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THE MAXIMAL OPERATORS OF LOGARITHMIC MEANS OF ONE-DIMENSIONAL VILENKIN-FOURIER SERIES

GEORGE TEPHNADZE

Abstract. The main aim of this paper is to investigate (H_p, L_p)-type inequalities for maximal operators of logarithmic means of one-dimensional bounded Vilenkin-Fourier series.

1. Introduction

In one-dimensional case the weak type inequality

$$\mu\left(\sigma^{*}f>\lambda\right)\leq\frac{c}{\lambda}\left\|f\right\|_{1}\quad(\lambda>0)$$

can be found in Zygmund [20] for the trigonometric series, in Schipp [11] for Walsh series and in Pál, Simon [10] for bounded Vilenkin series. Again in one-dimensional, Fujji [3] and Simon [12] verified that σ^* is bounded from H_1 to L_1. Weisz [17] generalized this result and proved the boundedness of σ^* from the martingale space H_p to the space L_p for $p > 1/2$. Simon [13] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ due to Goginava ([7], see also [2]).

Riesz's logarithmic means with respect to the trigonometric system was studied by a lot of authors. We mention, for instance, the paper by Szász [14] and Yabuta [19]. This means with respect to the Walsh and Vilenkin systems was discussed by Simon[13] and Gát[4].

Móricz and Siddiqi[9] investigated the approximation properties of some special Nörlund means of Walsh-Fourier series of L_p function in norm. The case when $q_k = 1/k$ is excluded, since the methods of Móricz and Siddiqi are not applicable to Nörlund logarithmic means. In [5] Gát and Goginava proved some convergence and divergence properties of the Nörlund logarithmic means of functions in the class of continuous functions and in the Lebesgue space L_1. Among there, they gave a negative answer to the question of Móricz and

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Siddiqi [9]. Gát and Goginava [6] proved that for each measurable function $\phi(u) = \circ\left(u\sqrt{\log u}\right)$ there exists an integrable function f, such that

$$\int_{G_m} \phi(|f(x)|)\, d\mu\,(x) < \infty$$

and there exist a set with positive measure, such that the Walsh-logarithmic means of the function diverge on this set.

The main aim of this paper is to investigate (H_p, L_p)-type inequalities for the maximal operators of Riesz and Nörlund logarithmic means of one-dimensional Vilenkin-Fourier series. We prove that the maximal operator R^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. We also shows that when $0 < p \leq 1/2$ there exists a martingale $f \in H_p$ for which

$$\|R^* f\|_{L_p} = +\infty.$$

For the Nörlund logarithmic means we prove that when $0 < p \leq 1$ there exists a martingale $f \in H_p$ for which

$$\|L^* f\|_{L_p} = +\infty.$$

Analogical theorems for Walsh-Paley system is proved in [8].

2. Definitions and notation

Let N_+ denote the set of the positive integers, $N := N_+ \cup \{0\}$. Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$ the addition group of integers modulo m_k.

Define the group G_m as the complete direct product of the groups Z_{m_i} with the product of the discrete topologies of the groups Z_{m_j}.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on G_{m_k} with $\mu(G_m) = 1$.

If the sequence m is bounded then G_m is called a bounded Vilenkin group, else it is called an unbounded one. *In this paper we discuss bounded Vilenkin groups only.* The elements of G_m are represented by sequences

$$x := (x_0, x_1, \ldots, x_j, \ldots) \ \left(x_i \in Z_{m_j}\right).$$

It is easy to give a base for the neighborhood of G_m

$$I_0(x) : = G_m$$
$$I_n(x) : = \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in N)$$

Denote $I_n := I_n(0)$ for $n \in N_+$.

If we define the so-called generalized number system based on m in the following way:

$$M_0 := 1,\ M_{k+1} := m_k M_k \quad (k \in N),$$

then every $n \in N$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$ where $n_j \in Z_{m_j}$ $(j \in N_+)$ and only a finite number of n_js differ from zero.

Next, we introduce on G_m an orthonormal system which is called the Vilenkin system. At first, define the complex valued function $r_k(x) : G_m \to C$, the generalized Rademacher functions as

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad \left(i^2 = -1, x \in G_m, k \in N\right).$$

Now, define the Vilenkin system $\psi := (\psi_n : n \in N)$ on G_m as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) \quad (n \in N).$$

Specifically, we call this system the Walsh-Paley one if $m \equiv 2$. The Vilenkin system is orthonormal and complete in $L_2(G_m)$ [1, 15].

Now, we introduce analogues of the usual definitions in Fourier-analysis. If $f \in L_1(G_m)$ we can establish the Fourier coefficients, the partial sums of the Fourier series, the Fejér means, the Dirichlet kernels with respect to the Vilenkin system ψ in the usual manner:

$$\widehat{f}(k) : = \int_{G_m} f \overline{\psi}_k d\mu \quad (k \in N),$$

$$S_n f : = \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k \quad (n \in N_+, S_0 f := 0),$$

$$\sigma_n f : = \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in N_+),$$

$$D_n : = \sum_{k=0}^{n-1} \psi_k \qquad (n \in N_+).$$

Recall that

$$D_{M_n}(x) = \begin{cases} M_n, & \text{if } x \in I_n \\ 0, & \text{if } x \notin I_n. \end{cases}$$

The norm (or quasinorm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}} \quad (0 < p < \infty).$$

The $\sigma-$algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ is denoted by $\digamma_n (n \in N)$. Denote by $f = \left(f^{(n)}, n \in N\right)$ a martingale with respect to $\digamma_n (n \in N)$ (for details see e.g. [16]).

The maximal function of a martingale f is defined by

$$f^* = \sup_{n \in N} \left| f^{(n)} \right|.$$

In case $f \in L_1(G_m)$, the maximal functions are also be given by

$$f^*(x) = \sup_{n\in N} \frac{1}{\mu(I_n(x))} \left| \int_{I_n(x)} f(u)\, d\mu(u) \right|.$$

For $0 < p < \infty$ the Hardy martingale spaces $H_p(G_m)$ consist of all martingale for which

$$\|f\|_{H_p} := \|f^*\|_{L_p} < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $(S_{M_n}(f) : n \in N)$ is a martingale.

If $f = \left(f^{(n)}, n \in N\right)$ is martingale then the Vilenkin-Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)}(x)\, \overline{\Psi}_i(x)\, d\mu(x).$$

The Vilenkin-Fourier coefficients of $f \in L_1(G_m)$ are the same as those of the martingale $(S_{M_n}(f) : n \in N)$ obtained from f.

In the literature, there is the notion of Riesz's logarithmic means of the Fourier series. The n-th Riesz's logarithmic means of the Fourier series of an integrable function f is defined by

$$R_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{k},$$

where

$$l_n := \sum_{k=1}^{n} (1/k).$$

Let $\{q_k : k > 0\}$ be a sequence of nonnegative numbers. The n-th Nö rlund means for the Fourier series of f is defined by

$$\frac{1}{Q_n} \sum_{k=1}^{n} q_{n-k} S_k f,$$

where

$$Q_n := \sum_{k=1}^{n} q_k.$$

If $q_k = 1/k$, then we get Nörlund logarithmic means

$$L_n f(x) := \frac{1}{l_n} \sum_{k=1}^{n} \frac{S_k f(x)}{n-k}$$

It is a kind of "reverse" Riesz's logarithmic mean. In this paper we call these means logarithmic means.

For the martingale f we consider the following maximal operators of

$$R^{*}f(x):=\sup_{n\in N}\left|R_{n}f\left(x\right)\right|,$$
$$L^{*}f(x):=\sup_{n\in N}\left|L_{n}f\left(x\right)\right|,$$
$$\sigma^{*}f\left(x\right):=\sup_{n\in N}\left|\sigma_{n}f(x)\right|.$$

A bounded measurable function a is a p-atom, if there exists a dyadic interval I, such that

a) $\int_{I}ad\mu=0$,
b) $\|a\|_{\infty}\leq\mu\left(I\right)^{-1/p}$,
c) $\operatorname{supp}(a)\subset I$.

3. Formulation of main results

Theorem 1. *Let $p>1/2$. Then the maximal operator R^{*} is bounded from the Hardy space H_{p} to the space L_{p}.*

Theorem 2. *Let $0<p\leq1/2$. Then there exists a martingale $f\in H_{p}$ such that*

$$\|R^{*}f\|_{p}=+\infty.$$

Corollary 1. *Let $0<p\leq1/2$. Then there exists a martingale $f\in H_{p}$ such that*

$$\|\sigma^{*}f\|_{p}=+\infty.$$

Theorem 3. *Let $0<p\leq1$. Then there exists a martingale $f\in L_{p}$ such that*

$$\|L^{*}f\|_{p}=+\infty.$$

4. Auxiliary propositions

Lemma 1. [18] *A martingale $f=\left(f^{(n)},n\in N\right)$ is in $H_{p}\left(0<p\leq1\right)$ if and only if there exists a sequence $(a_{k},k\in N)$ of p-atoms and a sequence $(\mu_{k},k\in N)$ of a real numbers such that for every $n\in N$*

$$\sum_{k=0}^{\infty}\mu_{k}S_{M_{n}}a_{k}=f^{(n)},\tag{1}$$

$$\sum_{k=0}^{\infty}\left|\mu_{k}\right|^{p}<\infty.$$

Moreover,

$$\|f\|_{H_{p}}\backsim\inf\left(\sum_{K=0}^{\infty}\left|\mu_{k}\right|^{p}\right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (1).

5. Proof of the theorems

Proof of Theorem 1: Using Abel transformation we obtain

$$R_n f(x) = \frac{1}{l_n}\sum_{j=1}^{n-1}\frac{\sigma_j f(x)}{j+1} + \frac{\sigma_n f(x)}{l_n},$$

Consequently,

$$L^* f \leq c\sigma^* f. \tag{2}$$

On the other hand Weisz[17] proved that σ^* is bounded from the Hardy space H_p to the space L_p when $p > 1/2$. Hence, from (2) we conclude that R^* is bounded from the martingale Hardy space H_p to the space L_p when $p > 1/2$. □

Proof of Theorem 2: Let $\{\alpha_k : k \in N\}$ be an increasing sequence of the positive integers such that

$$\sum_{k=0}^{\infty} \alpha_k^{-p/2} < \infty, \tag{3}$$

$$\sum_{\eta=0}^{k-1} \frac{\left(M_{2\alpha_\eta}\right)^{1/p}}{\sqrt{\alpha_\eta}} < \frac{\left(M_{2\alpha_k}\right)^{1/p}}{\sqrt{\alpha_k}}, \tag{4}$$

$$\frac{\left(M_{2\alpha_{k-1}}\right)^{1/p}}{\sqrt{\alpha_{k-1}}} < \frac{M_{\alpha_k}}{\alpha_k^{3/2}}. \tag{5}$$

We note that such an increasing sequence $\{\alpha_k : k \in N\}$ which satisfies conditions (3)-(5) can be constructed.

Let

$$f^{(A)}(x) = \sum_{\{k;\ 2\alpha_k < A\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{m_{2\alpha_k}}{\sqrt{\alpha_k}}$$

and

$$a_k(x) = \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}}\left(D_{M_{(2\alpha_k+1)}}(x) - D_{M_{2\alpha_k}}(x)\right).$$

It is easy to show that

$$\|a_k\|_\infty \leq \frac{M_{2\alpha_k}^{1/p-1}}{m_{2\alpha_k}} M_{2\alpha_k+1} \leq \left(M_{2\alpha_k}\right)^{1/p} = (\mu(\text{supp } a_k))^{-1/p},$$

$$S_{M_A} a_k(x) = \begin{cases} a_k(x), & 2\alpha_k < A, \\ 0, & 2\alpha_k \geq A. \end{cases} \tag{6}$$

$$f^{(A)}(x) = \sum_{\{k;2\alpha_k<A\}} \lambda_k a_k = \sum_{k=0}^{\infty} \lambda_k S_{M_A} a_k(x),$$
$$\operatorname{supp}(a_k) = I_{2\alpha_k},$$
$$\int_{I_{2\alpha_k}} a_k d\mu = 0.$$

From (3) and Lemma 1 we conclude that $f = \left(f^{(n)}, n \in N\right) \in H_p$.
Let

$$q_A^s = M_{2A} + M_{2s} - 1, \quad A > S.$$

Then we can write

$$\begin{aligned} R_{q^s_{\alpha_k}} f(x) &= \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=1}^{q^s_{\alpha_k}} \frac{S_j f(x)}{j} \\ &= \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{j} + \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}} \frac{S_j f(x)}{j} = I + II. \end{aligned} \tag{7}$$

It is easy to show that

$$\widehat{f}(j) = \begin{cases} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}, & \text{if } j \in \{M_{2\alpha_k}, \ldots, \ M_{2\alpha_k+1} - 1\}, k = 0, 1, 2 \ldots, \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{2\alpha_k}, \ldots, \ M_{2\alpha_k+1} - 1\}. \end{cases} \tag{8}$$

Let $j < M_{2\alpha_k}$. Then from (4) and (8) we have

$$\begin{aligned} |S_j f(x)| &\le \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \left|\widehat{f}(v)\right| \\ &\le \sum_{\eta=0}^{k-1} \sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1} \frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}} \le c \sum_{\eta=0}^{k-1} \frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}} \le \frac{c M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}. \end{aligned} \tag{9}$$

Consequently,

$$|I| \le \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{|S_j f(x)|}{j} \le \frac{c}{\alpha_k} \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{1}{j} \le c \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}. \tag{10}$$

Let $M_{2\alpha_k}\leq j\leq q_{\alpha_k}^s$. Then we have the following

$$S_jf(x)=\sum_{\eta=0}^{k-1}\sum_{v=M_{2\alpha_\eta}}^{M_{2\alpha_\eta+1}-1}\widehat{f}(v)\psi_v(x)+\sum_{v=M_{2\alpha_k}}^{j-1}\widehat{f}(v)\psi_v(x) \tag{11}$$
$$=\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}}\Big(D_{M_{2\alpha_\eta+1}}(x)-D_{M_{2\alpha_\eta}}(x)\Big)$$
$$+\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\Big(D_j(x)-D_{M_{2\alpha_k}}(x)\Big).$$

This gives that

$$II=\frac{1}{l_{q_{\alpha_k}^s}}\sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s}\frac{1}{j}\left(\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}}\Big(D_{M_{2\alpha_\eta+1}}(x)-D_{M_{2\alpha_\eta}}(x)\Big)\right) \tag{12}$$
$$+\frac{1}{l_{q_{\alpha_k}^s}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\sum_{j=M_{2\alpha_k}}^{q_{\alpha_k}^s}\frac{\Big(D_j(x)-D_{M_{2\alpha_k}}(x)\Big)}{j}=II_1+II_2.$$

To discuss II_1, we use (4). Thus, we can write that

$$|II_1|\leq c\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p}}{\sqrt{\alpha_\eta}}\leq\frac{cM_{2\alpha_{k-1}}}{\sqrt{\alpha_{k-1}}}. \tag{13}$$

Since,

$$D_{j+M_{2\alpha_k}}(x)=D_{M_{2\alpha_k}}(x)+\psi_{M_{2\alpha_k}}(x)D_j(x),\qquad \text{when}\quad j<M_{2\alpha_k}, \tag{14}$$

for II_2 we have,

$$II_2=\frac{1}{l_{q_{\alpha_k}^s}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\sum_{j=0}^{M_{2s}}\frac{D_{j+M_{2\alpha_k}}(x)-D_{M_{2\alpha_k}}(x)}{j+M_{2\alpha_k}} \tag{15}$$
$$=\frac{1}{l_{q_{\alpha_k}^s}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\psi_{M_{2\alpha_k}}\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{j+M_{2\alpha_k}}.$$

We write

$$R_{q_{\alpha_k}^s}f(x)=I+II_1+II_2,$$

Then by (5), (7), (10) and (12)-(15) we have

$$\left|R_{q_{\alpha_k}^s}f(x)\right|\geq|II_2|-|I|-|II_1|\geq|II_2|-c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}$$
$$\geq\frac{c}{\alpha_k}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\left|\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{j+M_{2\alpha_k}}\right|-c\frac{M_{\alpha_k}}{\alpha_k^{3/2}}$$

Let $0 < p \leq 1/2$, $x \in I_{2s} \backslash I_{2s+1}$ for $s = [\alpha_k/2], \ldots, \alpha_k$. Then it is evident

$$\left| \sum_{j=0}^{M_{2s}-1} \frac{D_j(x)}{j+M_{2\alpha_k}} \right| \geq \frac{cM_{2s}^2}{M_{2\alpha_k}}$$

Hence, we can write

$$\begin{aligned}
\left| R_{q^s_{\alpha_k}} f(x) \right| &\geq \frac{c}{\alpha_k} \frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}} \frac{cM_{2s}^2}{M_{2\alpha_k}} - c\frac{M_{\alpha_k}}{\alpha_k^{3/2}} \\
&\geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} - c\frac{M_{\alpha_k}}{\alpha_k^{3/2}} \geq \frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_{G_m} |R^* f(x)|^p \, d\mu(x) &\geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s}\backslash I_{2s+1}} \left| R_{q^s_{\alpha_k}} f(x) \right|^p d\mu(x) \\
&\geq \sum_{s=[\alpha_k/2]}^{\alpha_k} \int_{I_{2s}\backslash I_{2s+1}} \left(\frac{cM_{2\alpha_k}^{1/p-2} M_{2s}^2}{\alpha_k^{3/2}} \right)^p d\mu(x) \\
&\geq c \sum_{s=[\alpha_k/2]}^{\alpha_k} \frac{M_{2\alpha_k}^{1-2p} M_{2s}^{2p-1}}{\alpha_k^{3p/2}} \\
&\geq \begin{cases} \frac{2^{\alpha_k(1-2p)}}{\alpha_k^{3p/2}}, & \text{when } 0 < p < 1/2 \\ c\alpha_k^{1/4}, & \text{when } p = 1/2 \end{cases} \to \infty \text{ , when } k \to \infty.
\end{aligned}$$

which completes the proof of the Theorem 2. $\square$

Proof of Theorem 3: We write

$$\begin{aligned}
(16) \qquad L_{q^s_{\alpha_k}} f(x) &= \frac{1}{l_{q_{\alpha_k,s}}} \sum_{j=1}^{q^s_{\alpha_k}} \frac{S_j f(x)}{q^s_{\alpha_k} - j} \\
&= \frac{1}{l_{q^s_{\alpha_k}}} \sum_{j=1}^{M_{2\alpha_k}-1} \frac{S_j f(x)}{q^s_{\alpha_k} - j} + \frac{1}{q^s_{\alpha_k}} \sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}} \frac{S_j f(x)}{q^s_{\alpha_k} - j} = III + IV.
\end{aligned}$$

Since (see 9)

$$|S_j f(x)| \leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}, \quad j < M_{2\alpha_k}.$$

For III we can write

$$(17) \qquad |III| \leq \frac{c}{\alpha_k} \sum_{j=0}^{M_{2\alpha_{k-1}}} \frac{1}{q^s_{\alpha_k} - j} \frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}} \leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}.$$

Using (11) we have

$$(18)\; IV=\frac{1}{l_{q^s_{\alpha_k}}}\sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}}\frac{1}{q_{\alpha_k,s}-j}\left(\sum_{\eta=0}^{k-1}\frac{M_{2\alpha_\eta}^{1/p-1}}{\sqrt{\alpha_\eta}}\left(D_{M_{2\alpha_\eta+1}}(x)-D_{M_{2\alpha_\eta}}(x)\right)\right)$$
$$+\frac{1}{l_{q^s_{\alpha_k}}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\sum_{j=M_{2\alpha_k}}^{q^s_{\alpha_k}}\frac{\left(D_j(x)-D_{M_{2\alpha_k}}(x)\right)}{q^s_{\alpha_k}-j}=IV_1+IV_2.$$

Applying (4) in IV_1 we have

$$|IV_1|\leq c\frac{M_{2\alpha_{k-1}}^{1/p}}{\sqrt{\alpha_{k-1}}}, \tag{19}$$

From (14) we obtain

$$IV_2=\frac{1}{l_{q_{\alpha_k,s}}}\frac{M_{2\alpha_k}^{1/p-1}}{\sqrt{\alpha_k}}\psi_{M_{2\alpha_k}}\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{M_{2s}-j}. \tag{20}$$

Let $x\in I_{2s}\backslash I_{2s+1}$. Then $D_j(x)=j,\ j<M_{2s}$. Consequently,

$$\sum_{j=0}^{M_{2s}-1}\frac{D_j(x)}{M_{2s}-j}=\sum_{j=0}^{M_{2s}-1}\frac{j}{M_{2s}-j}=\sum_{j=0}^{M_{2s}-1}\left(\frac{M_{2s}}{M_{2s}-j}-1\right)\geq csM_{2s}.$$

Then

$$|IV_2|\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s},\quad x\in I_{2s}\backslash I_{2s+1}. \tag{21}$$

Combining (5), (16)-(21) for $x\in I_{2s}\backslash I_{2s+1}$, $s=[\alpha_k/2],\ldots,\alpha_k$, and $0<p\leq 1$ we have

$$\left|L_{q^s_{\alpha_k}}f(x)\right|\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s}-c\frac{M_{\alpha_k}}{\alpha_k}\geq c\frac{M_{2\alpha_{k-1}}^{1/p-1}}{\alpha_k^{3/2}}sM_{2s}$$

Then

$$\begin{aligned}
\int_{G_m} |L^* f(x)|^p \, d\mu(x) &\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} |L^* f(x)|^p \, d\mu(x) \\
&\geq \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} \left| L_{q^s_{\alpha_k}} f(x) \right|^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} \int_{I_{2s}\setminus I_{2s+1}} \left(\frac{M^{1/p-1}_{2\alpha_{k-1}}}{\alpha_k^{3/2}} s M_{2s} \right)^p d\mu(x) \\
&\geq c \sum_{s=[m_k/2]}^{m_k} \frac{M^{1-p}_{2\alpha_{k-1}}}{\alpha_k^{p/2}} M^{p-1}_{2s} \\
&\geq \begin{cases} \frac{2^{\alpha_k(1-p)}}{\alpha_k^{p/2}}, & \text{when } 0<p<1, \\ c\sqrt{\alpha_k}, & \text{when } p=1, \end{cases} \to \infty, \text{ when } k \to \infty.
\end{aligned}$$

Theorem 3 is proved. □

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Institute of Mathematics,
Faculty of Exact and Natural Sciences,
Tbilisi State University,
Chavchavadze str. 1,
Tbilisi 0128,
Georgia
E-mail address: giorgitephnadze@gmail.com