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ON GENERALIZATIONS OF SOME COMMON FIXED POINT THEOREMS IN UNIFORM SPACES

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Abstract. In this paper, we establish some generalizations of some common fixed point theorems in uniform spaces for selfmappings by using the notions of A-distance and E-distance. A more general φ-contractive-type condition than those of Aamri and El Moutawakil [1] and Olatinwo [8] was employed to establish our results. These generalizations can be viewed as an improvement to some of the results of Aamri and El Moutawakil [1] and Olatinwo [8].

1. Introduction

Let X be a nonempty set and let Φ be a nonempty family of subsets of $X \times X$. The pair (X, Φ) is called a uniform space if it satisfies the following properties:

(i) if G is in Φ, then G contains the diagonal $\{(x, x)|x \in X\}$;
(ii) if G is in Φ and H is a subset of $X \times X$ which contains G, then H is in Φ;
(iii) if G and H are in Φ, then $G \bigcap H$ is in Φ;
(iv) if G is in Φ, then there exists H in Φ, such that, whenever (x, y) and (y, z) are in H, then (x, z) is in H;
(v) if G is in Φ, then $\{(y, x)|(x, y) \in G\}$ is also in Φ.

Φ is called the uniform structure of X and its elements are called entourages or neighbourhoods or surroundings.

If property (v) is omitted, then (X, Φ) is called a quasiuniform space. (For examples, see Bourbaki [4] and Zeidler [14].) Several researchers such as Berinde [3], Jachymski [5], Kada et al [6], Rhoades [9], Rus [11], Wang et al [13] and Zeidler [14] studied the theory of fixed point or common fixed point for contractive selfmappings in complete metric spaces or Banach spaces in general.

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Within the last two decades, Kang [7], Montes and Charris [10] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space.

Later, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

Aamri and El Moutawakil [1] introduced and employed the following contractive definition: Let $f, g\colon X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \quad \forall x, y \in X, \tag{1.1}$$

where $\psi\colon \Re^+ \to \Re^+$ is a nondecreasing function satisfying

(i) for each $t \in (0, +\infty)$, $0 < \psi(t)$,
(ii) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \in (0, +\infty)$.

ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$, $t \in \Re^+$.

Recently, Olatinwo [8] established some common fixed point theorems by employing the following contractive definition: Let $f, g\colon X \to X$ be selfmappings of X. There exist $L \geq 0$ and a comparison function $\psi\colon \Re^+ \to \Re^+$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi(p(g(x), g(y))), \quad \forall x, y \in X. \tag{1.2}$$

In this paper, we establish some common fixed point theorems by using a more general contractive condition than (1.1) and (1.2).

We also employ the concepts of an A-distance, an E-distance as well as the notion of comparison function in this paper.

2. Preliminaries

The following definitions shall be required in the sequel.

Let (X, Φ) be a uniform space. Without loss of generality, $(X, \tau(\Phi))$ denotes a topological space whenever topological concepts are mentioned in the context of a uniform space (X, Φ). (For instance, see Aamri and El Moutawakil [1]). Definitions $2.1 - 2.6$ are contained in Aamri and El Moutawakil [1].

Definition 2.1. If $H \in \Phi$ and $(x, y) \in H, (y, x) \in H$, x and y are said to be H-*close*. A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $H \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are H-close for $n, m \geq N$.

Definition 2.2. A function $p\colon X \times X \to \Re^+$ is said to be an A-*distance* if for any $H \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in H$.

Definition 2.3. A function $p\colon X \times X \to \Re^+$ is said to be an E-*distance* if

(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y)$, $\forall x, y \in X$.

Definition 2.4. A uniform space (X, Φ) is called a *Hausdorff uniform space* if and only if the intersection of all $H \in \Phi$ reduces to the diagonal $\{(x, x)|x \in X\}$, i.e. if $(x, y) \in H$ for all $H \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $H \in \Phi$ is said to be *symmetrical* if $H = H^{-1} = \{(y, x)|(x, y) \in H\}$.

Definition 2.5. Let (X, Φ) be a uniform space and p be an A-distance on X.

(i) X is said to be *S-complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$.
(ii) X is said to be *p-Cauchy complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.
(iii) $f\colon X \to X$ is said to be *p-continuous* if $\lim_{n\to\infty} p(x_n, x) = 0$ implies that $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.
(iv) $f\colon X \to X$ is *$\tau(\Phi)$-continuous* if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.
(v) X is said to be *p-bounded* if $\delta_p = \sup\{p(x, y)|x, y \in X\} < \infty$.

Definition 2.6. Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be *p-compatible* if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n\to\infty} p(f(x_n), u) = \lim_{n\to\infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

The following definition which is also required in the sequel to establish some common fixed point results is contained in Berinde [2], Berinde [3], Rus [11] and Rus et al [12].

Definition 2.7. A function $\psi\colon \Re^+ \to \Re^+$ is called a *comparison function* if

(i) ψ is monotone increasing;
(ii) $\lim_{n\to\infty} \psi^n(t) = 0$, $\forall t \geq 0$.

Remark 2.8. Every comparison function satisfies the condition $\psi(0) = 0$.

We also note that both conditions (i) and (ii) imply that $\psi(t) < t, \forall t > 0, t \in \Re^+$.

Our aim in this paper is to establish some common fixed point theorems by using a more general contractive condition than (1.1) and (1.2). Consequently, we shall employ the following contractive definition: Let $f, g\colon X \to X$ be selfmappings of X. There exist comparison functions $\psi_1\colon \Re^+ \to \Re^+$ and $\psi_2\colon \Re^+ \to \Re^+$ with $\psi_1(0) = 0$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq \psi_1(p(x, g(x))) + \psi_2(p(g(x), g(y))), \quad \forall x, y \in X. \tag{2.1}$$

Remark 2.9. The contractive condition (2.1) is more general than (1.2) in the sense that if $\psi_1(u) = Lu$ in (1.3), for $L \geq 0$, $u \in \Re^+$, then we obtain

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi_2(p(g(x), g(y))), \quad \forall x, y \in X,$$

which is the contractive condition employed by Olatinwo [8] in (1.2).

Again, if $L = 0$ in the above inequality, then we obtain (1.1), which was employed by Aamri and El Moutawakil [1].

Thus, our contractive condition (2.1) is a generalization of both the contractive definitions (1.1) and (1.2) of Aamri and El Moutawakil [1] and Olatinwo [8] respectively.

The following Lemma contained in Aamri and El Moutawakil [1], Kang [7] and Montes and Charris [10] shall be required in the sequel.

Lemma 2.10. *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be sequences in $\Re^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:*

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in N$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.*
(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in N$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.*
(c) *If $p(x_n, x_m) \leq \alpha_n, \forall m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).*

The following remark is contained in Aamri and El Moutawakil [1].

Remark 2.11. A sequence in X is p-Cauchy if it satisfies the usual metric property.

Example 2.12. In mathematical analysis, a uniform space is a set with a uniform structure. Therefore, uniform spaces are topological spaces with additional structure which is used to define uniform properties such as completeness, uniform continuity and uniform convergence. Uniform spaces generalize metric spaces and topological groups.

Every metric space (X, d) can be considered as a uniform space, since a metric is a pseudometric and therefore, the pseudometric definition provides X with a uniform structure.

However, different metric spaces can have the same uniform structure. An example is a constant multiple of a metric.

Using metrics, an example of distinct uniform structures with coinciding topologies can be constructed.

For instance, let $d_1(x, y) = |x - y|$ be the usual metric on $\Re$ and let $d_2(x, y) = |e^x - e^y|, \forall x, y \in \Re$.

Then, both metrics induce the usual topology on $\Re$, yet the uniform structures are distinct, since $\{(x, y) : |x - y| < 1\}$ is an entourage in the uniform structure for d_1 but not for d_2. Intuitively, this example can be seen as taking the usual uniformity and distorting it through the action of a continuous, yet non-uniformly continuous function.

Clearly, d_1 is an $A-$distance while d_2 is an $E-$distance on $\Re$. Indeed, d_2 is also an $A-$distance which also satisfies condition (p_2) of Definition 2.3 of an $E-$distance as follows:

For arbitrary $x, y, z \in \Re$, we have

$$d_2(x,y) = |e^x - e^y| \leq |e^x - e^z| + |e^z - e^y| = d_2(x,z) + d_2(z,y)$$

3. The main results

One of our main results in this paper is the existence result for the common fixed point of f and g given by:

Theorem 3.1. *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X such that X is p-bounded and S-complete. For arbitrary $x_0 \in X$, define a sequence $\{x_n\}_{n=0}^{\infty}$ iteratively by*

$$x_n = f(x_{n-1}), \quad n = 1, 2, \ldots$$

Suppose that f and g are commuting p-continuous or $\tau(\Phi)$-continuous self-mappings of X such that

(i) $f(X) \subseteq g(X)$,
(ii) $p(f(x_i), f(x_i)) = 0, \forall x_i \in X,\ i = 0, 1, 2, \ldots,$
(iii) $f, g\colon X \to X$ *satisfy the contractive condition* (2.1).

Suppose also that $\psi_1\colon \Re^+ \to \Re^+$ and $\psi_2\colon \Re^+ \to \Re^+$ are comparison functions with $\psi_1(0) = 0$. Then, f and g have a common fixed point.

Proof. For arbitrary $x_0 \in X$, select $x_1 \in X$ such that $f(x_0) = g(x_1)$. Similarly, for $x_1 \in X$, select $x_2 \in X$ such that $f(x_1) = g(x_2)$.

Continuing this process, we select $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$.

Now, we show that the sequence $\{f(x_n)\}_{n=0}^{\infty}$ so generated is a p-Cauchy sequence. Indeed, since $x_n = f(x_{n-1}),\ n = 1, 2, \ldots$, then by using the contractive condition (2.1) together with conditions (ii) and (iii) of the Theorem, we get

$$\begin{aligned}
p(f(x_n), f(x_{n+m})) &\leq \psi_1(p(x_n, g(x_n))) + \psi_2(p(g(x_n), g(x_{n+m}))) \\
&= \psi_1(p(f(x_{n-1}), f(x_{n-1}))) + \psi_2(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&= \psi_1(0) + \psi_2(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&= 0 + \psi_2(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&= \psi_2(p(f(x_{n-1}), f(x_{n+m-1}))) \\
&\leq \psi_2(\psi_1(p(x_{n-1}, g(x_{n-1}))) + \psi_2(p(g(x_{n-1}), g(x_{n+m-1})))) \\
&= \psi_2(\psi_1(p(f(x_{n-2}), f(x_{n-2}))) + \psi_2(p(f(x_{n-2}), f(x_{n+m-2})))) \\
&= \psi_2(\psi_1(0) + \psi_2(p(f(x_{n-2}), f(x_{n+m-2})))) \\
&= \psi_2(0 + \psi_2(p(f(x_{n-2}), f(x_{n+m-2})))) \\
&= \psi_2^2(p(f(x_{n-2}), f(x_{n+m-2}))) \\
&\leq \ldots \leq \psi_2^n(p(f(x_0), f(x_m))) \leq \psi_2^n(\delta_p(X))
\end{aligned}$$

which implies that

$$p(f(x_n), f(x_{n+m})) \leq \psi_2^n(\delta_p(X), \tag{3.1}$$

where $p(f(x_0), f(x_m))) \leq \delta_p(X)$ and $\delta_p(X) = \sup\{p(x, y)|x, y \in X\} < \infty$.

Using the definition of comparison function in (3.1) gives

$$\lim_{n\to\infty} \psi_2^n(\delta_p(X)) = 0$$

and hence,

$$p(f(x_n), f(x_{n+m})) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore, by using Lemma 2.10(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence.

But, X is S-complete. Hence, $\lim_{n\to\infty} p(f(x_n), u)) = 0$, for some $u \in X$.

Since, $x_n \in X$ implies that $f(x_{n-1}) = g(x_n)$, therefore, we have

$$\lim_{n\to\infty} p(g(x_n), u)) = 0.$$

Also, since f and g are p-continuous, then

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0.$$

But, f and g are commuting, therefore $fg = gf$. Hence,

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0.$$

By applying Lemma 2.10(a), we have that $f(u) = g(u)$.

Since, $f(u) = g(u)$ and $fg = gf$, then we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$.

We need to show that $p(f(u), f(f(u))) = 0$. Suppose on the contrary that $p(f(u), f(f(u))) \neq 0$. By using the contractive definition (2.1) and the condition that $\psi(t) < t, \forall t > 0$ in the Remark 2.8, we obtain

$$\begin{aligned}
p(f(u), f(f(u))) &\leq \psi_1(p(u, g(u))) + \psi_2(p(g(u), g(f(u)))) \\
&= \psi_1(p(f(u), f(u))) + \psi_2(p(f(u), f(f(u)))) \\
&= \psi_1(0) + \psi_2(p(f(u), f(f(u)))) \\
&= \psi_2(p(f(u), f(f(u)))) \\
&< p(f(u), f(f(u))),
\end{aligned}$$

which is a contradiction. Hence, $p(f(u), f(f(u))) = 0$.

By using condition (ii) of the Theorem, we have $p(f(u), f(u)) = 0$.

Therefore, since $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, by using Lemma 2.10(a), we get $f(f(u)) = f(u)$, which implies that $f(u)$ is a fixed point of f.

But, $f(u) = f(f(u)) = f(g(u)) = g(f(u))$, which shows that $f(u)$ is also a fixed point of g. Thus, $f(u)$ is a common fixed point of f and g.

The proof of that case when f and g are $\tau(\Phi)$-continuous is similar since S-completeness implies p-Cauchy completeness. This completes the proof. □

Remark 3.2. The existence result in Theorem 3.1 is a generalization of Theorem 3.1 of Aamri and El Moutawakil [1] as well as Theorem 3.1 of Olatinwo [8].

The uniqueness of the common fixed point of f and g is established by the next two Theorems.

Theorem 3.3. *Let $(X, \Phi), f, g, \psi_1, \psi_2, \{x_n\}_{n=0}^{\infty}$ be as defined in Theorem 3.1 above and p an E-distance on X. Then, f and g have a unique common fixed point.*

Proof. Since an E-distance function p is also an A-distance, then by Theorem 3.1 above, we know that f and g have a common fixed point. Suppose that there exist $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$.

We need to show that $u = v$. Suppose on the contrary that $u \neq v$, i.e. let $p(u, v) \neq 0$.

Then, by using the contrative definition (2.1) and the condition that $\psi(t) < t, \forall t > 0$ in the Remark 2.8, we obtain

$$\begin{aligned} p(u, v) &= p(f(u), f(v)) \\ &\leq \psi_1(p(u, g(u))) + \psi_2(p(g(u), g(v))) \\ &= \psi_1(p(u, u)) + \psi_2(p(u, v)) \\ &= \psi_1(0) + \psi_2(p(u, v)) \\ &= 0 + \psi_2(p(u, v)) = \psi_2(p(u, v)) < p(u, v) \end{aligned}$$

which is a contradiction. Hence, we have $p(u, v) = 0$.

Similarly, we have $p(v, u) = 0$. By applying condition (p_2) of Definition 2.3, we obtain $p(u, u) \leq p(u, v) + p(v, u)$, and hence $p(u, u) = 0$.

Since $p(u, u) = 0$ and $p(u, v) = 0$, then by using Lemma 2.10(a), we get $u = v$. This completes the proof. □

Remark 3.4. The uniqueness result in Theorem 3.3 is a generalization of Theorem 3.2 as well as Corollaries 3.1 and 3.2 of Aamri and El Moutawakil [1].

Also, the uniqueness result in Theorem 3.3 is a generalization of Theorem 3.3 of Olatinwo [8].

Theorem 3.5. *Let $(X, \Phi), p, \psi_1, \psi_2$ and $\{x_n\}_{n=0}^{\infty}$ be as defined in Theorem 3.1 above. Suppose that f and g are p-compatible, p-continuous or $\tau(\Phi)$-continuous selfmappings of X satisfying conditions (i), (ii) and (iii) of Theorem 3.1 above. Then, f and g have a unique common fixed point.*

Proof. By Theorem 3.1 above, we know that f and g have a common fixed point. Hence, for some $u \in X$, we have

$$\lim_{n\to\infty} p(f(x_n, u)) = \lim_{n\to\infty} p(g(x_n, u)) = 0.$$

Since f and g are p-continuous, then

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0.$$

Also, since f and g are p-compatible, then

$$\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0.$$

By applying condition (p_2) of Definition 2.3, we obtain

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)).$$

Letting $n \to \infty$ and using Lemma 2.10(a) yields

$$\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0.$$

Since $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 2.10(a), we obtain $f(u) = g(u)$.

The rest of the proof goes as in Theorem 3.1 and therefore it is omitted. This completes the proof. □

Remark 3.6. The uniqueness result in Theorem 3.5 is a generalization of Theorem 3.3 of Aamri and El Moutawakil [1]. Also, the uniqueness result in Theorem 3.5 is a generalization of Theorem 3.5 of Olatinwo [8].

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