Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
27 (2011), 307–323
www.emis.de/journals
ISSN 1786-0091

WEAKLY CYCLIC RICCI SYMMETRIC MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

A. A. SHAIKH, LOVEJOY S. K. DAS, AND SANJIB KUMAR JANA

Abstract. In this paper the definition of weakly cyclic Ricci symmetric manifolds admitting semi-symmetric metric connection is given and its applications to the general relativity and cosmology are investigated. The existence of such a manifold is proved by an example.

1. Introduction

The notion of weakly Ricci symmetric manifolds was introduced by Tamássy and Binh [11]. A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(Y)S(X, Z) + D(Z)S(X, Y), \tag{1}$$

where A, B, D are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. The 1-forms A, B and D are known as the associated 1-forms of the manifold. Such an n-dimensional manifold is denoted by $(WRS)_n$. As an equivalent notion of $(WRS)_n$, Chaki and Koley [3] introduced the notion of generalized pseudo Ricci symmetric manifold. If in (1) the 1-form A is replaced by $2A$ then the definition of a $(WRS)_n$ reduces to that of generalized pseudo Ricci symmetric manifold by Chaki and Koley [3].

Extending the notion of $(WRS)_n$, recently Shaikh and Jana [10] introduced the notion of *weakly cyclic Ricci symmetric* manifolds and studied its geometric properties with several non-trivial examples. A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly cyclic Ricci symmetric manifold if its Ricci tensor S

2010 *Mathematics Subject Classification.* 53B35, 53B05.
Key words and phrases. weakly cyclic Ricci symmetric manifolds, semi-symmetric metric connection, Riccati equation, Sturm-Liouville equation, Frobenius method.

of type (0, 2) is not identically zero and satisfies the following:

$$(2)\quad (\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\ = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(X,Y)$$

where A, B and D are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g. Such an n-dimensional manifold is denoted by $(WCRS)_n$. Every $(WRS)_n$ is a $(WCRS)_n$ but not conversely as shown by several examples in [10].

The notion of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [4] in 1924. Then in 1932 Hayden [5] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold has been given by K. Yano in 1970 [12].

Semi-symmetric metric connection plays an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jerusalem or Mekka or the North Pole, then this displacement is semi-symmetric and metric [9, p. 143]. Again during the mathematical congress in Moscow in 1934 one evening mathematicians invented the 'Moscow displacement'. The streets of Moscow are approximately straight lines through the Kremlin and concentric circles around it. If a person walk in the street always facing the Kremlin, then this displacement is semi-symmetric and metric [9, p. 143].

The object of the present paper is to study a $(WCRS)_n$ admitting a semi-symmetric metric connection. Section 2 is concerned with preliminaries and the definition of *weakly cyclic Ricci symmetric manifolds admitting semi-symmetric metric connection* (briefly $[(WCRS)_n, \widetilde{\nabla}]$) is given.

In general relativity the matter content of the spacetime is described by the energy-momentum tensor T which is to be determined from the physical considerations dealing with the distribution of matter and energy. Since the matter content of the universe is assumed to behave like a perfect fluid in the standard cosmological models, the physical motivation for studying Lorentzian manifolds is the assumption that a gravitational field may be effectively modeled by some Lorentzian metric defined on a suitable four dimensional manifold M. The Einstein equations are fundamental in the construction of cosmological models which imply that the matter determines the geometry of the spacetime and conversely the motion of matter is determined by the metric tensor of the space which is non-flat.

In section 3 of the paper we investigate the applications of $[(WCRS)_n, \widetilde{\nabla}]$ to the general relativity and cosmology. It is shown that a viscous fluid spacetime obeying Einstein's equation with a cosmological constant is a connected Lorentzian $[(WCRS)_4, \widetilde{\nabla}]$. Consequently $[(WCRS)_4, \widetilde{\nabla}]$ can be viewed as a model of the viscous fluid spacetime. Also it is observed that in a viscous fluid

$[(WCRS)_4, \widetilde{\nabla}]$ spacetime none of the isotropic pressure and energy density can be a constant and the matter content of the spacetime is a non-thermalised fluid under a certain condition.

The existence of $(WCRS)_n$ is given in [10] by several examples. Hence a natural question arises, does there exist a $[(WCRS)_n, \widetilde{\nabla}]$? In section 4 of the paper, the existence theorem of such a manifold is ensured by an example with a suitable metric.

2. Preliminaries

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with the metric tensor g and ∇ be the Riemannian connection of the manifold (M^n, g). A linear connection $\widetilde{\nabla}$ on (M^n, g) is said to be semi-symmetric [4] if the torsion tensor τ of the connection $\widetilde{\nabla}$ satisfies

$$\tau(X, Y) = \alpha(Y)X - \alpha(X)Y \tag{3}$$

for any vector field X, Y on M and α is a 1-form associated with the torsion tensor τ of the connection $\widetilde{\nabla}$ given by

$$\alpha(X) = g(X, \rho),$$

ρ being the vector field associated with the 1-form α. The 1-form α is called the associated 1-form of the semi-symmetric connection and the vector field ρ is called the associated vector field of the connection. A semi-symmetric connection $\widetilde{\nabla}$ is called a semi-symmetric metric connection [5] if in addition it satisfies

$$\widetilde{\nabla} g = 0. \tag{4}$$

The relation between the semi-symmetric metric connection $\widetilde{\nabla}$ and the Riemannian connection ∇ of (M^n, g) is given by [12]

$$\widetilde{\nabla}_X Y = \nabla_X Y + \alpha(Y)X - g(X, Y)\rho. \tag{5}$$

In particular, if the 1-form α vanishes identically then a semi-symmetric metric connection reduces to the Riemannian connection. The covariant differentiation of a 1-form ω with respect to $\widetilde{\nabla}$ is given by [12]

$$(\widetilde{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) + \omega(X)\alpha(Y) - \omega(\rho)g(X, Y).$$

If R and $\widetilde{R}$ are respectively the curvature tensor of the Levi-Civita connection ∇ and the semi-symmetric metric connection $\widetilde{\nabla}$ then we have [12]

$$\begin{aligned} \widetilde{R}(X, Y)Z = R(X, Y)Z - P(Y, Z)X \\ + P(X, Z)Y - g(Y, Z)LX + g(X, Z)LY, \end{aligned} \tag{6}$$

where P is a tensor field of type (0, 2) given by

$$P(X, Y) = g(LX, Y) = (\nabla_X \alpha)(Y) - \alpha(X)\alpha(Y) + \frac{1}{2}\alpha(\rho)g(X, Y) \tag{7}$$

for any vector field X and Y. From (6) it follows that

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-2)P(Y,Z) - ag(Y,Z), \tag{8}$$

where $\widetilde{S}$ and S denote respectively the Ricci tensor with respect to $\widetilde{\nabla}$ and ∇, $a = \operatorname{trace} P$. The tensor P of type $(0,2)$ given in (7) is not symmetric in general and hence from (8) it follows that the Ricci tensor $\widetilde{S}$ of the semi-symmetric metric connection $\widetilde{\nabla}$ is not so. But if we consider that the 1-form α, associated with the torsion tensor τ, is closed then it can be easily shown that the relation

$$(\nabla_X \alpha)(Y) = (\nabla_Y \alpha)(X) \text{ for all vector fields } X, Y$$

holds and hence the tensor $P(X,Y)$ is symmetric. Consequently, the Ricci tensor $\widetilde{S}$ is symmetric. Conversely, if $P(X,Y)$ is symmetric then from (7) it follows that the 1-form α is closed. This leads to the following:

Proposition 2.1 ([1]). *Let (M^n, g) $(n > 2)$ be a Riemannian manifold admitting a semi-symmetric metric connection $\widetilde{\nabla}$. Then the Ricci tensor $\widetilde{S}$ of $\widetilde{\nabla}$ is symmetric if and only if the 1-form α, associated with the torsion tensor τ, is closed.*

Contracting (8) with respect to Y and Z, it can be easily found that

$$\widetilde{r} = r - 2(n-1)a, \tag{9}$$

where $\widetilde{r}$ and r denote respectively the scalar curvature with respect to $\widetilde{\nabla}$ and ∇.

Definition 2.1. A Riemannian manifold (M^n, g) $(n > 2)$ is called *weakly cyclic Ricci symmetric manifold admitting semi-symmetric metric connection* if its Ricci tensor $\widetilde{S}$ of type $(0,2)$ is not identically zero and satisfies the condition

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(Y,Z) + (\widetilde{\nabla}_Y \widetilde{S})(Z,X) + (\widetilde{\nabla}_Z \widetilde{S})(X,Y) \\ = \widetilde{A}(X)\widetilde{S}(Y,Z) + \widetilde{B}(Y)\widetilde{S}(Z,X) + \widetilde{D}(Z)\widetilde{S}(X,Y),\end{aligned} \tag{10}$$

where $\widetilde{A}$, $\widetilde{B}$, $\widetilde{D}$ are 1-forms (not simultaneously zero) and $\widetilde{\nabla}$ denotes the semi-symmetric metric connection.

The 1-forms $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ are known as the associated 1-forms of the manifold. Such an n-dimensional manifold is denoted by $[(WCRS)_n, \widetilde{\nabla}]$.

In view of (5) and (8), it follows from (10) that

$$
\begin{aligned}
&(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) \\
&- [2\alpha(X) + \widetilde{A}(X)]S(Y,Z) - [2\alpha(Y) + \widetilde{B}(Y)]S(X,Z) \\
&- [2\alpha(Z) + \widetilde{D}(Z)]S(X,Y) \\
&= (n-2)[(\nabla_X P)(Y,Z) + (\nabla_Y P)(Z,X) + (\nabla_Z P)(X,Y) \\
&- \{2\alpha(X) + \widetilde{A}(X)\}P(Y,Z) - \{2\alpha(Y) + \widetilde{B}(Y)\}P(X,Z) \\
&- \{2\alpha(Z) + \widetilde{D}(Z)\}P(X,Y) + \{P(\rho,Z) + P(Z,\rho)\}g(X,Y) \\
&+ \{P(\rho,X) + P(X,\rho)\}g(Y,Z) + \{P(\rho,Y) + P(Y,\rho)\}g(X,Z)] \\
&- \{2\alpha(QX) + \widetilde{A}(X) - da(X)\}g(Y,Z) \\
&- \{2\alpha(QY) + \widetilde{B}(Y) - da(Y)\}g(X,Z) \\
&- \{2\alpha(QZ) + \widetilde{D}(Z) - da(Z)\}g(X,Y),
\end{aligned} \tag{11}
$$

where Q is the Ricci operator, *i.e.*, $S(X,Y) = g(QX,Y)$.

In particular, if the 1-form α vanishes identically, then (11) reduces to (2) with $\widetilde{A} = A$, $\widetilde{B} = B$ and $\widetilde{D} = D$. Hence the manifold $(WCRS)_n$ is a particular case of $[(WCRS)_n, \widetilde{\nabla}]$. Also the manifold $(WCRS)_n$ could be a $[(WCRS)_n, \widetilde{\nabla}]$ when it admits a semi-symmetric metric connection $\widetilde{\nabla}$ different from the Riemannian connection ∇.

We now prove the following Lemma.

Lemma 2.1. *If in a $[(WCRS)_n, \widetilde{\nabla}]$ the 1-form α, associated with the torsion tensor τ, is closed then its Ricci tensor $\widetilde{S}$ is of the form:*

$$\widetilde{S} = \widetilde{r}\gamma \otimes \gamma, \tag{12}$$

where γ is a non-zero 1-form defined by $\gamma(X) = g(X,\mu)$, μ being a unit vector field.

Proof. Interchanging Y and Z in (10) we obtain

$$
\begin{aligned}
&(\widetilde{\nabla}_X \widetilde{S})(Z,Y) + (\widetilde{\nabla}_Z \widetilde{S})(Y,X) + (\widetilde{\nabla}_Y \widetilde{S})(X,Z) \\
&\qquad = \widetilde{A}(X)\widetilde{S}(Z,Y) + \widetilde{B}(Z)\widetilde{S}(Y,X) + \widetilde{D}(Y)\widetilde{S}(X,Z).
\end{aligned}
$$

Since the 1-form α, associated with the torsion tensor τ, is closed, from Proposition 2.1 it follows that the Ricci tensor $\widetilde{S}$ is symmetric. Subtracting the last relation from (10) we get

$$[\widetilde{B}(Y) - \widetilde{D}(Y)]\widetilde{S}(X,Z) = [\widetilde{B}(Z) - \widetilde{D}(Z)]\widetilde{S}(X,Y),$$

where the symmetry property of $\widetilde{S}$ has been used.

Let us consider $\widetilde{E}(X) = g(X,\nu) = \widetilde{B}(X) - \widetilde{D}(X)$ for all vector fields X and ν is a vector field associated with the 1-form $\widetilde{E}$. Then the above relation reduces to

$$\widetilde{E}(Y)\widetilde{S}(Z,X) = \widetilde{E}(Z)\widetilde{S}(X,Y). \tag{13}$$

Contraction of (13) with respect to X and Z yields

$$\widetilde{r}\widetilde{E}(Y) = \widetilde{E}(\widetilde{Q}Y), \tag{14}$$

where $\widetilde{Q}$ is the Ricci operator associated with the Ricci tensor $\widetilde{S}$, i.e., $\widetilde{S}(X,Y) = g(\widetilde{Q}X,Y)$. Also from (13) we have

$$\widetilde{E}(\nu)\widetilde{S}(X,Y) = \widetilde{E}(Y)\widetilde{S}(X,\nu) = \widetilde{E}(Y)g(\widetilde{Q}X,\nu) = \widetilde{E}(Y)\widetilde{E}(\widetilde{Q}X),$$

which, in view of (14), yields

$$\widetilde{S}(X,Y) = \frac{\widetilde{r}}{\widetilde{E}(\nu)}\widetilde{E}(X)\widetilde{E}(Y) = \widetilde{r}\,\gamma(X)\gamma(Y), \tag{15}$$

where $\gamma(X) = g(X,\mu) = \frac{1}{\sqrt{\widetilde{E}(\nu)}}\widetilde{E}(X)$, μ being a unit vector field associated with the 1-form γ. □

Using (12) and (9) in (8) we obtain

$$S(Y,Z) = ag(Y,Z) + [r - 2(n-1)a]\gamma(Y)\gamma(Z) + (n-2)P(Y,Z), \tag{16}$$

provided that the 1-form α, associated with the torsion tensor τ, is closed.

A non-zero vector V on a manifold M is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g(V,V) < 0$ (resp. ≤ 0, $= 0$, > 0)([2], [8]).

Since μ is a unit vector field on the Riemannian manifold $[(WCRS)_n, \widetilde{\nabla}] = M$ with metric tensor g, it can be easily shown [8, p. 148] that $\widetilde{g} = g - 2\gamma\otimes\gamma$ is a Lorentz metric on M. Also, μ becomes timelike so the resulting Lorentz manifold is time-orientable.

3. General relativistic viscous fluid $[(WCRS)_4, \widetilde{\nabla}]$ spacetime

A viscous fluid spacetime is a connected Lorentz manifold (M^4, g) with signature $(-,+,+,+)$. In general relativity the key role is played by Einstein's equation

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = kT(X,Y) \tag{17}$$

for all vector fields X, Y, where S is the Ricci tensor of type (0, 2), r is the scalar curvature, λ is the cosmological constant, k is the gravitational constant and T is the energy-momentum tensor of type (0, 2). Let us consider the energy-momentum tensor T of a viscous fluid spacetime to the following form [7]

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + P(X,Y) \tag{18}$$

where σ, p are the energy density and isotropic pressure respectively and P denotes the anisotropic pressure tensor of the fluid, μ is the unit timelike vector field, called flow vector field of the fluid associated with the 1-form γ given by $g(X,\mu) = \gamma(X)$ for all X. Then by virtue of (18), (17) can be written as

$$S(X,Y) = \left(\frac{r}{2} + kp - \lambda\right) g(X,Y) + k(\sigma + p)\gamma(X)\gamma(Y) + kP(X,Y), \tag{19}$$

which, in view of (16), shows that the spacetime under consideration is a $[(WCRS)_4, \widetilde{\nabla}]$ with μ as the unit timelike flow vector field of the fluid and P as the anisotropic pressure tensor. Hence we can state the following:

Theorem 3.1. *A viscous fluid spacetime obeying Einstein's equation with a cosmological constant is a connected Lorentzian $[(WCRS)_4, \widetilde{\nabla}]$ with μ as the flow vector field of the fluid and P as the anisotropic pressure tensor, provided that the 1-form α, associated with the torsion tensor τ, is closed.*

Now by virtue of (19), (16) yields

$$\begin{aligned}\left(\frac{r}{2} + kp - \lambda - a\right) g(X,Y) + (k\sigma + kp + 6a - r)\gamma(X)\gamma(Y)& \\ + (k-2)P(X,Y) = 0.&\end{aligned} \tag{20}$$

Setting $X = Y = \mu$ in (20) we obtain by virtue of (9) that

$$\sigma = \frac{1}{2k}[3\widetilde{r} - 2\lambda + 4a - 2(k-2)b], \tag{21}$$

where $b = P(\mu,\mu)$. Again contracting (20) we find by virtue of (9) that

$$p = \frac{1}{6k}[6\lambda - 3\widetilde{r} - 2(k+4)a - 2(k-2)b]. \tag{22}$$

This leads to the following theorem.

Theorem 3.2. *In a viscous fluid $[(WCRS)_4, \widetilde{\nabla}]$ spacetime obeying Einstein's equation with a cosmological constant λ none of the isotropic pressure and energy density can be a constant, provided that the 1-form α, associated with the torsion tensor τ corresponding to the semi-symmetric metric connection $\widetilde{\nabla}$, is closed.*

Now since $\sigma > 0$ and $p > 0$, we have from (21) and (22) that

$$\lambda < \frac{3\widetilde{r} + 4a - 2(k-2)b}{2} \quad \text{and} \quad \lambda > \frac{3\widetilde{r} + 2(k+4)a + 2(k-2)b}{6}$$

and hence

$$\frac{3\widetilde{r} + 2(k+4)a + 2(k-2)b}{6} < \lambda < \frac{3\widetilde{r} + 4a - 2(k-2)b}{2} \tag{23}$$

and

$$\widetilde{r} > \frac{k-2}{3}(a + 4b). \tag{24}$$

This leads to the following theorem.

Theorem 3.3. *In a viscous fluid* $[(WCRS)_4, \widetilde{\nabla}]$ *spacetime obeying Einstein's equation, the cosmological constant* λ *satisfies the relation* (23) *and the scalar curvature* $\widetilde{r}$ *satisfies the relation* (24), *provided that the 1-form* α, *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$, *is closed.*

We now discuss whether a viscous fluid $[(WCRS)_4, \widetilde{\nabla}]$ spacetime with μ as the unit timelike flow vector field can admit heat flux or not. Therefore, if possible, let the energy-momentum tensor T be of the following form [7]

$$T(X,Y) = pg(X,Y) + (\sigma + p)\gamma(X)\gamma(Y) + \gamma(X)\eta(Y) + \gamma(Y)\eta(X),$$

where $\eta(X) = g(X,V)$ for all vector fields X; V being the heat flux vector field ; σ, p are the energy density and isotropic pressure respectively. Thus we have $g(\mu, V) = 0$, i.e., $\eta(\mu) = 0$. Hence by virtue of the last relation, (9) and (16), (17) yields

$$\begin{aligned} (25)\quad 2P(X,Y) + \left(\lambda - \frac{\widetilde{r}}{2} - kp - 2a\right) g(X,Y) + [\widetilde{r} - k(p+\sigma)]\gamma(X)\gamma(Y) \\ - k[\gamma(X)\eta(Y) + \gamma(Y)\eta(X)] = 0. \end{aligned}$$

Setting $Y = \mu$ in (25) we obtain

$$2P(X,\mu) + \left(\lambda - \frac{3}{2}\widetilde{r} + k\sigma - 2a\right)\gamma(X) + k\eta(X) = 0. \tag{26}$$

Putting $X = \mu$ in (26) we obtain

$$2b - \left(\lambda - \frac{3}{2}\widetilde{r} + k\sigma - 2a\right) = 0.$$

Using the last relation in (26) we obtain

$$\eta(X) = -\frac{2}{k}[P(X,\mu) + b\gamma(X)], \text{ since} k \neq 0. \tag{27}$$

Thus we have the following theorem.

Theorem 3.4. *A viscous fluid* $[(WCRS)_4, \widetilde{\nabla}]$ *spacetime obeying Einstein's equation with a cosmological constant* λ *admits heat flux given by* (27), *provided that* $P(X,\mu) + b\gamma(X) \neq 0$ *for all* X *and the 1-form* α, *associated with the torsion tensor* τ *corresponding to the semi-symmetric metric connection* $\widetilde{\nabla}$, *is closed.*

From (27) it follows that

$$V = -\frac{2}{k}(N + bI)\mu,$$

where $P(X,Y) = g(NX,Y)$ for all vector fields X, Y. This implies that $V = 0$ if and only if $-b$ is the eigenvalue of P corresponding to the eigenvector μ. Hence we can state the following theorem.

Theorem 3.5. *A viscous fluid $[(WCRS)_4, \widetilde{\nabla}]$ spacetime, with closed 1-form α associated with the torsion tensor τ corresponding to the semi-symmetric metric connection $\widetilde{\nabla}$, can not admit heat flux if and only if $-b$ is the eigenvalue of P corresponding to the eigenvector μ.*

4. Example of $[(WCRS)_n, \widetilde{\nabla}]$

This section deals with an example of $[(WCRS)_n, \widetilde{\nabla}]$.

Example. Let $M^5 = \mathbf{R}^5$ be a manifold endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2] + (dx^4)^2 + (dx^5)^2, \ (i, j = 1, 2, \ldots, 5). \tag{28}$$

Then the only non-vanishing components of the Christoffel symbols are

$$\Gamma^1_{11} = \frac{1}{2} = \Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = -\Gamma^1_{22} = \Gamma^3_{23} = \Gamma^3_{32}, \ \Gamma^1_{33} = -\frac{1}{2}e^{x^2} = \Gamma^2_{33}.$$

Using the above relations, we can find the non-vanishing components of the Ricci tensor and their covariant derivatives are as follows:

$$\begin{aligned} S_{22} &= \frac{1}{2}, \\ S_{33} &= \frac{1}{2}e^{x^2}, \\ S_{22,1} &= -\frac{1}{2}, \\ S_{33,1} &= -\frac{1}{2}e^{x^2}, \\ S_{12,2} &= -\frac{1}{4} = S_{21,2}, \\ S_{13,3} &= -\frac{1}{4}e^{x^2} = S_{31,3}, \\ S_{23,3} &= -\frac{1}{4}e^{x^2} = S_{32,3}, \end{aligned} \tag{29}$$

where ',' denotes the covariant differentiation with respect to the Levi-Civita connection ∇ and S_{ij} denote the components of the Ricci tensor S in terms of local coordinate system. It can be easily shown that the scalar curvature of the manifold is $r = e^{-x^1}$, which is non-vanishing and non-constant. Therefore (M^5, g) is a Riemannian manifold of non-vanishing scalar curvature. We shall now show that this M^5 is a $(WCRS)_5$, i.e., it satisfies the defining condition (2). Let us now consider the associated 1-forms as follows:

$$A_i = B_i = D_i = \begin{cases} -2 & \text{for } i = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{30}$$

where A_i, B_i and D_i are the components of the 1-forms A, B and D respectively in terms of local coordinate system. In terms of local coordinate system, (2) can be written as follows:

$$S_{ij,k} + S_{jk,i} + S_{ki,j} = A_k S_{ij} + B_i S_{jk} + D_j S_{ki}, \; i, j, k = 1, 2, \ldots, 5. \tag{31}$$

If $i = j = 2$ and $k = 1$, then by virtue of (29) and (30) we get the following relations for the right hand side (R.H.S.) and left hand side (L.H.S.) of (31):

$$\begin{aligned}\text{R.H.S. of (31)} &= A_1 S_{22} + B_2 S_{12} + D_2 S_{21}\\ &= -1\\ &= S_{22,1} + S_{12,2} + S_{21,2}\\ &= \text{L.H.S. of (31)}.\end{aligned}$$

Hence (31) holds for $i = j = 2$ and $k = 1$. For other values of i, j and k, in a similar manner it can be shown that the relation (31) is true. Therefore, (M^5, g) is a $(WCRS)_5$ which is neither Ricci recurrent nor $(WRS)_5$.

We shall now show that the manifold under consideration is a $[(WCRS)_5, \widetilde{\nabla}]$, where $\widetilde{\nabla}$ is the linear connection determined by (5) and is different to the Levi-Civita connection ∇. Also it may be mentioned that the statement holds for a connection $\widetilde{\nabla}$ to be specified later on. If, in terms of local coordinate system, the components of the connection ∇ and $\widetilde{\nabla}$ are respectively denoted by Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$, then the relation (5) can be written as

$$\widetilde{\Gamma}^i_{jk} = \Gamma^i_{jk} + \alpha_j \delta^i_k - \alpha_k \delta^i_j. \tag{32}$$

We now define the 1-form α as follows:

$$\alpha(\partial_i) = \alpha_i = \begin{cases} \psi(x^1) & \text{for } i = 1, \\ 0 & \text{otherwise}, \end{cases} \tag{33}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ at any point $x \in M$ and ψ is a continuously differentiable function of x^1 defined over the interval $[l, m]$. We now determine ψ precisely so that the equation (10) is satisfied for the manifold $[(WCRS)_5, \widetilde{\nabla}]$ under the metric given by (28). As the above process is same with the determination of α completely, hence the construction of our example will be completed. In view of (7), (32) and (33) we obtain the non-vanishing components of the Christoffel symbol $\widetilde{\Gamma}^i_{jk}$ and P_{ij} as follows:

$$\begin{gathered}\widetilde{\Gamma}^1_{11} = \frac{1}{2} = \widetilde{\Gamma}^3_{23} = \widetilde{\Gamma}^3_{32} = -\widetilde{\Gamma}^1_{22}, \; \widetilde{\Gamma}^2_{12} = \frac{1}{2} + \psi = \widetilde{\Gamma}^3_{13},\\ \widetilde{\Gamma}^2_{21} = \frac{1}{2} - \psi = \widetilde{\Gamma}^3_{31}, \; \widetilde{\Gamma}^1_{33} = -\frac{1}{2} e^{x^2} = \widetilde{\Gamma}^2_{33},\\ \widetilde{\Gamma}^4_{14} = \psi = -\widetilde{\Gamma}^4_{41}, \; \widetilde{\Gamma}^5_{15} = \psi = -\widetilde{\Gamma}^5_{51},\\ P_{11} = \dot{\psi} - \frac{1}{2}\psi - \psi^2\end{gathered} \tag{34}$$

where '.' denotes the total differentiation with respect to x^1. In view of (8), (29) and (34) we get the non-zero components of the Ricci tensor $\widetilde{S}_{ij}$ as follows:

$$(35)\quad \begin{aligned}&(a)\ \widetilde{S}_{11} = -(e^{x^1}+3)(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(b)\ \widetilde{S}_{22} = \tfrac{1}{2} - e^{x^1}(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(c)\ \widetilde{S}_{33} = \tfrac{1}{2}e^{x^2} - e^{x^1+x^2}(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(d)\ \widetilde{S}_{44} = -(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(e)\ \widetilde{S}_{55} = -(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2)\end{aligned}$$

and consequently we obtain

$$(36)\quad \begin{aligned}&(a)\ \widetilde{S}_{11,1} = 3(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2) - (e^{x^1}+3)(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(b)\ \widetilde{S}_{12,2} = \tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{3}{2} + \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(c)\ \widetilde{S}_{21,2} = -\tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{3}{2} - \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(d)\ \widetilde{S}_{13,3} = e^{x^2}[\tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{3}{2} + \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2)],\\ &(e)\ \widetilde{S}_{31,3} = -e^{x^2}[\tfrac{1}{2}\psi + \tfrac{1}{4} + (\tfrac{3}{2} - \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2)],\\ &(f)\ \widetilde{S}_{14,4} = -\psi(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(g)\ \widetilde{S}_{41,4} = \psi(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(h)\ \widetilde{S}_{15,5} = -\psi(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(i)\ \widetilde{S}_{51,5} = \psi(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(j)\ \widetilde{S}_{22,1} = -\tfrac{1}{2} - e^{x^1}(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(k)\ \widetilde{S}_{33,1} = -\tfrac{1}{2}e^{x^2} - e^{x^1+x^2}(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(l)\ \widetilde{S}_{44,1} = -(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(m)\ \widetilde{S}_{55,1} = -(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\end{aligned}$$

where ',' denotes the covariant differentiation with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ and '.' denotes the total differentiation. We shall now show that our considered manifold M^5 with the metric given by (28) and $\widetilde{\nabla}$ for ψ verifying (36) is a $[(WCRS)_5, \widetilde{\nabla}]$, i.e., it satisfies (10). In terms of local coordinate system, we consider the components of the 1-forms $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ as follows:

$$(37)\qquad \widetilde{A}_i = \widetilde{B}_i = \widetilde{D}_i = \begin{cases} \frac{\dot{P}_{11}}{P_{11}} & \text{for } i = 1\\ 0 & \text{otherwise,}\end{cases}$$

at any point $x \in M$, where $P_{ij}(i,j = 1,2,\ldots,5)$ are the components of the tensor P defined in (7). In terms of local coordinate system, the equation (10) can be written as follows:

$$(38)\qquad \widetilde{S}_{ij,k} + \widetilde{S}_{jk,i} + \widetilde{S}_{ki,j} = \widetilde{A}_k\widetilde{S}_{ij} + \widetilde{B}_i\widetilde{S}_{jk} + \widetilde{D}_j\widetilde{S}_{ki},\ i,j,k = 1,2,\ldots,5.$$

In view of (35)-(37), (38) reduces to the following ordinary differential equation:

$$(39)\qquad \dot{\psi} - \frac{1}{2}\psi - \psi^2 + \frac{6e^{x^1}+17}{4(e^{x^1}+3)} = 0,$$

which is the generalized Riccati's equation [6, p. 23]. Now applying the transformation $\psi = -\frac{\dot{u}(x^1)}{u(x^1)}$ in (39) we obtain

$$\ddot{u} - \frac{1}{2}\dot{u} - \frac{6e^{x^1} + 17}{4(e^{x^1} + 3)}u = 0. \tag{40}$$

Further taking $t = e^{-x^1}$ in (40), we get

$$t\frac{d^2u}{dt^2} + \frac{3}{2t}\frac{du}{dt} + \frac{6 + 17t}{4(1 + 3t)t^2}u = 0, \tag{41}$$

which is the Sturm-Liouville equation. Here $t = 0$ is the regular singular point. Hence to obtain the general solution of (41) near the regular singular point $t = 0$ we use the Frobenius method [6, p. 396]. Using this method we obtain the series solution of (41) as

$$u = \sum_{m=0}^{\infty}(c_m t^{m+1} + d_m t^{m-\frac{3}{2}}),$$

where the values of c_m and d_m in terms of c_0 and d_0 respectively are obtained from the following equations

$$c_m(2m^2 + 5m + 6) - \frac{1}{6}\sum_{j=0}^{m-1}(-3)^{m-j}c_j = 0,$$
$$d_m(2m^2 - 5m + 6) - \frac{1}{6}\sum_{j=0}^{m-1}(-3)^{m-j}d_j = 0$$

and c_0, d_0 are arbitrary non-zero constants such that $c_0 \neq d_0$. Thus in terms of x^1, we obtain the value of u as follows:

$$u = \sum_{m=0}^{\infty}\left[c_m e^{-(m+1)x^1} + d_m e^{-(m-\frac{3}{2})x^1}\right]. \tag{42}$$

In view of (42), the general solution of (40) is obtained as

$$\psi(x^1) = -\frac{\sum_{m=0}^{\infty}\left[(m+1)c_m e^{-(m+1)x^1} + \left(m - \frac{3}{2}\right)d_m e^{-(m-\frac{3}{2})x^1}\right]}{\sum_{m=0}^{\infty}[c_m e^{-mx^1} + d_m e^{-(m-\frac{5}{2})x^1}]}. \tag{43}$$

This completes the determination of ψ, i.e., the coefficients of the connection and hence our considered manifold is a $[(WCRS)_5, \widetilde{\nabla}]$. Thus we can state the following theorem.

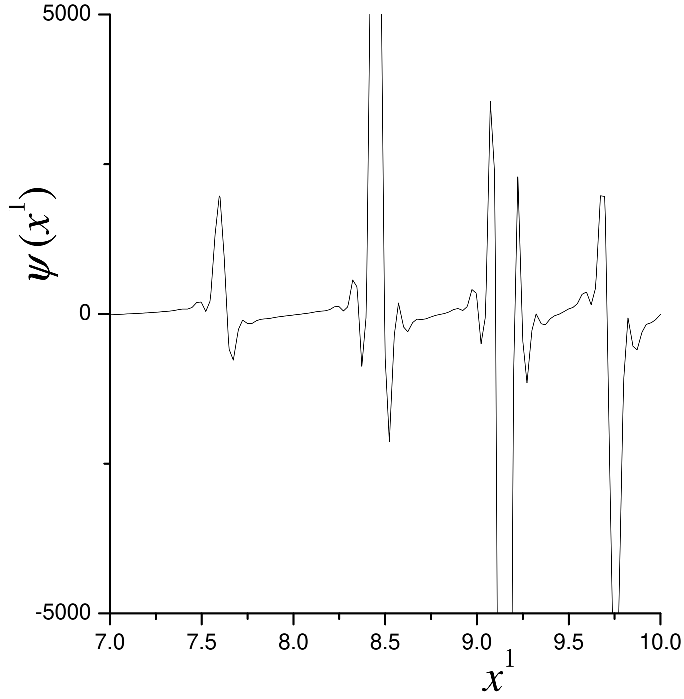

FIGURE 1

Theorem 4.1. *Let* (M^5, g) *be the Riemannian manifold equipped with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2] + (dx^4)^2 + (dx^5)^2, \quad (i, j = 1, 2, \dots, 5) \tag{44}$$

and $\widetilde{\nabla}$ *with* ψ *given by* (43). *Then* (M^5, g) *is a* $[(WCRS)_5, \widetilde{\nabla}]$.

We now investigate the solution of $\psi(x^1)$ in (43)) such that $\psi(x^1)$ is continuous in the interval $[l, m]$ for some choice of l and m. We take a particular solution of (40) for which we have assumed $c_0 = -1$ and $d_0 = 1$. We now draw the graph of the particular solution of (40) as follows: Here we observe that the function $\psi = \psi(x^1)$ has the apparent vertical asymptotes near $x^1 = 8.4$, corresponding to a zero of the denominator of $\psi(x^1)$. Thus for this particular solution, the interval of continuity of $\psi(x^1)$ can be taken as any closed interval excluding the points where $\psi(x^1)$ has vertical asymptotes, for example $[l, m]$ such that $8.2 < l < m < 8.5$.

Therefore, we have completely determined $\psi(x^1)$ as well as the coefficients of the semi-symmetric metric connection such that the defining condition (10)) holds for the considered metric given in (28).

The metric g on the manifold $M^5 = \mathbf{R}^3 \times \mathbf{R}^2$ is a product of the metric

$$g_1 = e^{x^1}[(dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2] \tag{45}$$

on $\mathbf{R}^3$ by the flat metric $g_2 = (dx^4)^2 + (dx^5)^2$ on $\mathbf{R}^2$. Proceeding as in the above, it can be shown that the manifold $M^3 = (\mathbf{R}^3, g_1)$ is a $(WCRS)_3$.

In a similar manner, as in the above, we obtain the non-vanishing components of the Ricci tensor $\widetilde{S}_{ij}$ as follows:

$$\begin{aligned} &(a)\ \widetilde{S}_{11} = -(e^{x^1}+1)(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(b)\ \widetilde{S}_{22} = \tfrac{1}{2} - e^{x^1}(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(c)\ \widetilde{S}_{33} = \tfrac{1}{2}e^{x^2} - e^{x^1+x^2}(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2) \end{aligned} \tag{46}$$

and consequently we obtain

$$\begin{aligned} &(a)\ \widetilde{S}_{11,1} = \dot{\psi} - \tfrac{1}{2}\psi - \psi^2 - (e^{x^1}+1)(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(b)\ \widetilde{S}_{12,2} = \tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{1}{2} + \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(c)\ \widetilde{S}_{21,2} = -\tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{1}{2} - \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2),\\ &(d)\ \widetilde{S}_{13,3} = e^{x^2}[\tfrac{1}{2}\psi - \tfrac{1}{4} - (\tfrac{1}{2} + \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2)],\\ &(e)\ \widetilde{S}_{31,3} = -e^{x^2}[\tfrac{1}{2}\psi + \tfrac{1}{4} + (\tfrac{1}{2} - \psi e^{x^1})(\dot{\psi} - \tfrac{1}{2}\psi - \psi^2)],\\ &(f)\ \widetilde{S}_{22,1} = -\tfrac{1}{2} - e^{x^1}(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}),\\ &(h)\ \widetilde{S}_{33,1} = -\tfrac{1}{2}e^{x^2} - e^{x^1+x^2}(\ddot{\psi} - \tfrac{1}{2}\dot{\psi} - 2\psi\dot{\psi}), \end{aligned} \tag{47}$$

where ',' denotes the covariant differentiation with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ and '.' denotes the total differentiation. In terms of local coordinate system, we consider the components of the 1-forms $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ as follows:

$$\widetilde{A}_i = \widetilde{B}_i = \widetilde{D}_i = \begin{cases} \frac{2(e^{3x^1}+2e^{2x^1}+e^{x^1}-1)}{(1-e^{2x^1})(e^{x^1}+2)} & \text{for } i = 1\\ 0 & \text{otherwise,} \end{cases} \tag{48}$$

at any point $x \in M$. In terms of local coordinate system, the equation (10) can be written as follows:

$$\widetilde{S}_{ij,k} + \widetilde{S}_{jk,i} + \widetilde{S}_{ki,j} = \widetilde{A}_k\widetilde{S}_{ij} + \widetilde{B}_i\widetilde{S}_{jk} + \widetilde{D}_j\widetilde{S}_{ki},\ i, j, k = 1, 2, 3. \tag{49}$$

In view of (46)–(48), (49) reduces to the following ordinary differential equation:

$$(e^{x^1}+1)(\ddot{\psi} - \frac{1}{2}\dot{\psi} - 2\psi\dot{\psi}) + (2e^{x^1}+1)(\dot{\psi} - \frac{1}{2}\psi - \psi^2)[1 + 2(\dot{\psi} - \frac{1}{2}\psi - \psi^2)] = 0,$$

which, on integration, gives the following first order non-linear ordinary differential equation:

$$\dot{\psi} - \frac{1}{2}\psi - \psi^2 - \frac{1}{ce^{x^1}(e^{x^1}+1) - 2} = 0, \tag{50}$$

where c is an arbitrary constant. Now if we assume $c = 1$ and apply the transformation $\psi = -t\frac{du}{dt}$ where $t = e^{x^1}$, then from the last equation we obtain

$$\frac{d^2u}{dt^2} + \frac{1}{2t}\frac{du}{dt} + \frac{1}{t^2(t^2+t-2)}u = 0, \tag{51}$$

which is the Sturm-Liouville equation. Proceeding as in the above we find the general solution of (51) near the regular singular point $t = 0$ as

$$u = \sum_{m=0}^{\infty}(c_m t^{m+1} + d_m t^{m-\frac{1}{2}}),$$

where the values of c_m and d_m in terms of c_0 and d_0 respectively are obtained from the following equations

$$2m(2m+3)c_m - m(2m-1)c_{m-1} - (m-1)(2m-3)c_{m-2} = 0,\ c_1 = \frac{1}{10}c_0$$

$$2m(2m-3)d_m - (m-2)(2m-3)d_{m-1} - (m-3)(2m-5)d_{m-2} = 0,$$
$$d_1 = -\frac{1}{2}d_0,\ m \geq 2,$$

c_0, d_0 are arbitrary non-zero constants such that $c_0 \neq d_0$. Thus in terms of x^1, we obtain the value of u as follows:

$$u = \sum_{m=0}^{\infty}\left[c_m e^{(m+1)x^1} + d_m e^{(m-\frac{1}{2})x^1}\right]. \tag{52}$$

In view of (52), the general solution of (50)) is obtained as

$$\psi(x^1) = -\frac{\displaystyle\sum_{m=0}^{\infty}\left[(m+1)c_m e^{(m+1)x^1} + \left(m-\frac{1}{2}\right)d_m e^{(m-\frac{1}{2})x^1}\right]}{\displaystyle\sum_{m=0}^{\infty}[c_m e^{(m+1)x^1} + d_m e^{(m-\frac{1}{2})x^1}]}. \tag{53}$$

This completes the determination of ψ, i.e., the coefficients of the connection and hence our considered manifold is a $[(WCRS)_3, \widetilde{\nabla}]$. Thus we can state the followingtheorem.

Theorem 4.2. *Let $M^3 = (\mathbf{R}^3, g_1)$ be the Riemannian manifold equipped with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 + (dx^2)^2 + e^{x^2}(dx^3)^2],\ (i,j = 1,2,3)$$

and $\widetilde{\nabla}$ with ψ given by (53). *Then (M^3, g) is a $[(WCRS)_3, \widetilde{\nabla}]$.*

We now investigate the solution of $\psi(x^1)$ in (53) such that $\psi(x^1)$ is continuous in the interval $[l, m]$ for some choice of l and m. Assuming $c_0 = -1$ and $d_0 = 1$ we now draw the graph of the particular solution of (50) as follows: Here we observe that the function $\psi = \psi(x^1)$ has the apparent vertical asymp-

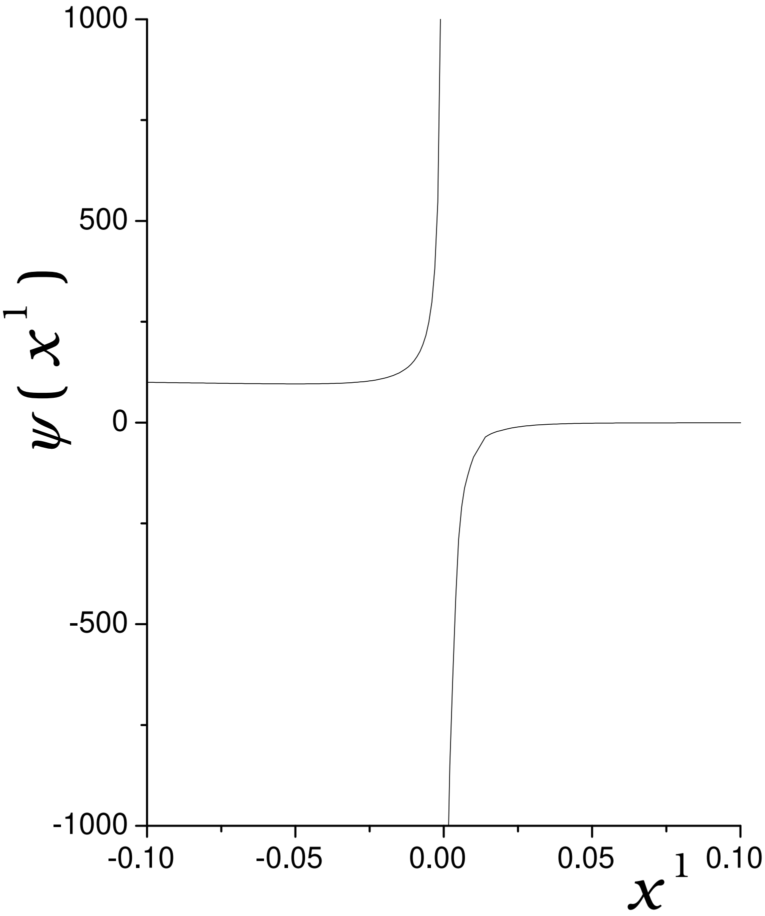

FIGURE 2

totes near $x^1 = 0.0$, corresponding to a zero of the denominator of $\psi(x^1)$. Thus for this particular solution, the interval of continuity of $\psi(x^1)$ can be taken as any closed interval excluding the points where $\psi(x^1)$ has vertical asymptotes, for example $[l, m]$ such that $0 < l < m < 0.1$.

Therefore, we have completely determined $\psi(x^1)$ as well as the coefficients of the semi-symmetric metric connection such that the defining condition (10) holds for the considered metric given in (45).

Therefore, the result of Theorem 4.1 is also valid for the manifold $M^3 = (\mathbf{R}^3, g_1)$ as shown in the Theorem 4.2.

REFERENCES

[1] B. Barua and A. K. Ray. Some properties of semisymmetric metric connection in a Riemannian manifold. *Indian J. Pure Appl. Math.*, 16(7):736–740, 1985.

[2] J. K. Beem and P. E. Ehrlich. *Global Lorentzian geometry*, volume 67 of *Monographs and Textbooks in Pure and Applied Math.* Marcel Dekker Inc., New York, 1981.

[3] M. C. Chaki and S. Koley. On generalized pseudo Ricci symmetric manifolds. *Period. Math. Hungar.*, 28(2):123–129, 1994.

[4] A. Friedmann and J. A. Schouten. Über die Geometrie der halbsymmetrischen Übertragungen. *Math. Z.*, 21(1):211–223, 1924.

[5] H. Hayden. Sub-spaces of a space with torsion. *Proc. Lond. Math. Soc., II. Ser.*, 34:27–50, 1932.

[6] E. L. Ince. *Ordinary Differential Equations*. Dover Publications, New York, 1944.

[7] M. Novello and M. J. Reboucas. The stability of a rotating universe. *The Astrophysics Journal*, 225:719–724, 1978.

[8] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983. With applications to relativity.

[9] J. A. Schouten. *Ricci-calculus. An introduction to tensor analysis and its geometrical applications*. Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Bd X. Springer-Verlag, Berlin, 1954. 2d. ed.

[10] A. A. Shaikh and S. K. Jana. On weakly cyclic Ricci symmetric manifolds. *Ann. Polon. Math.*, 89(3):273–288, 2006.

[11] L. Tamássy and T. Q. Binh. On weak symmetries of Einstein and Sasakian manifolds. *Tensor (N.S.)*, 53(Commemoration Volume I):140–148, 1993. International Conference on Differential Geometry and its Applications (Bucharest, 1992).

[12] K. Yano. On semi-symmetric metric connection. *Rev. Roumaine Math. Pures Appl.*, 15:1579–1586, 1970.

Received March 29, 2009.

A. A. Shaikh,
Department of Mathematics,
Burdwan-713 104,
West Bengal, India
E-mail address: aask2003@yahoo.co.in

Lovejoy S. K. Das,
Department of Mathematics,
Ken State University,
Tascarawas Campus 330,
University Drive North East,
New Philadelphia,
Ohio 44663-9403, USA
E-mail address: ldas@kent.edu

Sanjib Kumar Jana,
Department of Mathematics,
Burdwan-713 104,
West Bengal, India