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SOME APPLICATIONS OF DIFFERENTIAL SUBORDINATION OF MULTIVALENT FUNCTIONS ASSOCIATED WITH THE WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTION

M. K. AOUF, A. SHAMANDY, R. M. EL-ASHWAH, AND E. E. ALI

Abstract. Making use of the principle of differential subordination, we investigate some inclusion relationships of certain subclasses of multivalent analytic functions associated with the Wright generalized hypergeometric function.

1. Introduction

Let $A_n(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc

$$U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

For convenience, we write $A_1(p) = A(p)$. If f and g are analytic in U, we say that f is subordinate to g, written symbolically as follows: $f \prec g$, if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$ $(z \in U)$. In particular, if the function g is univalent in U, then we have the following equivalence (cf. [2, 14], see also [15, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For functions $f \in A_n(p)$, given by (1.1), and $g \in A_n(p)$ given by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{k+p} z^{k+p} \quad (p, n \in \mathbb{N}), \tag{1.2}$$

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then the Hadamard product (or convolution) of f and g is defined by

$$(f*g)(z)=z^p+\sum_{k=n}^{\infty}a_{k+p}b_{k+p}z^{k+p}=(g*f)(z)\quad (p,n\in\mathbb{N};\ z\in U). \tag{1.3}$$

Let $\alpha_1, A_1, \ldots, \alpha_q, A_q$ and $\beta_1,\ B_1, \ldots, \beta_s, B_s \quad (q, s\in\mathbb{N})$ be positive real parameters such that

$$1+\sum_{i=1}^{s}B_i-\sum_{i=1}^{q}A_i\geq 0.$$

The Wright generalized hypergeometric function [31] (see also [28])

$$\begin{aligned}{}_q\Psi_s\left[(\alpha_1,A_1),\ldots,(\alpha_q,A_q);(\beta_1,B_1),\ldots,(\beta_s,B_s);z\right]&\\ ={}_q\ \Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]&\end{aligned}$$

is defined by

$${}_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]=\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{q}\Gamma(\alpha_i+nA_i)}{\prod_{i=1}^{s}\Gamma(\beta_i+nB_i)}\cdot\frac{z^n}{n!}\quad (z\in U).$$

If $A_i=1(i=1,\ldots,q)$ and $B_i=1(i=1,\ldots,s)$, we have the relationship:

$$\Omega_q\Psi_s\left[(\alpha_i,1)_{1,q};(\beta_i,1)_{1,s};z\right]={}_q\ F_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)\ ,$$

where ${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ is the generalized hypergeometric function (see [28]) and

$$\Omega=\frac{\prod_{i=1}^{s}\Gamma(\beta_i)}{\prod_{i=1}^{q}\Gamma(\alpha_i)}. \tag{1.4}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [23, 24]).

By using the generalized hypergeometric function Dziok and Srivastava [7] introduced a linear operator. In [8] Dziok and Raina and in [1] Aouf and Dziok extended the linear operator by using the Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s^p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$ by

$${}_q\Phi_s^p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]=\Omega z^p_q\Psi_s\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]$$

and consider the following linear operator

$$\theta_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]:A_n(p)\to A_n(p),$$

defined by the convolution

$$\theta_p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s}\right]f(z)={}_q\ \Phi_s^p\left[(\alpha_i,A_i)_{1,q};(\beta_i,B_i)_{1,s};z\right]*f(z)\ .$$

We observe that, for a function f of the form (1.1), we have

$$\theta_p\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}\right] f(z) = z^p + \sum_{k=n}^{\infty} \Omega\sigma_n(\alpha_1) a_{k+p} z^{k+p}, \tag{1.5}$$

where Ω is given by (1.4) and $\sigma_n(\alpha_1)$ is defined by

$$\sigma_n(\alpha_1) = \frac{[\Gamma(\alpha_1 + A_1 n)\dots\Gamma(\alpha_q + A_q n)]}{\Gamma(\beta_1 + B_1 n)\dots\Gamma(\beta_s + B_s n) n!}. \tag{1.6}$$

If, for convenience, we write

$$\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] = \theta_p\left[(\alpha_1, A_1), \dots, (\alpha_q, A_q); (\beta_1, B_1), \dots, (\beta_s, B_s)\right] f(z),$$

then one can easily verify from the definition (1.5) that

$$zA_1(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z))' = \alpha_1 \theta_{p,q,s}\left[\alpha_1 + 1, A_1, B_1\right] f(z)$$

$$-(\alpha_1 - pA_1)\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z) \ (A_1 > 0). \tag{1.7}$$

The linear operator $\theta_{1,q,s}\left[\alpha_1, A_1, B_1\right] = \theta\left[\alpha_1\right]$ was introduced by Dziok and Raina [8] and studied by Aouf and Dziok [1].

We note that, for $f \in A_n(p), A_i = 1 (i = 1, \dots, q), B_i = 1 (i = 1, \dots, s)$ and by specializing the parameters α_i $(i = 1, \dots, q)$, β_i $(i = 1, \dots, s)$, q and s we obtain the following operators studied by various authors:

(i) $\theta_{p,q,s}\left[\alpha_1\right] f(z) = H_{p,q,s}(\alpha_1) f(z)$ (see Patel et al.[22]);

(ii) $\theta_{p,2,1}\left[a, 1; c\right] f(z) = L_p(a, c) f(z) (a > 0, c > 0)$ (see Carlson and Shaffer [3] and Saitoh [25]);

(iii) $\theta_{p,2,1}\left[\mu + p, 1; 1\right] f(z) = D^{\mu+p-1} f(z) (\mu > -p)$, where $D^{\mu+p-1} f(z)$ is the $(\mu + p - 1) - th$ order Ruscheweyh derivative of a function $f \in A_n(p)$ (see Kumar and Shukla [10]);

(iv) $\theta_{p,2,1}\left[1 + p, 1; 1 + p - \mu\right] f(z) = \Omega_z^{(\mu,p)} f(z)$, where the operator $\Omega_z^{(\mu,p)} f(z)$ is defined by (see Srivastava and Aouf [27])

$$\Omega_z^{(\mu,p)} f(z) = \frac{\Gamma(1 + p - \mu)}{\Gamma(1 + p)} z^\mu D_z^\mu f(z) (0 \le \mu < 1; p \in \mathbb{N}),$$

where $\Omega_z^\mu f(z)$ is the fractional derivative operator (see, for details, [6] and [18] and [19]);

(v) $\theta_{p,2,1}\left[\delta + p, 1; \delta + p + 1\right] f(z) = F_{\delta,p}(f)(z)$, where $F_{\delta,p}(f)$ is the generalized Bernardi-Libera-Livingston operator (see [5]), defined by

$$F_{\delta,p}(f)(z) = \frac{\delta + p}{z^\delta} \int_0^z t^{\delta-1} f(t) dt (\delta > -p; p \in \mathbb{N});$$

(vi) $\theta_{p,2,1}\left[p + 1, 1; m + p\right] f(z) = I_{m,p} f(z) (m \in \mathbb{Z}; m > -p)$, where the operator $I_{m,p}$ is the $(m + p - 1) - th$ Noor operator, considered by Liu and Noor [12];

(vii) $\theta_{p,2,1}\left[\lambda + p, c; a\right] f(z) = I_p^\lambda(a, c) f(z) (a, c \in \mathbb{R} \backslash \mathbb{N}_0^-; \lambda > -p)$, where $I_p^\lambda(a, c) f(z)$ is the Cho-Kwon-Srivastava operator (see [4]).

For fixed parameters A and $B(-1 \leq B < A \leq 1)$, we say that a function $f \in A_n(p)$ is in the class $Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$, if it satisfies the following subordination condition:

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}(p \in \mathbb{N}). \tag{1.8}$$

In view of the definition of subordination, (1.8) is equivalent to the following condition:

$$\left| \frac{\frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{z^{p-1}} - p}{B\frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{z^{p-1}} - pA} \right| < 1 \quad (z \in U). \tag{1.9}$$

For convenience, we write $Q^n_{p,q,s}(\alpha_1, A_1, B; 1-\frac{2\theta}{p}, -1) = Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$, where $Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$ denote the class of functions in $A_n(p)$ satisfying the following inequality:

$$\operatorname{Re} \frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{z^{p-1}} > \theta \quad (0 \leq \theta < p; p \in \mathbb{N};\ z \in U). \tag{1.10}$$

2. Preliminaries

To establish our main results, we shall need the following lemmas.

Lemma 1 ([9]). *Let the function h be analytic and convex (univalent) in U with $h(0) = 1$ Suppose also the function φ given by*

$$\varphi(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \ldots, \tag{2.1}$$

is analytic in U. If

$$\varphi(z) + \frac{1}{\gamma} z\varphi'(z) \prec h(z), \tag{2.2}$$

where $\gamma \neq 0$ and $\operatorname{Re}\gamma \geq 0$. Then

$$\varphi(z) \prec \Psi(z) = \frac{\gamma}{n} z^{-\frac{\gamma}{n}} \int_0^z t^{\frac{\gamma}{n}-1} h(t)dt \prec h(z), \tag{2.3}$$

and Ψ is the best dominant of (2.2).

With a view to stating a well-known result (Lemma 2 below), we denote by $P(\delta)$ the class of functions Φ given by

$$\Phi(z) = 1 + c_1 z + c_2 z^2 + \ldots\ , \tag{2.4}$$

which are analytic in U and satisfy the following inequality:

$$\operatorname{Re}\Phi(z) > \delta(0 \leq \delta < 1; z \in U).$$

Lemma 2 ([20]). *Let the function* Φ*, given by (2.4), be in the class* $P(\delta)$*. Then*

$$\operatorname{Re}\Phi(z) \geq 2\delta - 1 + \frac{2(1-\delta)}{1+|z|} \; (0 \leq \delta < 1; z \in U).$$

Lemma 3 ([29]). *For* $0 \leq \gamma_1, \gamma_2 < 1$*,*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3) \; (\gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2)).$$

The result is the best possible.

Lemma 4 ([26]). *Let* Φ *be analytic in* U *with*

$$\Phi(0) = 1 \text{ and } \operatorname{Re}\Phi(z) > \frac{1}{2} \quad (z \in U).$$

Then, for any function F *analytic in* U*,* $(\Phi * F)(U)$ *is contained in the convex hull of* $F(U)$*.*

Lemma 5 ([17]). *Let* φ *be analytic in* U *with* $\varphi(0) = 1$ *and* $\varphi(z) \neq 0$ *for* $0 < |z| < 1$*, and let* $A, B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leq 1$*.*

(i) Let $B \neq 0$ *and* $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ *satisfy either*

$$\left|\frac{\gamma(A-B)}{B} - 1\right| \leq 1 or \left|\frac{\gamma(A-B)}{B} + 1\right| \leq 1.$$

If φ *satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec \frac{1+Az}{1+Bz},$$

then

$$\varphi(z) \prec (1+Bz)^{\gamma(\frac{A-B}{B})}$$

and this is the best dominant.

(ii) Let $B = 0$ *and* $\gamma \in \mathbb{C}^*$ *be such that* $|\gamma A| < \pi$*. If* φ *satisfies*

$$1 + \frac{z\varphi'(z)}{\gamma\varphi(z)} \prec 1 + Az \quad ,$$

then

$$\varphi(z) \prec e^{\gamma A z}$$

and this is the best dominant.

For real or complex numbers a, b and $c (c \notin \mathbb{Z}_0^-)$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c} \cdot \frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)} \cdot \frac{z^2}{2!} + \cdots .$$

We note that the above series converges absolutely for $z \in U$ and hence represents an analytic function in U (see, for details, [30, Chapter 14]).

Each of the identities (asserted by Lemma 6 below) is well-known (cf., e.g., [30, Chapter 14].

Lemma 6 ([30]). *For real or complex parameters a, b and $c(c \notin \mathbb{Z}_0^-)$,*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;z) \quad (\mathrm{Re}(c) > \mathrm{Re}(b) > 0); \tag{2.5}$$

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z); \tag{2.6}$$

and

$${}_2F_1(a,b;c;z) = (1-z)^{-a} {}_2F_1(a, c-b; c; \frac{z}{z-1}). \tag{2.7}$$

3. Main Results

Unless otherwise mentioned we shall assume through this paper that $-1 \leq B < A \leq 1$, λ, $A_1 > 0$ and $p, n \in \mathbb{N}$.

Theorem 1. *Let the function f defined by (1.1) satisfy the following subordination condition*

$$(1-\lambda)\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} + \lambda \frac{(\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z))'}{pz^{p-1}} \prec \frac{1+Az}{1+Bz}. \tag{3.1}$$

Then

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} \prec Q(z) \prec \frac{1+Az}{1+Bz}, \tag{3.2}$$

where the function Q given by

$$Q(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda n}+1;\frac{Bz}{1+Bz}) & (B \neq 0), \\ 1 + \frac{A\alpha_1}{A_1\lambda n + \alpha_1} z & (B = 0), \end{cases} \tag{3.3}$$

is the best dominant of (3.2). Furthermore

$$\mathrm{Re}\,\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \rho \quad (z \in U), \tag{3.4}$$

where

$$\rho(z) = \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1-B)^{-1}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda n}+1;\frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{A\alpha_1}{A_1\lambda n + \alpha_1} & (B = 0), \end{cases} \tag{3.5}$$

the estimate in (3.4) is the best possible.

Proof. Consider the function φ defined by

$$\varphi(z) = \frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} \quad (z \in U). \tag{3.6}$$

Then φ is of the form (2.1) and is analytic in U. Applying the identity (1.7) in (3.6) and differentiating the resulting equation with respect to z, we get

$$(1-\lambda)\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} + \lambda\frac{\left(\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}}$$
$$= \varphi(z) + \frac{A_1\lambda}{\alpha_1} z\varphi'(z) \prec \frac{1+Az}{1+Bz}\,.$$

Now, by using Lemma 1 for $\gamma = \frac{\alpha_1}{A_1\lambda}$, we obtain

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} \prec Q(z) \prec \frac{\alpha_1}{A_1\lambda n} z^{\frac{-\alpha_1}{A_1\lambda n}} \int\limits_0^z t^{\frac{\alpha_1}{A_1\lambda n}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \dfrac{A}{B} + \left(1-\dfrac{A}{B}\right)(1-Bz)^{-1}_2F_1\left(1,1;\dfrac{\alpha_1}{A_1\lambda n}+1;\dfrac{Bz}{1+Bz}\right) & (B \neq 0), \\ 1 + \dfrac{\alpha_1 A}{A_1\lambda n + \alpha_1} z & (B = 0), \end{cases}$$

by change of variables followed by use of the identities (2.5), (2.6) and (2.7) (with $a = 1, c = b+1, b = \dfrac{\alpha_1}{A_1\lambda n}$). This proves the assertion (3.2) of Theorem 1.

Next, in order to prove the assertion (3.4) of Theorem 1, it suffices to show that

$$\inf_{|z|<1} \{\operatorname{Re} Q(z)\} = Q(-1). \tag{3.7}$$

Indeed we have, for $|z| \leq r < 1$,

$$\operatorname{Re}\frac{1+Az}{1+Bz} \geq \frac{1-Ar}{1-Br}.$$

Upon setting

$$g(\zeta, z) = \frac{1+A\zeta z}{1+B\zeta z} \text{and } dv(\zeta) = \frac{\alpha_1}{A_1\lambda n}\zeta^{\frac{\alpha_1}{A_1\lambda n}-1}d\zeta \quad (0 \leq \zeta \leq 1),$$

which is a positive measure on the closed interval $[0, 1]$, we get

$$Q(z) = \int\limits_0^1 g(\zeta, z)dv(\zeta),$$

so that

$$\operatorname{Re} Q(z) \geq \int_0^1 \left(\frac{1 - A\zeta r}{1 - B\zeta r}\right) dv(\zeta) = Q(-r) \quad (|z| \leq r < 1).$$

Letting $r \to 1^-$ in the above inequalities, we obtain the assertion (3.4) of Theorem 1. Finally, the estimate in (3.4) is the best possible as the function Q is the best dominant of (3.2). □

Taking $\lambda = 1, A = 1 - \frac{2\sigma}{p}(0 \leq \sigma < p)$ and $B = -1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *The following inclusion property holds true for the class* $Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$:

$$Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; \theta) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; \beta(p, n, \alpha_1, A_1, \theta)) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta),$$

where

$$\beta(p, n, \alpha_1, A_1, \theta) = \theta + (p - \theta)\left\{ {}_2F_1(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{1}{2}) - 1 \right\}.$$

The result is the best possible.

Taking $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *The following inclusion property holds true for the function class* $Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$ *:*

$$Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; A, B) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; 1 - \frac{2\theta}{p}, -1) \subset Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B), 0 \leq \theta < p,$$

where

$$\theta = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1}_2 F_1(1, 1; \frac{\alpha_1}{A_1 n} + 1; \frac{B}{B-1}) & (B \neq 0), \\ 1 - \frac{\alpha_1 A}{A_1 n + \alpha_1} & (B = 0). \end{cases}$$

The result is the best possible.

Theorem 2. *If* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)(0 \leq \theta < 1)$*, then*

$$\operatorname{Re} \frac{(1-\lambda)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))' + \lambda(\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1] f(z))'}{pz^{p-1}} > \theta(|z| < R), \tag{3.8}$$

where

$$R = \left\{ \frac{\sqrt{\alpha_1^2 + \lambda^2 A_1^2 n^2} - \lambda A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

Proof. Since $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)$, we write

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1; \theta\right] f(z)\right)'}{pz^{p-1}} = \theta + (1-\theta)u(z) \ (z \in U). \tag{3.9}$$

Then, clearly, u is of the form (2.1), is analytic in U, and has a positive real part in U. Making use of the identity (1.7) in (3.9) and differentiating the resulting equation with respect to z, we obtain

(3.10)
$$\frac{1}{(1-\theta)}\left\{\frac{(1-\lambda)(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z))' + \lambda(\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f(z))'}{pz^{p-1}} - \theta\right\}$$
$$= u(z) + \frac{A_1\lambda}{\alpha_1} zu'(z).$$

Now, by applying the well-known estimate [13]

$$\frac{\left|zu'(z)\right|}{\operatorname{Re} u(z)} \leq \frac{2nr^n}{1-r^{2n}} \quad (|z| = r < 1)$$

in (3.10), we get

(3.11)
$$\frac{1}{(1-\theta)}\operatorname{Re}\left\{\frac{(1-\lambda)(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z))' + \lambda(\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f(z))'}{pz^{p-1}} - \theta\right\}$$
$$\geq \operatorname{Re} u(z)\left(1 - \frac{2A_1\lambda nr^n}{\alpha_1(1-r^{2n})}\right).$$

It is easily seen that the right-hand side of (3.11) is positive provided that $r < R$, where R is given as in Theorem 2. This proves the assertion (3.8) of Theorem 2.

In order to show that the bound R is the best possible, we consider the function $f \in A_n(p)$ defined by

$$\frac{\left(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z)\right)'}{pz^{p-1}} = \theta + (1-\theta)\frac{1+z^n}{1-z^n} \quad (0 \leq \theta < 1; z \in U).$$

Noting that

$$\frac{1}{(1-\theta)}\left\{\frac{(1-\lambda)(\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f(z))' + \lambda(\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f(z))'}{pz^{p-1}} - \theta\right\}$$
$$= \frac{\alpha_1 - \alpha_1 z^{2n} - 2A_1\lambda nz^n}{\alpha_1(1-z^n)^2} = 0,$$

for $z = R\exp(\frac{i\pi}{n})$. This completes the proof of Theorem 2. □

Putting $\lambda = 1$ in Theorem 2 , we obtain the following result.

Corollary 3. *If* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; \theta)(0 \leq \theta < 1)$, *then* $f \in Q^n_{p,q,s}(\alpha_1 + 1, A_1, B_1; \theta)$ *for* $|z| < \widetilde{R}$, *where*

$$\widetilde{R} = \left\{ \frac{\sqrt{\alpha_1^2 + A_1^2 n^2} - A_1 n}{\alpha_1} \right\}^{\frac{1}{n}}.$$

The result is the best possible.

For a function $f \in A_n(p)$, *the generalized Bernardi-Libera-Livingston integral operator* $F_{\delta,p}$ *is defined by*

$$\begin{aligned} F_{\delta,p}(f)(z) &= \frac{\delta + p}{z^p} \int_0^z t^{\delta-1} f(t) dt \\ &= \left(z^p + \sum_{k=n}^{\infty} \frac{\delta + p}{\delta + p + k} z^{p+k} \right) * f(z) \ (\delta > -p) \\ &= z^p {}_2F_1(1, \delta + p, \delta + p + 1; z) * f(z). \end{aligned} \tag{3.12}$$

Theorem 3. *Let* $f \in Q^n_{p,q,s}(\alpha_1, A_1, B_1; A, B)$ *and let the operator* $F_{\delta,p}(f)$ *defined by (3.12). Then*

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \prec \theta(z) \prec \frac{1 + Az}{1 + Bz}, \tag{3.13}$$

where the function θ *given by*

$$\theta(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} {}_2F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{Bz}{Bz+1}) & (B \neq 0), \\ 1 + \frac{(p+\delta)A}{p+\delta+n} z & (B = 0). \end{cases} \tag{3.14}$$

is the best dominant of (3.13). Furthermore,

$$\operatorname{Re}\left\{ \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \right\} > \xi^* \quad (z \in U), \tag{3.15}$$

where

$$\xi^* = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 - B)^{-1} {}_2F_1(1, 1; \frac{p+\delta}{n} + 1; \frac{B}{B+1}) & (B \neq 0) \\ 1 - \frac{(p+\delta)}{p+\delta+n} A & (B = 0). \end{cases} \tag{3.16}$$

The result is the best possible.

Proof. From (1.7) and (3.12) it follows that

$$\begin{aligned} z(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))' &= (p + \delta)(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)) - \\ &\quad \delta(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'. \end{aligned} \tag{3.17}$$

By setting

$$\varphi(z) = \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))'}{pz^{p-1}} \quad (z \in U), \tag{3.18}$$

we note that $\varphi(z)$ is of the form (2.1) and is analytic in U. Using the identity (3.17) in (3.18), and then differentiating the resulting equation with respect to z, we obtain

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} = \varphi(z) + \frac{z\varphi'(z)}{p+\delta} \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 3 follows by employing the techniques that we used in proving Theorem 1 above. □

Remark 1. We observe that.

$$\frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] F_{\delta,p}(f)(z))}{pz^{p-1}} = \frac{\delta+p}{pz^{\delta+p}} \int_0^z t^\delta (\theta_{p,q,s}[\alpha_1, A_1, B_1] f(t))' dt \tag{3.19}$$

$$(f \in A_n(p); z \in U).$$

In view of (3.19), Theorem 3 for $A = 1 - 2\mu (0 \le \mu < 1)$ and $B = -1$ yields the following corollary.

Corollary 4. *If $\delta > 0$ and if $f \in A_n(p)$satisfies the following inequality*

$$\operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{pz^{p-1}} > \mu \ (0 \le \mu < 1; z \in U),$$

then

$$\operatorname{Re} \frac{\delta+p}{pz^{p+\delta}} \int_0^z t^\delta (\theta_{p,q,s}[\alpha_1, A_1, B_1] f(t))' dt >$$

$$\mu + (1-\mu) \left[{}_2F_1 \left(1, 1; \frac{p+\delta}{n} + 1; \frac{1}{2} \right) - 1 \right] (z \in U).$$

The result is the best possible.

Theorem 4. *Let $f \in A_n(p)$. Suppose also that $g \in A_n(p)$ satisfies the following inequality:*

$$\operatorname{Re} \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)}{z^p} > 0 \ (z \in U).$$

If

$$\left| \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} - 1 \right| < 1 \ (z \in U),$$

then

$$\operatorname{Re} \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} > 0 \quad (|z| < R_0),$$

where

$$R_0 = \frac{\sqrt{9n^2 + 4p(p+n)} - 3n}{2(p+n)}.$$

Proof. Letting

$$w(z) = \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} - 1 = k_n z^n + k_{n+1} z^{n+1} + \ldots, \tag{3.20}$$

we note that w is analytic in U, with

$$w(0) = 0 \text{and } |w(z)| \leq |z|^n \quad (z \in U).$$

Then, by applying the familiar Schwarz lemma [16], we obtain

$$w(z) = z^n \Psi(z),$$

where the function Ψ is analytic in U and $|\Psi(z)| \leq 1 (z \in U)$. Therefore, (3.20) leads us to

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) = \theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)(1 + z^n \Psi(z)) \ (z \in U). \tag{3.21}$$

Differentiating (3.21) logarithmically with respect to z, we obtain

$$\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} = \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} + \frac{z^n\{n\Psi(z) + z\Psi'(z)\}}{1 + z^n \Psi(z)}. \tag{3.22}$$

Putting $\varphi(z) = \dfrac{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)}{z^p}$,we see that the function $\varphi(z)$ is of the form (2.1), is analytic in U, $\operatorname{Re}\varphi(z) > 0 (z \in U)$ and

$$\frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] g(z)} = \frac{z\varphi'(z)}{\varphi(z)} + p,$$

so that we find from (3.22) that

$$\operatorname{Re} \frac{(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \geq p - \left|\frac{z\varphi'(z)}{\varphi(z)}\right| - \left|\frac{z^n\left\{n\Psi(z) + z\Psi'(z)\right\}}{1 + z^n \Psi(z)}\right| \quad (z \in U). \tag{3.23}$$

Now, by using the following known estimates [13] (see also [21]) :

$$\left|\frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{2nr^{n-1}}{1 - r^{2n}} \text{ and } \left|\frac{n\Psi(z) + z\Psi'(z)}{1 + z^n \Psi(z)}\right| \leq \frac{n}{1 - r^n} \quad (|z| = r < 1),$$

in (3.21), we obtain

$$\operatorname{Re} \frac{z(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z))'}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \geq \frac{p - 3nr^n - (p+n)r^{2n}}{1 - r^{2n}} \ (|z| = r < 1),$$

which is certainly positive, provided that $r < R_0$, R_0 being given as in Theorem 4. □

Theorem 5. *Let $-1 \leq D_j < C_j \leq 1 (j = 1, 2)$. If each of the functions $f_j \in A_n(p)$ satisfies the following subordination condition:*

$$(1-\lambda)\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f_j(z)}{z^p}+\lambda\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f_j(z)}{z^p} \prec \frac{1+C_j z}{1+D_j z}, \tag{3.24}$$

then

$$(1-\lambda)\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] G(z)}{z^p}+\lambda\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] G(z)}{z^p} \prec \frac{1+(1-2\eta)z}{1-z}, \tag{3.25}$$

where

$$G(z) = \theta_{p,q,s}\left[\alpha_1, A_1, B_1\right](f_1 * f_2)(z)$$

and

$$\eta = 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left[1 - \frac{1}{2}{}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{1}{2})\right].$$

The result is the best possible when $D_1 = D_2 = -1$.

Proof. Suppose that each of the functions $f_j \in A_n(p)(j = 1, 2)$ satisfies the condition (3.24). Then, by letting

$$\varphi_j(z) = (1-\lambda)\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f_j(z)}{z^p}+\lambda\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] f_j(z)}{z^p} \quad (j = 1, 2), \tag{3.26}$$

we have

$$\varphi_j(z) \in P(\gamma_j) \quad \left(\gamma_j = \frac{1-C_j}{1-D_j}; j = 1, 2\right).$$

By making use of identity (1.7) in (3.26), we observe that

$$\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f_j(z) = \frac{\alpha_1}{A_1\lambda} z^{p-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1}\varphi_j(t)dt \quad (j = 1, 2),$$

which in view of the definition of G given already with (3.25) yields

$$\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] G(z) = \frac{\alpha_1}{A_1\lambda} z^{p-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1}\varphi_0(t)dt, \tag{3.27}$$

where, for convenience,

$$\begin{aligned}\varphi_0(z) &= (1-\lambda)\frac{\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] G(z)}{z^p}+\frac{\theta_{p,q,s}\left[\alpha_1+1, A_1, B_1\right] G(z)}{z^p}\\ &= \frac{\alpha_1}{A_1\lambda} z^{-\frac{\alpha_1}{A_1\lambda}} \int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1}(\varphi_1 * \varphi_2)(t)dt.\end{aligned} \tag{3.28}$$

Since $\varphi_1 \in P(\gamma_1)$ and $\varphi_2 \in P(\gamma_2)$, it follows from Lemma 3 that

$$(\varphi_1 * \varphi_2)(z) \in P(\gamma_3)\ (\gamma_3 = 1 - 2(1-\gamma_1)((1-\gamma_2)). \tag{3.29}$$

Now, by using (3.29) in (3.28) and then appealing to Lemma 2 and Lemma 4, we get

$$\begin{aligned}
\operatorname{Re}\{\varphi_0(z)\} &= \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}\operatorname{Re}\{(\varphi_1 * \varphi_2)\}(uz)du \\
&\geq \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|})du \\
&> \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u})du \\
&= 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left(1 - \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}(1+u)^{-1}du\right) \\
&= 1 - \frac{4(C_1-D_1)(C_2-D_2)}{(1-D_1)(1-D_2)}\left[1 - \frac{1}{2}{}_2F_1(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{1}{2})\right] \\
&= \eta (z \in U).
\end{aligned}$$

When $D_1 = D_2 = -1$, we consider the functions $f_j(z) \in A_n(p) (j = 1, 2)$, which satisfy the hypothesis (3.24) of Theorem 5 and are defined by

$$\theta_{p,q,s}[\alpha_1, A_1, B_1] f_j(z) = \frac{\alpha_1}{A_1\lambda} z^{-\frac{\alpha_1}{A_1\lambda}}\int_0^z t^{\frac{\alpha_1}{A_1\lambda}-1}(\frac{1+C_j t}{1-t})dt \quad (j = 1, 2).$$

Thus it follows from (3.28) and Lemma 2 that

$$\begin{aligned}
\varphi_0(z) &= \frac{\alpha_1}{A_1\lambda}\int_0^1 u^{\frac{\alpha_1}{A_1\lambda}-1}\left\{1 - (1+C_1)(1+C_2) + \frac{(1+C_1)(1+C_2)}{1-uz}\right\}du \\
&= 1 - (1+C_1)(1+C_2) + (1+C_1)(1+C_2)(1-z)^{-1}{}_2F_1\left(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{z}{z-1}\right) \\
&\to 1 - (1+C_1)(1+C_2) + \frac{1}{2}(1+C_1)(1+C_2){}_2F_1\left(1,1;\frac{\alpha_1}{A_1\lambda}+1;\frac{1}{2}\right)
\end{aligned}$$

as $z \to 1^-$, which completes the proof of Theorem 5. □

Remark 2. Taking $A_i = 1 (i = 1, \ldots, q), B_i = 1 (i = 1, \ldots, s)$ and $j = 1$ in Theorem 5, we obtain the result obtained by Liu [11, Theorem 2.4].

Putting $A_i = 1 (i = 1, \dots, q), B_i = 1 (i = 1, \dots, s), C_j = 1 - 2\theta_j (0 \le \theta_j < 1), D_j = 1 (j = 1, 2), q = s + 1, \alpha_1 = \beta_1 = p, \alpha_j = 1 (j = 2, 3, \dots, s + 1)$ and $\beta_j = 1 (j = 2, 3, \dots, s)$ in Theorem 5, we obtain the following result.

Corollary 5. *If the functions $f_j \in A_n(p) (j = 1, 2)$ satisfy the following inequality:*

$$\operatorname{Re}\left\{(1-\lambda)\frac{f_j(z)}{z^p} + \lambda \frac{f_j'(z)}{pz^{p-1}}\right\} > \theta_j \ \ (0 \le \theta_j < 1; j = 1, 2; z \in U), \tag{3.30}$$

then

$$\operatorname{Re}\left\{(1-\lambda)\frac{(f_1 * f_2)(z)}{z^p} + \lambda \frac{z(f_1 * f_2)'(z)}{pz^{p-1}}\right\} > \eta_0 \ \ (z \in U),$$

where

$$\eta_0 = 1 - 4(1-\theta_1)(1-\theta_2)\left[1 - \frac{1}{2}{}_2F_1(1, 1; \frac{p}{\lambda} + 1; \frac{1}{2})\right].$$

The result is the best possible.

Theorem 6. *Let the function f be defined by (1.1) be in the class $Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B]$ and let $g \in A_n(p)$ satisfy the following inequality:*

$$\operatorname{Re}\frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U).$$

Then

$$(f * g)(z) \in Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B].$$

Proof. We have

$$\frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1](f * g)(z)\right)'}{pz^{p-1}} = \frac{\left(\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)\right)'}{pz^{p-1}} * \frac{g(z)}{z^p} \ (z \in U).$$

Since

$$\operatorname{Re}\frac{g(z)}{z^p} > \frac{1}{2} \quad (z \in U)$$

and the function

$$\frac{1 + Az}{1 + Bz}$$

is convex (univalent) in U, it follows from (1.8) and Lemma 4 that $(f * g) \in Q^n_{p,q,s}[\alpha_1, A_1, B_1; A, B]$. □

Theorem 7. *Let $\alpha_1 > 0, \nu \in \mathbb{C}^*$ and let $A, B \in \mathbb{C}$ with $A \neq B$ and $|B| \le 1$. Suppose that*

$$\left|\frac{\frac{\nu\alpha_1}{A_1}(A - B)}{B} - 1\right| \le 1$$

or

$$\left|\frac{\frac{\nu\alpha_1}{A_1}(A - B)}{B} + 1\right| \le 1$$

if $B \neq 0$, and

$$|\nu\pi| \leq \frac{A_1\pi}{\alpha_1},$$

if $B = 0$.

If $f \in A_n(p)$ with $\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z) \neq 0$ for all $z \in U^* = U\backslash\{0\}$, then

$$\frac{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1+Az}{1+Bz}$$

implies

$$\left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p}\right)^{\nu} \prec q_1(z),$$

where

$$q_1(z) = \begin{cases} (1+Bz)^{\frac{v\alpha_1}{A_1}(\frac{A-B}{B})}, & \text{if } B \neq 0, \\ e^{\frac{\nu\alpha_1}{A_1}Az}, & \text{if } B = 0, \end{cases}$$

is the best dominant. (All the powers are the principal ones).

Proof. Let us put

$$\varphi(z) = \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{z^p}\right)^{\nu} \quad (z \in U), \tag{3.31}$$

where the power is the principal one, then $\varphi(z)$ is analytic in U, $\varphi(0) = 1$ and $\varphi(z) \neq 0$ for $z \in U$. Taking the logarithmic derivatives in both sides of (3.31), multiplying by z and using the identity (1.7), we have

$$1 + \frac{z\varphi'(z)}{\frac{\nu\alpha_1}{A_1}\varphi(z)} = \frac{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} \prec \frac{1+Az}{1+Bz}.$$

Now the assertions of Theorem 7 follows by using Lemma 5 with $\gamma = \frac{\nu\alpha_1}{A_1}$. This completes the proof of Theorem 7. □

Remark 3. Putting $A_i = 1 (i = 1, \ldots, q), B_i = 1 (i = 1, \ldots, s), A = 1 - 2\rho, 0 \leq \rho < 1$ and $B = -1$in Theorem 7, we obtain the result obtained by Liu [11, Theorem 5].

Putting $A = 1 - \frac{2\eta}{p}(0 \leq \eta < p), B = -1, A_i = 1(i = 1, \ldots, q), B_i = 1(i = 1, \ldots, s), n = 1, q = s+1, \alpha_1 = \beta_1 = p, \alpha_j = 1(j = 2, \ldots, s+1)$, and $\beta_j = 1(j = 2, \ldots s)$ in Theorem 7, we obtain the following corollary.

Corollary 6. *Assume that $\nu \in \mathbb{C}^*$ satisfies either*

$$|2\nu(\eta - p) - 1| \leq 1 \text{ or } |2\nu(\eta - p) + 1| \leq 1.$$

If the function $f \in A(p)$ with $f(z) \neq 0$ for $z \in U^*$ satisfy the following inequality:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \eta (0 \leq \eta < p),$$

then

$$\left(\frac{f(z)}{z^p}\right)^{\nu} \prec q_2(z) \qquad (z \in U)\ ,$$

where

$$q_2(z) = (1-z)^{-2\nu(p-\eta)}(z \in U),$$

is the best dominant.

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Department of Mathematics,
Faculty of Science,
Mansoura University,
Mansoura 35516, Egypt
E-mail address: mkaouf127@yahoo.com (Aouf)

E-mail address: shamandy16@hotmail.com (Shamandy)

E-mail address: r_elashwah@yahoo.com (El-Ashwah)

E-mail address: ekram_008eg@yahoo.com (Ali)