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FIXED POINT THEOREMS FOR A GENERAL CLASS OF ALMOST CONTRACTIONS IN METRIC SPACES

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Abstract. In this paper, we prove the existence of fixed points and common fixed points for a general class of almost contraction mappings in metric spaces. This class of almost contractions is an extension and generalization of several contractive conditions in the literature. Our main results are established without forcing the metric space to be complete.

1. Introduction

Let (X, d) be a complete metric space. A self-mapping T of X with

$$d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X, \tag{1.1}$$

where a is a constant satisfying $0 \leq a < 1$, is called a contraction. The popular Banach's contraction principle, which is one of the most important results in non-linear analysis, states that any contraction mapping $T\colon X \to X$ has a unique fixed point in X; and that the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to the unique fixed point.

As beautiful as the Banach's contraction principle is, one setback suffered by the theorem is that the contractive condition (1.1) implies that T must be continuous on X. A series of fruitful efforts have been made, over the years, to address this setback. Several weaker contractive conditions were introduced which do not force T to be continuous on X. In 1968, Kannan [14] replaced (1.1) with the following: there exists a constant $b \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty)) \text{for all } x, y \in X. \tag{1.2}$$

A similar condition to (1.2) was given in 1972 by Chatterjea [11] as follows.

$$d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx)) \text{ for all } x, y \in X, \tag{1.3}$$

where constant c satisfies $c \in [0, \frac{1}{2})$.

Several other results abound in the literature which are devoted to obtaining fixed points without necessarily requiring the continuity of T. For more on

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these various earlier definitions of contractive mappings, the reader may see [3], [8] [19] and others.

One of such classes of contractions, introduced by V. Berinde in [6], which is of interest to us in this paper is the almost contractions. $T\colon X \to X$ is said to be an almost contraction if there exists a constant $\delta \in [0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta \cdot d(x, y) + Ld(y, Tx) \text{ for all } x, y \in X, \tag{1.4}$$

It is well known that an almost contraction needs not have a unique fixed point. For example, if (X, d) is the closed unit interval $[0, 1]$ with the usual metric, and T is the identity map on X, the condition (1.4) above is clearly satisfied whenever $L \geq 1 - \delta$ but the fixed point set of T is the entire interval $[0, 1]$.

A detailed study of almost contractions can be found in [17]. Interested reader may also see Berinde [4, 5, 6, 7, 8] and Osilike [16] for various convergence and stability results obtained using the class of almost contractions and its weaker forms.

In this paper, we shall establish fixed point results involving a general class of almost contractions in metric spaces.

2. Preliminary results

In 2008, Akram, Zafar and Siddiqui[2] introduced a general class of contractions called A-contractions. They gave the following definition.

Let $\mathbf{R}_+$ denote the set of all nonnegative real numbers and A the set of all functions $\alpha\colon \mathbf{R}^{\mathbf{3}}_+ \to \mathbf{R}_+$ satisfying the following conditions.

(i) α is continuous on the set $\mathbf{R}^{\mathbf{3}}_+$
(ii) $a \leq kb$ for some $k \in [0, 1)$ whenever $a \leq \alpha(a, b, b)$ or $a \leq \alpha(b, a, b)$ or $a \leq \alpha(b, b, a)$ for all $a, b \in \mathbf{R}_+$.

Definition 1. A self-mapping T of a metric space X is said to be an A-contraction if it satisfies

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)) \tag{2.1}$$

for all $x, y \in X$ and some $\alpha \in A$.

The authors demonstrated how the A-contractions is a proper super class of several other existing classes of contractions. They obtained a unique fixed point in a complete metric space X and showed that the Picard iteration converges to the unique fixed point.

From the foregoing, it becomes natural to define a class of almost A-contractions by replacing condition (2.1) in Definition 1 above with (2.2) in the following.

Definition 2. A self-mapping T of a metric space X is an almost A-contraction if for some $\alpha \in A$ and a constant $L \geq 0$, the following condition is satisfied:

$$d(Tx, Ty) \leq \alpha(d(x, y), d(x, Tx), d(y, Ty)) + Ld(y, Tx) \text{ for all } x, y \in X. \tag{2.2}$$

Theorem 1. *Let T be an almost A-contraction on a complete metric space X. Then*

(i) T has a fixed point in X;

(ii) For any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$ converges to some fixed point u of T;

(iii) The following estimate holds

$$d(x_n, u) \leq \frac{k^n}{1-k} d(x_0, x_1), \ n = 0, 1, 2, \ldots \tag{2.3}$$

for some $k \in [0, 1)$.

Proof. For any $x_0 \in X$, since T is an almost A-contraction, there exist $\alpha \in A$ and a constant $L \geq 0$ such that, with $x = x_{n-1}$ and $y = x_n$, (2.2) gives

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq \alpha(d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)) + Ld(x_n, Tx_{n-1}) \\ &= \alpha(d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1})) \end{aligned}$$

This implies that

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n) \quad n = 1, 2, \ldots$$

for some $k \in [0, 1)$. By induction, we obtain

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1), \quad n = 0, 1, 2, \ldots.$$

Therefore, for a natural number $m > n$ where $n = 0, 1, 2, \ldots$, using the triangle inequality, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq k^n d(x_0, x_1) + k^{n+1} d(x_0, x_1) + \cdots + k^{m-1} d(x_0, x_1) \\ &\leq k^n d(x_0, x_1)[1 + k + \ldots + k^{m-n-1} + \cdots] \\ &= \frac{k^n}{1-k} d(x_0, x_1) \end{aligned}$$

In other words, for any $m > n$, $n = 0, 1, 2, \ldots$,

$$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, x_1). \tag{2.4}$$

This shows that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and since X is complete, there exists $u \in X$ with $lim_{n \to \infty} x_n = u$. In addition, by the continuity of α,

$$\begin{aligned} d(u, Tu) &\leq d(u, x_n) + d(Tx_{n-1}, Tu) \\ &\leq d(u, x_n) + \alpha(d(x_{n-1}, u), d(x_{n-1}, x_n), d(u, Tu)) + Ld(u, x_n), \end{aligned}$$

This gives

$$d(u,Tu) \leq d(u,u) + \alpha(d(u,u), d(u,u), d(u,Tu)) + Ld(u,u), \text{ as } n \to \infty.$$

That is, $d(u,Tu) \leq \alpha(0,0,d(u,Tu))$, which implies that $d(u,Tu) \leq 0$. Thus, $Tu = u$. Therefore, u is a fixed point of T.

This shows that for any $x_0 \in X$, the Picard iteration converges to a fixed point $u \in X$. Finally, we obtain (iii) by letting $m \to \infty$ in (2.4). $\square$

In the part *(ii)* of Theorem 1 of [2], it was shown that for some constant $r \in [0, \frac{1}{2})$ we have

$$r(d(Tx,x) + d(Ty,y)) = \alpha(d(x,y), d(Tx,x), d(Ty,y))$$

whenever $\alpha \in A$. Thus we obtain the following corollary.

Corollary 1. *Let T be a self-mapping of a complete metric space X satisfying*

$$d(Tx,Ty) \leq r(d(Tx,x) + d(Ty,y)) + Ld(y,Tx) \textit{ for all } x,y \in X,$$

for some constant $r \in [0, \frac{1}{2})$. Then

(i) T has a fixed point in X;

(ii) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$ converges to some fixed point u of T; and

(iii) The following estimate holds

$$d(x_n,u) \leq \frac{k^n}{1-k} d(x_0,x_1), \ n = 0,1,2,\ldots,$$

for some $k = \frac{r}{1-r} \in [0,1)$.

In part *(i)* of the same theorem of [2], it was also shown that for $0 \leq h < 1, 0 \leq r < \frac{1}{2}$, the following inequality holds.

$$h\sqrt{d(Tx,x)d(Ty,y)} \leq r(d(Tx,x) + d(Ty,y)).$$

Therefore, from Corollary 1, we have

Corollary 2. *Let T be a self-mapping of a complete metric space X satisfying*

$$d(Tx,Ty) \leq h\sqrt{d(Tx,x)d(Ty,y)} + Ld(y,Tx) \textit{ for all } x,y \in X,$$

for some constant $h \in [0,1)$. Then

(i) T has a fixed point in X;

(ii) for any $x_0 \in X$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$ converges to some fixed point u of T; and

(iii) The following estimate holds

$$d(x_n,u) \leq \frac{k^n}{1-k} d(x_0,x_1), \ n = 0,1,2,\ldots,$$

for some $k = \frac{h}{2}$.

We shall illustrate Theorem 1. with the following example.

Example. Let $X = [0,1]$ with the usual metric $d(x,y) = |x-y|$. Define $T\colon X \to X$ by

$$Tx = \begin{cases} 0 & if \;\; x = 0 \\ \frac{1}{2} & if \;\; x \in (0,1]. \end{cases}$$

Observe that the fixed point set of T, $F_T = \{0, \frac{1}{2}\}$.

Define $\alpha\colon \mathbf{R}_+^3 \to \mathbf{R}_+$ by $\alpha(a_1,a_2,a_3) = \frac{1}{4}(a_1+a_2+a_3)$ for all $a_1,a_2,a_3 \in \mathbf{R}_+$. It is easy to see that $\alpha \in A$ (see [1]).

When $x = y = 0$, or when $x > 0$ and $y > 0$, (2.2) is clearly satisfied. Indeed, the left hand side of the inequality is 0.

Moreover, when $x = 0$ and $y > 0$, (2.2) holds whenever $L \geq \frac{1}{8}$.

Finally, when $x > 0$ and $y = 0$, (2.2) is satisfied provided $L \geq \frac{1}{4}$.

Remark. It is clear that the mapping T as defined in the example above, fails to satisfy inequality (1.4). If we choose $x = 0$ and $y \in (0,1]$, then by the contractive condition (1.4) we get

$$\left|0 - \frac{1}{2}\right| \leq \delta|0-y| + L|y-0|$$

from which, by letting $y \to 0$ we get the contradiction $\frac{1}{2} < 0$. Thus, the class of almost A-contractions is larger than that of almost contractions.

3. Fixed points for almost A-contractions in arbitrary metric space.

Our main interest in this section is to obtain fixed points for almost A-contractions without necessarily requiring X to be complete.

Inspired by section 5 of [13], let T be a self-mapping of an arbitrary metric space X. We assume that a function $f\colon X \to \mathbf{R}_+$ defined by $f(x) = d(x,Tx)$, for all $x \in X$, attains its minimum in X. The following result shows existence of a fixed point of T without the completeness condition.

Theorem 2. *Let T be an almost A-contraction on a metric space X. Assume there exists a point $u \in X$ such that*

$$f(u) = \inf\{f(x) : x \in X\}, \tag{3.1}$$

where $f(x) = d(x,Tx)$ for all $x \in X$. Then u is a fixed point of T.

Proof. Define a sequence $\{T^n x\}$ in X by $T^{n+1}x = TT^n x$, for $n = 0,1,2,\ldots$. Since T is an almost A-contraction, then for some constant $k \in [0,1)$ and any $x \in X$, we have

$$d(T^n x, T^{n+1}x) \leq k^n d(x,Tx), \quad n = 0,1,2,\ldots.$$

Suppose $d(u,Tu) > 0$, then for a positive integer j, we obtain

$$d(T^j u, T^{j+1}u) \leq k^j d(u,Tu).$$

That is, $f(T^j u) \leq k^j f(u)$. This contradicts (3.1). Hence $d(u, Tu) = 0$, so that $Tu = u$. □

Finally in the following theorem, two metrics d and ρ are defined on a nonempty set X. Completeness is limited to only (X, d) while (X, ρ) is left as an arbitrary metric space.

Theorem 3. *Let X be a nonempty set with metrics d and ρ. Let S and T be a self-mappings of X such that*

(i) $d(x, y) \leq \rho(x, y)$, for all $x, y \in X$;
(ii) (X, d) is complete;
(iii) T is continuous on (X, d);
(iv) The following condition holds for some $\alpha \in A$, and $L \geq 0$;

$$\rho(Sx, Ty) \leq \alpha(\rho(x, y), \rho(x, Sx), \rho(y, Ty)) + L\rho(y, Sx)\rho(x, Ty).$$

Then S and T have a common fixed point $u \in X$.

Proof. For any $x_0 \in X$, define sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_n = \begin{cases} Sx_{n-1} & \text{when } n \text{ is odd;} \\ Tx_{n-1} & \text{when } n \text{ is even.} \end{cases}$$

Then, when n is odd, condition (iv) yields

$$\begin{aligned} \rho(x_1, x_2) &= \rho(Sx_0, Tx_1) \\ &\leq \alpha(\rho(x_0, x_1), \rho(x_0, Sx_0), \rho(x_1, Tx_1)) + L\rho(x_1, Sx_0)\rho(x_0, Tx_1)) \\ &= \alpha(\rho(x_0, x_1), \rho(x_0, x_1), \rho(x_1, x_2)) + L\rho(x_1, x_1)\rho(x_0, x_2)) \\ &= \alpha(\rho(x_0, x_1), \rho(x_0, x_1), \rho(x_1, x_2)) \end{aligned}$$

That is, $\rho(x_1, x_2) \leq k\rho(x_0, x_1)$. Similarly, when n is even, we have $\rho(x_2, x_3) \leq k\rho(x_1, x_2)$.

Therefore for all $n \in \mathbf{N}$, (iv) gives $\rho(x_n, x_{n+1}) \leq k\rho(x_{n-1}, x_n)$. This inductively gives

$$\rho(x_n, x_{n+1}) \leq k^n \rho(x_0, x_1).$$

As in the proof of Theorem 1, the sequence $\{x_n\}$ is Cauchy in (X, ρ). The sequence is also Cauchy in (X, d) since condition (i) holds. Furthermore, it converges to a point $u \in X$ on the account of condition (ii).

Using (iii), we have $Tu = T(\lim_{n\to\infty} T^n x) = \lim_{n\to\infty} T^{n+1} x = u$. That is u is a fixed point of T in X.

Finally, by (iv),

$$\begin{aligned} \rho(Su, u) = \rho(Su, Tu) &\leq \alpha(\rho(u, u), \rho(u, Su), \rho(u, Tu)) + L\rho(u, Su)\rho(u, Tu) \\ &= \alpha(0, \rho(Su, u), 0) \end{aligned}$$

Therefore, $\rho(Su, u) = 0$. Hence u is a common fixed point of S and T. □

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