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SUFFICIENT CONDITIONS FOR THE $T(T_0)$-SOLVABILITY OF FINITE GROUPS

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Abstract. Let G be a finite group. We say that G is a T_0-group if its Frattini quotient group $G/\Phi(G)$ is a T-group, where by a T-group we mean a group in which every subnormal subgroup is normal. In this paper, we investigate the structure of the group G if G is the product of two solvable T-groups (T_0-groups) H and K such that H permutes with every subgroup of K and K permutes with every subgroup of H (that is, H and K are mutually permutable) and that $(|G : H|, |G : K|) = 1$. Some structure theorems are also discussed.

1. Introduction

Throughout this paper, all groups are assumed to be finite. The terminology and notions employed agree with standard usage, as in Doerk and Hawkes [8]. In addition, the set of distinct primes dividing $|G|$ will be denoted by$\pi(G)$.

A T-group is a group G in which normality is a transitive relation, that is, if $H \trianglelefteq K \trianglelefteq G$, then $H \trianglelefteq G$. T-groups were studied by Gaschütz [10] and he proved that every finite solvable T-group is a subgroup closed T-group (the group and all of its subgroups are T-groups). Recently, van der Waall and Fransman [16] introduced the concept of a T_0-group as a generalization of a T-group. A group G is said to be a T_0-group if $G/\Phi(G)$ is a T-group. It is clear that the class of T_0-groups contains the classes of T-groups and nilpotent groups. In contrast to the fact of Gaschü tz and the fact that every nilpotent group is a subgroup closed, it does not hold in general that a finite solvable T_0-group is a subgroup closed T_0-group, see; [16, Example 3.7, p. 66], see also Example 2.1 of Asaad and Heliel [2]. In [2], the authors determined the structure of a minimal non T_0-group (non T_0-group all of its proper subgroups are T_0-groups).

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Recall that a subgroup H of a group G is said to be permutable in G if $HK = KH$ for all subgroups K of G, and H is said to be S-permutable if $HP = PH$ for all Sylow subgroups P of G. Kegel [13] showed that an S-permutable subgroup of G is subnormal. From this it follows that S-permutability is a transitive relation in G (i.e. H is S-permutable in G whenever it is S-permutable in some S-permutable subgroup of G), precisely when every subnormal subgroup of G is S-permutable. Groups with this property are called PST-groups.

A group G is said to be factorized if it can be expressed as a product of two of its subgroups H and K, as follows: $G = HK$. A well-known theorem by Kegel and Wielandt asserts that G is solvable provided that H and K are nilpotent. This theorem has been the motivation for a number of results in the literature on factorized groups. For example, taking into account the fact that the product of two normal supersolvable subgroups of G is not necessarily supersolvable, Baer [5] proved that if G is the product of two normal supersolvable subgroups and G' (the commutator subgroup of G) is nilpotent, then G is supersolvable. Friesen [9] proved that if G is the product of two normal supersolvable groups of coprime indices, then G is supersolvable. Asaad and Shaalan [4] proved that if $G = HK$ is a mutually permutable product of the supersolvable subgroups H and K such that $(|H|, |K|) = 1$, then G is supersolvable. They also proved that if G is a product of the supersolvable subgroups H and K, then G is supersolvable if H and K are totally permutable (H and K are totally permutable if every subgroup of H permutes with every subgroup of K and vice-versa). Heliel [11] proved that if $G = HK$ is a mutually permutable product of the subgroups H and K such that $(|H|, |K|) = 1$, then G is a solvable T-group if and only if H and K are solvable T-groups. Recently, in [6], Ballester-Bolinches and Cossey proved that if $G = HK$ is a totally permutable product of the solvable PST-groups H and K such that $(|G : H|, |G : K|) = 1$, then G is a solvable PST-group. In [14], Ramadan, Heliel and Enjy Ahmed received some new results in the same line. The reader is referred to [7] for more details and results of totally and mutually permutable product of groups. The purpose of this paper is to continue the above mentioned investigations.

2. Preliminaries

Lemma 2.1 (See [2])**.** *G is a subgroup closed T-group if and only if G is a (supersolvable) solvable T-group.*

Let G be a group and let $p_1 > p_2 > \cdots > p_r$ be the distinct primes dividing $|G|$. Then G is said to satisfy the Sylow tower property (or G has a Sylow tower of supersolvable type) if there exist $P_1, P_2, \ldots, P_r$ such that each P_i is a Sylow p_i-subgroup of G and $P_1P_2 \ldots P_k \trianglelefteq G$ for $k = 1, 2, \ldots, r$.

Lemma 2.2 (See [4])**.** *Assume that the group $G = HK$ is a mutually permutable product of the subgroups H and K. If H and K satisfy the Sylow tower property, then G satisfies the Sylow tower property.*

Lemma 2.3 (See Gaschütz [10], also [15, p. 406]). *If H is a normal Hall subgroup of G such that G/H is a T-group and all subnormal subgroups of H are normal in G, then G is a T-group.*

Lemma 2.4 (See [2]).

(i) *If G is a solvable T_0-group, then G is supersolvable.*
(ii) *A subgroup closed T_0-group is supersolvable.*

Lemma 2.5 (See [1]). *If H and K are solvable subgroups of a group G with $|G : H| = p$ and $|G : K| = q$, where p and q are distinct primes in $\pi(G)$, then G is solvable.*

Lemma 2.6 (See [2]). *If G is a minimal non T_0-group, then:*

(i) *G' is nilpotent.*
(ii) *$|\pi(G)| = 2$.*

Lemma 2.7 (See [3]). *Suppose that H and K are solvable T-groups of a group G with $|G : H| = p$ and $|G : K| = q$, where p and q are distinct primes in $\pi(G)$ and p is the largest prime such that $p \not\equiv 1 \mod (q)$. Then G is a solvable T-group.*

3. Results

We first prove the following result.

Theorem 3.1. *Assume that the group $G = HK$ is a mutually permutable product of the subgroups H and K such that $(|G : H|, |G : K|) = 1$. Then G is a solvable T-group iff H and K are solvable T-groups.*

Proof. Suppose first that H and K are solvable T-groups. By Lemma 2.1 and Gaschűtz [10], we have that both H and K are supersolvable. Therefore, by Lemma 2.2, G has a Sylow tower of supersolvable type and hence P is normal in G, where P is a Sylow p-subgroup of G and p is the largest prime dividing the order of G. We treat the following two cases:

Case 1. p divides $|G : K|$.

Then p does not divide $|G : H|$ and we have that $P \leqslant H^x$ for some x in G. Since H^x has the same properties as H, we can replace H^x by H and hence we can assume, without loss of generality, that $P \leqslant H$. Now, as $P \trianglelefteq G$, we have $G/P = (H/P)(KP/P)$, where H/P and KP/P are mutually permutable subgroups of coprime indices in G/P. Furthermore, $KP/P \cong K/P \cap K$ is a solvable T-group as K is a T-group. By induction on $|G|$, G/P is a solvable T-group. Let P_1 be an arbitrary subgroup of P. Clearly, P_1 is subnormal in H and so normal in H as H is a T-group. By hypothesis, P_1K is a subgroup of G and so P_1K possesses a Sylow tower of supersolvable type. Hence $P_1 \trianglelefteq P_1K$ and, since $P_1 \trianglelefteq H$, it follows that $P_1 \trianglelefteq G$. Now, by applying Lemma 2.3, we have that G is a solvable T-group.

Case 2. p does not divide $|G : K|$.

Then p divides $|G:H|$ or p does not divide $|G:H|$. If p divides $|G:H|$, then $P \leq K$ and we can easily prove that G is a solvable T-group as in case 1. If p does not divide $|G:H|$, then P is contained in K and H. By induction on $|G|$, G/P is a solvable T-group. Let P_1 be an arbitrary subgroup of P. Then P_1 is subnormal and therefore normal in H and K as H and K are T-groups. Applying Lemma 2.3 again, G is a solvable T-group.

Conversely, if G is a solvable T-group, then by Lemma 2.1, H and K are solvable T-groups. This completes the proof of the theorem. □

As an immediate consequences, we have the following corollaries.

Corollary 3.2. *Assume that H and K are normal subgroups of a group G whose indices are relatively prime. Then G is a solvable T-group iff H and K are solvable T-groups.*

Corollary 3.3. *Assume that H and K are normal subgroups of a group G such that $G = HK$ and $(|H|, |K|) = 1$. Then G is a solvable T-group iff H and K are solvable T-groups.*

Corollary 3.4 ([11])**.** *Assume that H and K are mutually permutable subgroups of a group G such that $G = HK$ and $(|H|, |K|) = 1$. Then G is a solvable T-group iff H and K are solvable T-groups.*

Corollary 3.5. *If H and K are normal solvable T_0-groups of a group G whose indices are relatively prime, then G is a solvable T_0-group.*

Proof. Consider two cases:

Case 1. $\Phi(G) \neq 1$.

Clearly, $G = HK$ and hence $G/\Phi(G) = (H\Phi(G)/\Phi(G))(K\Phi(G)/\Phi(G))$, where $H\Phi(G)/\Phi(G)$ and $K\Phi(G)/\Phi(G)$ are normal solvable T_0-groups of the $G/\Phi(G)$ whose indices are relatively prime. By induction on $|G|$, $G/\Phi(G)$ is a solvable T_0-group and hence G is a solvable T_0-group.

Case 2. $\Phi(G) = 1$.

Since H and K are normal subgroups of G, we have that both $\Phi(H)$ and $\Phi(K)$ are contained in $\Phi(G) = 1$ which just means that H and K are solvable T-groups. By Corollary 3.2, G is a solvable T-group, hence G is also a solvable T_0-group. This completes the proof of the corollary. □

Corollary 3.6 ([11])**.** *Assume that H and K are normal subgroups of a group G such that $G = HK$ and $(|H|, |K|) = 1$. Then G is a solvable T_0-group iff H and K are solvable T_0-groups.*

Now we prove the following theorem.

Theorem 3.7. *Assume that H and K are subgroup closed T_0-groups of a group G with $|G:H| = p$ and $|G:K| = q$, where p and $q \neq p$ stand for primes. Then G is a subgroup closed T_0-group or G' is nilpotent and $\pi(G) = \{p, q\}$.*

Proof. By Lemma 2.4(ii), H and K are supersolvable (in particular, solvable) groups and, by Lemma 2.5, it follows that $G = HK$ is a solvable group. Let M be an arbitrary maximal subgroup of G. Then, as G is solvable, M has a prime power index in G. We argue that M is a subgroup closed T_0-group. If M is conjugate to H or K, then M is a subgroup closed T_0-group. Thus, we may assume that M is neither conjugate to H nor K. Then, by [8, p. 57, Theorem 16.2], $G = MH = MK$. Hence, $|G : H| = |M : M \cap H| = p$ and $|G : K| = |M : M \cap K| = q$, where $M \cap H$ and $M \cap K$ are subgroup closed T_0-groups of M. By induction on $|G|$, M is a subgroup closed T_0-group. Since M is an arbitrary maximal subgroup of G, we have that all proper subgroups of G are T_0-groups. If G is a T_0-group, then G is a subgroup closed T_0-group and we are done. If G is not a T_0-group, then G is a minimal non T_0-group and, by Lemma 2.6, we have that G' is nilpotent and $\pi(G) = \{p, q\}$ which completes the proof. □

The motivation for the next result is as follows: Van der Waall and Fransman [16] proved that if G is a subgroup closed T_0-group which all of its Sylow subgroups are T-groups, then G is a subgroup closed T-group (solvable T-group). Now, we extend this result and give a sufficient condition for the T-solvability of G as follows:

Theorem 3.8. *Assume that G is a solvable T_0-group which all of its Sylow subgroups are elementary abelian. Then G is a solvable T-group (subgroup closed T-group).*

Proof. Assume that the result is false and let G be a counterexample of minimal order. Since G is a T_0-group, it follows that $G/\Phi(G)$ is a T-group. Our choice of G implies that $\Phi(G) \neq 1$. By Lemma 2.4(i), G is supersolvable. Then, for the largest prime p dividing the order of G, $P \trianglelefteq G$, where P is a Sylow p-subgroup of G. If q divides $\Phi(G)$, $q \neq p$; let Q be a Sylow q-subgroup of $\Phi(G)$. Since Q is characteristic in $\Phi(G)$, we have that $Q \trianglelefteq G$ and therefore G/Q is a solvable T_0-group all of its Sylow subgroups are elementary abelian. By the minimality of G, G/Q is a solvable T-group and so each subgroup of PQ/Q is normal in G/Q. Let L be an arbitrary subgroup of P. Then $LQ/Q \trianglelefteq G/Q$ and so $LQ \trianglelefteq G$. But LQ is supersolvable, then L is characteristic in LQ $(p > q)$ and, since $LQ \trianglelefteq G$, we have $L \trianglelefteq G$. Since $G/P \cong K$, where K is a p'-Hall subgroup in G, is a solvable T-group by our minimal choice of G, it follows, by Lemma 2.3, that G is a solvable T-group; a contradiction. Thus $\Phi(G) < P$. By Maschke's theorem [8, p. 38], $P = \Phi(G) \times P_1$, where P_1 is K-invariant subgroup of G. Since P is abelian and P_1 is K-invariant subgroup of G, we have $P_1 \trianglelefteq G$ and therefore $G = PK = \Phi(G)(P_1K) = P_1K$ which is impossible; a final contradiction completing the proof of the theorem. □

We need the following result.

Proposition 3.9. *Let M be a T-group of a supersolvable group G, where G is not of prime power order. If $|G : M| = p$, where p is the largest prime in $\pi(G)$ such that $p \not\equiv 1 \mod (q)$ for all $q \in \pi(G) - \{p\}$, then G is a solvable T-group.*

Proof. We prove the result by induction on the order of G. Let H be a maximal subgroup of G. Since G is supersolvable, it follows that $|G : H| = q$ for some prime $q \neq p$. Clearly, H is not conjugate to M and, since G is solvable, it follows, by a well-known result of Ore [8, p. 57, Theorem 16.2], that $G = MH$. Since $|G : M| = |H : M \cap H| = p$ and $M \cap H$ is a T-group, we have that H is a solvable T-group by induction on the order of G. Now, we have that M and H are solvable T-groups of a group G with $|G : M| = p$ and $|G : H| = q$, where p and q are distinct primes in $\pi(G)$ and p is the largest prime in $\pi(G)$ such that $p \not\equiv 1 \mod (q)$. Applying Lemma 2.7 yields that G is a solvable T-group completing the proof. □

Now, we can prove the following theorem.

Theorem 3.10. *Let M be a T_0-group of a supersolvable group G. If $|G : M| = p$, where p is the largest prime in $\pi(G)$ such that $p \not\equiv 1 \mod (q)$ for all $q \in \pi(G) - \{p\}$. Then G is a solvable T_0-group.*

Proof. Assume that the result is false and let G be a counter-example of minimal order. Then G is not of prime power order since if G is of prime power order, we have that G is nilpotent and so a T_0-group; a contradiction. We argue that $\Phi(G) = 1$. If not, $M/\Phi(G)$ is a T_0-group and $|G : M| = |G/\Phi(G) : M/\Phi(G)| = p$, where $p \not\equiv 1 \mod q$ for all $q \in \pi(G/\Phi(G)) - \{p\}$. By the minimality of G, we have that $G/\Phi(G)$ is a solvable T_0-group, whence G is also a solvable T_0-group; a contradiction. Thus, $\Phi(G) = 1$. Since, G is solvable and $\Phi(G) = 1$, it follows, by [12, p. 279, Satz 4.5], that the Fitting subgroup $F(G) = L_1 \times L_2 \times \cdots \times L_r$, where $L_s (s = 1, 2, \ldots, r)$ are (abelian) minimal normal subgroups of G. As G is supersolvable, we have that all chief factors of G are of prime orders and hence $|L_s|$ =prime. Now, we argue that $L_s \leq M$ for all s $(s = 1, 2, \ldots, r)$. If not, then there exists $L_s \nleq M$ and $G = L_s M$. Clearly, $L_s \cap M = 1$ and $|L_s| = p$. If $C_G(L_s) \neq G$, then $G/C_G(L_s) \subseteq Aut(L_s)$ which implies that $p \equiv 1 \mod (q)$ for some $q \in \pi(G) - \{p\}$; a contradiction. Thus, $C_G(L_s) = G$ which implies that $L_s \leq Z(G)$ and so $M \trianglelefteq G$. Since M is a T_0-group and $\Phi(M) \leq \Phi(G) = 1$, we have that M is a T-group. Applying Proposition 3.9, we have that G is a solvable T-group, whence also a solvable T_0-group; a contradiction. Thus, we may assume that $L_s \leq M$ for every s $(s = 1, 2, \ldots, r)$ and hence $F(G) \leq M$. Since G is supersolvable, it follows that G' is nilpotent and so $G' \leq F(G) \leq M$ which implies easily that $M \trianglelefteq G$. Again, as $\Phi(M) = 1$, M is a T-group and, by applying Proposition 3.9, G is a solvable T_0-group; a final contradiction completing the proof of the theorem. □

Remark 3.11. The condition that $p \not\equiv 1 \mod (q)$ in Proposition 3.9 and Theorem 3.10 can not be omitted. For example, let $G = S_3 \times C_3$, where S_3 is the symmetric group of degree 3 and $C_3 = \langle c : c^3 = 1\rangle$. Take $M = S_3$. Then M is a T-group (T_0-group) and $|G : M| = 3$, $3 \equiv 1 \mod (2)$, but G is not a solvable T-group (T_0-group).

Remark 3.12. The converse of Theorem 3.10 is not true. For example, set $G = D_8 \times E$, where D_8 is the dihedral group of order 8 and E is a nonabelian group of order 3^3. Clearly, D_8 and E are solvable T_0-groups and $(|D_8|, |E|) = 1$. Thus, Corollary 3.6 implies that G is a solvable T_0-group. Now, let $M = D_8 \times L$, where $|L| = 3^2$. Then M is a maximal solvable T_0-group, $|G : M| = 3$ and $3 \equiv 1 \mod (2)$.

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