

We define two sequences $y=(y_k)$ and $z=(z_k)$ such that

$$u_i(\|A_i(\Delta_n^m y_k)-s_ie_i\|) = \begin{cases} u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|), & \text{if } u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)>1,\\ 0, & \text{if } u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)\le 1,\end{cases}$$

and

$$u_i(\|A_i(\Delta_n^m z_k)-s_ie_i\|) = \begin{cases} 0, & \text{if } u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)>1,\\ u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|), & \text{if } u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)\le 1.\end{cases}$$

Hence

$$u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)=u_i(\|A_i(\Delta_n^m y_k)-s_ie_i\|)+u_i(\|A_i(\Delta_n^m z_k)-s_ie_i\|).$$

Now,

$$u_i(\|A_i(\Delta_n^m y_k)-s_ie_i\|)\le u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|)$$

and

$$u_i(\|A_i(\Delta_n^m z_k)-s_ie_i\|)\le u_i(\|A_i(\Delta_n^m x_k)-s_ie_i\|).$$

Since $\Delta_n^m N_\theta(E,A,u)$ is normal, $y=(y_k),\ z=(z_k)\in\Delta_n^m N_\theta(E,A,u)$. Let $\sup_i M_i(2)=T$. Then

$$
\begin{aligned}
&\frac{1}{h_r}\sum_{i\in I_r}u_i\left[M_i\left(\frac{\|A_i(\Delta_n^m x_k)-s_ie_i\|}{\rho^{(i)}}\right)\right]\\
=&\frac{1}{h_r}\sum_{i\in I_r}u_i\left[M_i\left(\frac{\|A_i(\Delta_n^m y_k)-s_ie_i\|+\|A_i(\Delta_n^m z_k)-s_ie_i\|}{\rho^{(i)}}\right)\right]\\
\le&\frac{1}{h_r}\sum_{i\in I_r}u_i\left[\frac{1}{2}M_i\left(2\frac{\|A_i(\Delta_n^m y_k)-s_ie_i\|}{\rho^{(i)}}\right)+\frac{1}{2}M_i\left(2\frac{\|A_i(\Delta_n^m z_k)-s_ie_i\|}{\rho^{(i)}}\right)\right]\\
<&\frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}u_iK_1\left(\frac{\|A_i(\Delta_n^m y_k)-s_ie_i\|}{\rho^{(i)}}\right)M_i(2)\\
&+\frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}u_iK_2\left(\frac{\|A_i(\Delta_n^m z_k)-s_ie_i\|}{\rho^{(i)}}\right)M_i(2)\\
\le&\frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}u_iK_1\left(\frac{\|A_i(\Delta_n^m y_k)-s_ie_i\|}{\rho^{(i)}}\right)\sup M_i(2)\\
&+\frac{1}{2}\frac{1}{h_r}\sum_{i\in I_r}u_iK_2\left(\frac{\|A_i(\Delta_n^m z_k)-s_ie_i\|}{\rho^{(i)}}\right)\sup M_i(2)\\
\to&0 \text{ as } r\to\infty.
\end{aligned}
$$

Hence $x=(x_k)\in\Delta_n^m N_\theta(E,A,u,\mathcal{M})$. This completes the proof of the theorem. □

Theorem 3.2. *Let $A=(a_{ik})$ be an infinite matrix of complex numbers, $\mathcal{M}=(M_i)$ be a Musielak-Orlicz function satisfying Δ_2-condition and $u=(u_i)$ be any sequence of strictly positive real numbers. If*

$$\lim_{v\to\infty}\inf_i\frac{u_iM_i(\frac{v}{\rho^{(i)}})}{\frac{v}{\rho^{(i)}}}>0 \ \textit{for some}\ \rho^{(i)}>0,$$

then $\Delta_n^m N_\theta(E,A,u)=\Delta_n^m N_\theta(E,A,u,\mathcal{M})$.

Proof. If $\lim_{v\to\infty}\inf_i\frac{u_iM_i(\frac{v}{\rho^{(i)}})}{\frac{v}{\rho^{(i)}}}>0$ for some $\rho^{(i)}>0$, then there exists a number $\gamma>0$ such that

$$u_iM_i(\frac{v}{\rho^{(i)}})\ge u_i\gamma(\frac{v}{\rho^{(i)}});\ \text{for all } v>0 \text{ and some } \rho^{(i)}>0.$$

Let $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$ and $(x_k) \xrightarrow{\Delta_n^m} s$, where $s = (s_1, s_2, \ldots) \in E, e_i \in \mathbb{C}$. Then clearly

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} u_i\left[M_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\right] &\geq \frac{1}{h_r}\sum_{i\in I_r} u_i\left[\gamma\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\right]\\ &= \gamma\frac{1}{h_r}\sum_{i\in I_r} u_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\end{aligned}$$

Hence $x = (x_k) \in \Delta_n^m N_\theta(E, A, u)$. This completes the proof. □

4. Some attractive inclusions between the spaces $|\Delta_n^m \sigma_1(A, u)|$ and $\Delta_n^m N_\theta(E, A, u)$

A sequence $x = (x_k)$ is said to be Δ_n^m-lacunary strong (A, u)-convergent with respect to a Musielak-Orlicz function $\mathcal{M} = (M_i)$ if there is a number $s = (s_1, s_2, \ldots) \in E$ such that $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

We have generalized the strongly Cesaro-summable sequence spaces into Δ_n^m-strongly Cesaro-summable vector-valued sequence spaces as

$$\begin{aligned}|\Delta_n^m \sigma_1(A, u)| = \Bigg\{ x = (x_k) :\ &\text{there exists } L = (L_1, L_2, \ldots) \in E, e_i \in \mathbb{C}\\ &\text{such that } \frac{1}{n}\sum_{i=1}^{n} u_i\|A_i(\Delta_n^m x_k) - L_i e_i\| \to 0\Bigg\},\end{aligned}$$

where $A = (a_{nk})$ is a Cesaro matrix, i.e.,

$$a_{nk} = \begin{cases} \frac{1}{n}, & \text{if } 1 \leq k \leq n,\\ 0, & \text{if } k \geq n.\end{cases}$$

Then it can be shown that $|\Delta_n^m \sigma_1(A, u)|$ is a paranormed space with respect to the paranorm

$$\|x\| = \|x_1\| + \sup_n\left(\frac{1}{n}\sum_{i=1}^{n} u_i\|A_i(\Delta_n^m x_k)\|\right).$$

In this section of the paper we study relation between the spaces $|\Delta_n^m \sigma(A, u)|$ and $\Delta_n^m N_\theta(E, A, u)$.

Lemma 4.1. $|\Delta_n^m \sigma_1(A, u)| \subset \Delta_n^m N_\theta(E, A, u)$ *if and only if* $\liminf_r q_r > 1$.

Proof. First, suppose that $\liminf_r q_r > 1$. Then there exist $\delta > 0$ such that $1+\delta \le q_r$ for all $r \ge 1$. Let $x \in |\Delta_n^m \sigma_1(A,u)|^0$. Then

$$\begin{aligned}\frac{1}{h_r}\sum_{i\in I_r} u_i\|A_i(\Delta_n^m x_k)\| &= \frac{1}{h_r}\sum_{i=1}^{k_r} u_i\|A_i(\Delta_n^m x_k)\| - \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} u_i\|A_i(\Delta_n^m x_k)\| \\ &= \frac{k_r}{h_r}\left(\frac{1}{k_r}\sum_{i=1}^{k_r} u_i\|A_i(\Delta_n^m x_k)\|\right) \\ &\quad -\frac{k_{r-1}}{h_r}\left(\frac{1}{k_{r-1}}\sum_{i=1}^{k_{r-1}} u_i\|A_i(\Delta_n^m x_k)\|\right).\end{aligned}$$

Now, $h_r = k_r - k_{r-1}$. So we have

$$\frac{k_r}{h_r} = \frac{k_r}{k_r - k_{r-1}} = \frac{q_r}{q_r-1} = 1 + \frac{1}{q_r-1} \le 1 + \frac{1}{\delta} = \frac{\delta+1}{\delta}.$$

Also

$$\frac{k_{r-1}}{h_r} = \frac{k_{r-1}}{k_r - k_{r-1}} = \frac{1}{q_r-1} \le \frac{1}{\delta}.$$

Since $x \in |\Delta_n^m \sigma_1(A,u)|^0$, then

$$\frac{1}{h_r}\sum_{i=1}^{k_r} u_i\|A_i(\Delta_n^m x_k)\| \to 0 \text{ and } \frac{1}{h_r}\sum_{i=1}^{k_{r-1}} u_i\|A_i(\Delta_n^m x_k)\| \to 0,$$

and hence

$$\frac{1}{h_r}\sum_{i\in I_r} u_i\|A_i(\Delta_n^m x_k)\| \to 0$$

i.e., $x = (x_k) \in \Delta_n^m N_\theta^0(E,A,u)$. By linearity, it follows that $|\Delta_n^m \sigma_1(A,u)| \subset \Delta_n^m N_\theta(E,A,u)$.

Next, suppose that $\liminf_r q_r = 1$. Since θ is lacunary we can select a subsequence k_{r_j} of θ such that

$$\frac{k_{r_j}}{k_{r_j-1}} < 1 + \frac{1}{j} \text{ and } \frac{k_{r_j-1}}{k_{r_{j-1}}} > j$$

where $r_j \ge r_{j-1}+2$. Define $x = (x_i)$ by

$$u_i\Delta_n^m x_i = \begin{cases} e_i, & \text{if } i \in I_{r_j}, \text{ for some } j = 1,2,\ldots, \\ 0, & \text{otherwise,}\end{cases}$$

where $\|e_i\| = 1$ and let $A = I$, then for any $L = (L_1, L_2, \cdots) \in E, e_i \in \mathbb{C}$,

$$\frac{1}{h_{r_j}}\sum_{i\in I_r} u_i\left(\frac{\|A_i(\Delta_n^m x_k) - L\|}{\rho^{(i)}}\right) = \frac{\|e_i - L_i e_i\|}{\rho^{(i)}} = \frac{\|1 - L_i\|}{\rho^{(i)}} \text{ for } j = 1,2,\ldots$$

and

$$\frac{1}{h_r}\sum_{i\in I_r} u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)=\frac{\|e_i\|}{\rho^{(i)}}=\frac{1}{\rho^{(i)}}.$$

So, $x=(x_k)\notin \Delta_n^m N_\theta(E,A,u)$. But, $x=(x_k)$ is strongly Cesaro-summable, since if t is sufficiently large integer we can find the unique j for which $k_{r_j-1}<t\leq k_{r_{j+1}-1}$ and hence

$$\frac{1}{t}\sum_{i=1}^{t} u_i\left(\|A_i(\Delta_n^m x_k)\|\right)<\frac{1}{k_{r_j-1}}\sum_{i=1}^{t}1\leq \frac{1}{k_{r_j-1}}k_{r_j}\leq \frac{k_{r_j-1}+h_{r_j}}{k_{r_j-1}}<\frac{1}{j}+\frac{1}{j}=\frac{2}{j}$$

as $t\to\infty$, it follows that also $j\to\infty$. Hence $x=(x_k)\in|\Delta_n^m\sigma_1(A,u)|$. □

Lemma 4.2. $\Delta_n^m N_\theta(E,A,u)\subset|\Delta_n^m\sigma_1(A,u)|$ *if and only if* $\limsup_r q_r<\infty$.

Proof. First, suppose that if $\limsup_r q_r<\infty$, there exists $M>0$ such that $q_r<M$ for all $r\geq 1$. Let $x=(x_k)\in\Delta_n^m N_\theta(E,A,u)$ and $\epsilon>0$. Then

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)=0 \text{ for some } \rho>0.$$

Then we can find $R>0$ and $K>0$ such that

$$\sup_{j\geq R}\frac{1}{h_j}\sum_{I_j} u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)<\epsilon$$

and

$$\frac{1}{h_j}\sum_{I_j} u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)<K \text{ for all } i=1,2,\ldots.$$

Then if t is any integer with

$$k_{r-1}\leq t\leq k_r, \text{ where } r>R,$$

then

$$\begin{aligned}
&\frac{1}{t}\sum_{j=1}^{t}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&\quad\leq\frac{1}{k_{r-1}}\sum_{i=1}^{k_r}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&\quad=\frac{1}{k_{r-1}}\left(\sum_{I_1}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)+\sum_{I_2}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)+\cdots\right.\\
&\qquad\left.+\sum_{I_r}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\right)\\
&\quad=\frac{k_1}{k_{r-1}}\frac{1}{h_1}\sum_{I_1}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)+\frac{k_2-k_1}{k_{r-1}}\frac{1}{h_2}\sum_{I_2}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)+\cdots\\
&\qquad+\frac{k_R-k_{R-1}}{k_{r-1}}\frac{1}{h_R}\sum_{I_R}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&\qquad+\frac{k_{R+1}-k_R}{k_{r-1}}\frac{1}{h_{R+1}}\sum_{I_{R+1}}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&\qquad+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}\frac{1}{h_r}\sum_{I_r}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&\leq\frac{k_R}{k_{r-1}}\sup_{i\geq 1}\frac{1}{h_i}\sum_{I_i}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)+\frac{k_r-k_R}{k_{r-1}}\frac{1}{h_r}\sum_{I_r}u_i\left(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\right)\\
&<K\frac{k_R}{k_{r-1}}+\epsilon\left(q_r-\frac{k_R}{k_{r-1}}\right)<K\frac{k_R}{k_{r-1}}+\epsilon q_r<K\frac{k_R}{k_{r-1}}+\epsilon M.
\end{aligned}$$

Since $k_{r-1}\to\infty$ as $r\to\infty$, it follows that

$$\frac{1}{t}\sum_{j=1}^{t}u_i\Big(\frac{\|A_i(\Delta_n^m x_k)\|}{\rho^{(i)}}\Big)\to 0$$

and hence $x=(x_k)\in|\Delta_n^m\sigma_1(A,u)|$.

Next, suppose that $\limsup_r q_r=\infty$. Now, we construct a sequence in $\Delta_n^m N_\theta(E,A,u)$ that is not Cesaro Δ_n^m-summable. By the idea of Freedman [6] we can construct a subsequence k_{r_j} of the lacunary sequence $\theta=(k_r)$ such that $q_{r_j}>j$, and then define a bounded difference sequence $x=(x_i)$ by

$$u_i\Delta_n^m x_i=\begin{cases}e_i, & \text{if } k_{r_j-1}<i<2k_{r_j-1},\\ 0, & \text{otherwise,}\end{cases}$$

where $\|e_i\| = 1$. Let $A = I$ and $\rho = 1$. Then

$$\frac{1}{h_{r_j}} \sum_{I_{r_j}} u_i \left(\|A_i(\Delta_n^m x_k)\|\right) = \frac{2k_{r_j-1} - k_{r_j-1}}{k_{r_j} - k_{r_j-1}} = \frac{k_{r_j-1}}{k_{r_j} - k_{r_j-1}} < \frac{1}{j-1}$$

and if $r \neq r_j$,

$$\frac{1}{h_{r_j}} \sum_{I_{r_j}} u_i \left(\|A_i(\Delta_n^m x_k)\|\right) = 0.$$

Thus $x = (x_k) \in \Delta_n^m N_\theta(E, A, u)$. For the above sequence and for $i = 1, 2, \ldots, k_{r_j}$

$$\begin{aligned}\frac{1}{k_{r_j}} \sum_{i} u_i \left(\|A_i(\Delta_n^m x_k) - e_i\|\right) &> \frac{1}{k_{r_j}}(2k_{r_j-1} - k_{r_j-1}) \\ &= 1 - \frac{2}{q_{r_j}} > 1 - \frac{2}{j},\end{aligned}$$

this converges to 1, but for $i = 1, 2, \ldots, 2k_{r_j-1}$

$$\frac{2}{k_{r_{j-1}}} \sum_{i} u_i \left(\|A_i(\Delta_n^m x_k)\|\right) \geq \frac{k_{r_j-1}}{2k_{r_j-1}} = \frac{1}{2}.$$

It proves that $x = (x_k) \notin |\Delta_n^m \sigma_1(A, u)|$, since any sequence in $|\Delta_n^m \sigma_1(A, u)|$ consisting of 0's and e_i's has an limit only 0 or 1. $\square$

Theorem 4.3. *Let θ be a lacunary sequence. Then*

$$|\Delta_n^m \sigma_1(A, u)| = \Delta_n^m N_\theta(E, A, u)$$

if and only if

$$1 \leq \liminf_r q_r \leq \limsup_r q_r < \infty.$$

The proof of this theorem follows from Lemma 4.1 and Lemma 4.2.

5. Statistical Convergence

The notion of statistical convergence was introduced by Fast [5] and Schoenberg [23] independently. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [7], Connor [3], Salat [21], Mursaleen [15], Isık [8], Savaş [22], Malkosky and Savaş [14], Kolk [10], Maddox [12], Tripathy and Sen [24] and many others. In recent years, generalizations of statistical convergence have appeared in the study of strong integral summability and the structure of ideals of bounded continuous functions on locally compact spaces. Statistical convergence and its generalizations are also connected with subsets of the stone-Cech compactification of natural numbers. Moreover, statistical convergence is closely related to the concept of convergence in probability. The notion depends on the density of subsets of the set $\mathbb{N}$ of natural numbers.

A subset E of $\mathbb{N}$ is said to have the natural density $\delta(E)$ if the following limit exists:

$$\delta(E) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_E(k),$$

where χ_E is the characteristic function of E. It is clear that any finite subset of $\mathbb{N}$ has zero natural density and $\delta(E^c) = 1 - \delta(E)$.

A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for every $\epsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \Big| \big\{ k \leq n : |x_k - L| \geq \epsilon \big\} \Big| = 0.$$

Bilgin [2] also introduced the concept of statistical convergence in $N_0(A, F)$ and proved some inclusion relation.

Let θ be a lacunary sequence and $A = (a_{ik})$ be an infinite matrix of complex numbers. Then a sequence $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$ is said to be Δ_n^m-lacunary (A, u)-statistically convergent to a number $s = (s_1, s_2, \ldots) \in E, e_i \in \mathbb{C}$ if for any $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} u_i |\Delta_n^m A_0(\epsilon)| = 0,$$

where

$$\Delta_n^m A_0(\epsilon) = \left\{ i \in I_r : u_i M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}} \right) \geq \epsilon \right\}.$$

We denote it as $(x_k) \overset{\Delta_n^m - \text{ stat}}{\longrightarrow} s$. The vertical bar denotes the cardinality of the set. The set of all Δ_n^m-lacunary (A, u)-statistical convergent sequences is denoted by $\Delta_n^m S_\theta(A, u)$.

In this section we study some relations between the spaces $|\Delta_n^m S_\theta(A, u)|$ and $\Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

Theorem 5.1. *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function and (M_i) be pointwise convergent. Then*

$$\Delta_n^m N_\theta(E, A, u, \mathcal{M}) \subset \Delta_n^m S_\theta(A, u)$$

if and only if

$$\lim_i u_i M_i(\frac{v}{\rho^{(i)}}) > 0 \text{ for some } v > 0,\ \rho^{(i)} > 0.$$

Proof. Let $\epsilon > 0$ and $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

Let $(x_k) \overset{\Delta_n^m}{\longrightarrow} s$, where $s = (s_1, s_2, \ldots) \in E$, $e_i \in \mathbb{C}$. Since $\lim_i u_i M_i(\frac{v}{\rho^{(i)}}) > 0$, there exists a number $c > 0$ such that

$$u_i M_i(\frac{v}{\rho^{(i)}}) \geq c \text{ for } v > \epsilon.$$

Let

$$I_r^1 = \left\{ i \in I_r : u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}} \right) \right] \geq \epsilon \right\}.$$

Then

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in I_r} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}} \right) \right] &\geq \frac{1}{h_r} \sum_{i \in I_r^1} u_i \left[M_i \left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}} \right) \right] \\ &\geq c \frac{1}{h_r} |\Delta_n^m A_0(\epsilon)|. \end{aligned}$$

Hence, it follows that $x = (x_k) \in \Delta_n^m S_\theta(A, u)$.

Conversely, let us assume that the condition does not hold. Then there is a number $v > 0$ such that $\lim_i u_i M_i(\frac{v}{\rho^{(i)}}) = 0$ for some $\rho > 0$. Now, we select a lacunary sequence $\theta = (k_r)$ such that $u_i M_i(\frac{v}{\rho^{(i)}}) < 2^{-r}$ for any $i > k_r$.

Let $A = I$, define the sequence $x = (x_k)$ by putting

$$\Delta_n^m x_i = \begin{cases} v, & \text{if } k_{r-1} < i \leq \frac{k_r + k_{r-1}}{2}, \\ 0, & \text{if } \frac{k_r + k_{r-1}}{2} < i \leq k_r. \end{cases}$$

Therefore,

$$\begin{aligned} \frac{1}{h_r} \sum_{i \in I_r} M_i \left(\frac{\|\Delta_n^m x_i\|}{\rho^{(i)}} \right) &= \frac{1}{h_r} \sum_{k_{r-1} < i \leq \frac{k_r + k_{r-1}}{2}} M_i \left(\frac{v}{\rho^{(i)}} \right) \\ &< \frac{1}{h_r} \frac{1}{2^{r-1}} \left[\frac{k_r + k_{r-1}}{2} - k_{r-1} \right] \\ &= \frac{1}{2^r} \to 0 \text{ as } r \to \infty. \end{aligned}$$

Thus, we have $x = (x_k) \in \Delta_n^m N_\theta^0(E, A, u, \mathcal{M})$. But,

$$\begin{aligned} &\lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ i \in I_r : M_i \left(\frac{\|\Delta x_i\|}{\rho^{(i)}} \right) \geq \epsilon \right\} \right| = \\ &\quad = \lim_{r \to \infty} \frac{1}{h_r} \left| \left\{ i \in (k_{r-1}, \frac{k_r + k_{r-1}}{2}) : M_i \left(\frac{v}{\rho^{(i)}} \right) \geq \epsilon \right\} \right| \\ &\quad = \lim_{r \to \infty} \frac{1}{h_r} \frac{k_r - k_{r-1}}{2} = \frac{1}{2}. \end{aligned}$$

So $x = (x_k) \notin \Delta_n^m S_\theta(A, u)$. □

Theorem 5.2. *Let $\mathcal{M} = (M_i)$ be a Musielak-Orlicz function and $u = (u_i)$ be any sequence of strictly positive real numbers. Then*

$$\Delta_n^m S_\theta(A, u) \subset \Delta_n^m N_\theta(E, A, u, \mathcal{M})$$

if and only if

$$\sup_v \sup_i u_i M_i(\frac{v}{\rho^{(i)}}) < \infty.$$

Proof. Let $x = (x_k) \in \Delta_n^m S_\theta(A,u)$ and $(x_k) \overset{\Delta_n^m-\text{ stat}}{\longrightarrow} s$. Suppose $h(v) = \sup_i u_i M_i(\frac{v}{\rho^{(i)}})$ and $h = \sup_v h(v)$. Let

$$I_r^2 = \left\{ i \in I_r : M_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right) < \epsilon \right\}.$$

Now, $M_i(v) \leq h$ for all $i,\ v > 0$. So

$$\begin{aligned}
\frac{1}{h_r}\sum_{i\in I_r} u_i \left[M_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\right] = & \\
=&\frac{1}{h_r}\sum_{i\in I_r^1} u_i \left[M_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\right] \\
&+ \frac{1}{h_r}\sum_{i\in I_r^2} u_i \left[M_i\left(\frac{\|A_i(\Delta_n^m x_k) - s_i e_i\|}{\rho^{(i)}}\right)\right] \\
\leq& h\frac{1}{h_r}|\Delta_n^m A_0(\epsilon)| + h(\epsilon).
\end{aligned}$$

Hence, as $\epsilon \to 0$, it follows that $x = (x_k) \in \Delta_n^m N_\theta(E, A, u, \mathcal{M})$.

Conversely, suppose that

$$\sup_v \sup_i u_i M_i(\frac{v}{\rho^{(i)}}) = \infty.$$

Then we have

$$0 < v_1 < v_2 < \ldots < v_{r-1} < v_r < \ldots,$$

so that $M_{k_r}(\frac{v_r}{\rho^{(i)}}) \geq h_r$ for $r \geq 1$. Let $A = I$. We set a sequence $x = (x_i)$ by

$$u_i \Delta_n^m x_i = \begin{cases} v_r, & \text{if } i = k_r \text{ for some } r = 1, 2, \ldots, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{ i \in I_r : u_i \left[M_i\left(\frac{\|\Delta_n^m x_i\|}{\rho^{(i)}}\right)\right] \geq \epsilon \right\}\right| = \lim_{r\to\infty}\frac{1}{h_r} = 0.$$

Hence $(x_k) \overset{\Delta_n^m-\text{ stat}}{\longrightarrow} 0$ and hence $x = (x_k) \in \Delta_n^m S_\theta(A,u)$.

But,

$$\begin{aligned}
\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r} u_i \left[M_i\left(\frac{\|A_i(\Delta_n^m x_i) - s_i e_i\|}{\rho^{(i)}}\right)\right] &= \lim_{r\to\infty}\frac{1}{h_r}\left[M_{k_r}\left(\frac{v_r - s_i e_i\|}{\rho^{(i)}}\right)\right] \\
&\geq \lim_{r\to\infty}\frac{1}{h_r} h_r = 1.
\end{aligned}$$

So, $x = (x_k) \notin \Delta_n^m N_\theta(E, A, u, \mathcal{M})$. ∎

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