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SEVERAL VARIANTS OF VIA TITU ANDREESCU TYPE AND POPOVICIU TYPE INEQUALITIES

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Abstract. In this paper, we will give several generalized variants of Via Titu Andreescu type and Popoviciu type inequalities.

1. Introduction

Let f be a convex function, i.e., for any $t \in [0,1]$ and any x, y in the domain of f,

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \tag{1}$$

then the following two known inequalities hold,

- *Via Titu Andreescu type inequality* (see [2] p. 6)

$$\begin{aligned} f(x_1) + f(x_2) + f(x_3) + f\left(\frac{x_1+x_2+x_3}{n}\right) \\ \geq \frac{4}{3}\left[f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_2+x_3}{2}\right) + f\left(\frac{x_3+x_1}{2}\right)\right], \end{aligned} \tag{2}$$

  where x_1, x_2, x_3 lie in the domain of the convex function f.
- *Popoviciu type inequality* (see [4])

$$\sum_{i=1}^{n} f(x_i) + \frac{n}{n-2} f\left(\frac{x_1+\cdots+x_n}{n}\right) \geq \frac{2}{n-2}\sum_{i<j} f\left(\frac{x_i+x_j}{2}\right),$$

  where f is a convex function on interval I and $x_1, \ldots, x_n \in I$.

Generalized Popoviciu type inequality

$$(n-1)[f(b_1) + \cdots + f(b_n)] \leq f(a_1) + \cdots + f(a_n) + n(n-2)f(a),$$

where $a = (a_1 + \cdots + a_n)/n$ and $b_i = (na - a_i)/(n-1)$, $i = 1, \ldots, n$, and $a_1, \ldots, a_n \in I$.

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Recently, Bougoffa [1] obtained the following variant of Via Titu Andreescu type and Popoviciu type inequalities.

Theorem B. *If f is a convex function and $x_1, \dots, x_n$ or $a_1, \dots, a_n$ lie in its domain, then the following inequalities hold,*

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{n-1}{n}\left[f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right) + f\left(\frac{x_n+x_1}{2}\right)\right] \tag{3}$$

and

$$(n-1)[f(b_1) + \cdots + f(b_n)] \leq n[f(a_1) + \cdots + f(a_n) - f(a)], \tag{4}$$

where $a = (a_1 + \cdots + a_n)/n$ and $b_i = (na - a_i)/(n-1)$, $i = 1, \dots, n$.

The aim of the present paper is to show the more generalized inequalities in Theorem B, which are stated and proved in Section 2.

2. Main results

Before showing our main results, we need recall the well-known Jensen's inequality

Lemma 2.1 (see [3]). *Let f be a convex function on an interval I and let $w_1, \dots, w_n$ be nonnegative real numbers whose sum is 1. Then for all $x_1, \dots, x_n \in I$,*

$$w_1 f(x_1) + \cdots + w_n f(x_n) \geq f(w_1 x_1 + \cdots + w_n x_n). \tag{5}$$

Theorem 2.1. *For any n and $1 \leq k \leq n-1$, let f be a convex function and $x_1, \dots, x_n$ lie in its domain and let $\{c_{i,j}\}_{1\leq i\leq n, 1\leq j\leq k+1}$ be nonnegative real numbers with $\sum_{j=1}^{k+1} c_{i,j} = 1$ for all $1 \leq i \leq n$. In addition, assume that $x_{n+1} = x_1, \dots, x_{n+k} = x_k$, then we have*

$$\sum_{i=1}^{n} a_i f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} a_i x_i\right) \geq \frac{n-1}{n}\sum_{i=1}^{n} f\left(\sum_{l=1}^{k+1} c_{i,l} x_{i+l-1}\right), \tag{6}$$

where

$$a_i := \begin{cases} \sum_{l=1}^{i} c_{l,i} + \sum_{l=i+1}^{k+1} c_{n-k+l-1,k-l+3}, & 1 \leq i \leq k \\ \sum_{l=1}^{k+1} c_{i-l+1,l} & k+1 \leq i \leq n. \end{cases} \tag{7}$$

Proof. For any $1 \leq i \leq n$ and $1 \leq k \leq n-1$, by the Jensen's inequality (note $c_{i,1} + \cdots + c_{i,k+1} = 1$), we have

$$f(c_{i,1}x_i + c_{i,2}x_{i+1} + \cdots + c_{i,k+1}x_{i+k}) \leq \sum_{l=1}^{k+1} c_{i,l} f(x_{i+l-1}).$$

Thus,

$$\begin{aligned}
\sum_{i=1}^{n} f\left(\sum_{l=1}^{k+1} c_{i,l} x_{i+l-1}\right) &\le \sum_{i=1}^{n}\sum_{l=1}^{k+1} c_{i,l} f(x_{i+l-1}) \\
&= \sum_{i=1}^{k}\left(\sum_{l=1}^{i} c_{l,i} + \sum_{l=i+1}^{k+1} c_{n-k+l-1,k-l+3}\right) f(x_i) + \sum_{i=k+1}^{n}\sum_{l=1}^{k+1} c_{i-l+1,l} f(x_i) \\
&\triangleq \sum_{i=1}^{n} a_i f(x_i).
\end{aligned}$$

Furthermore, noting the fact $\sum_{i=1}^{n} a_i = n$ and using the Jensen's inequality again,

$$\begin{aligned}
\sum_{i=1}^{n} a_i f(x_i) &= \frac{n}{n-1}\left[\sum_{i=1}^{n} a_i f(x_i) - \frac{1}{n}\sum_{i=1}^{n} a_i f(x_i)\right] \\
&\le \frac{n}{n-1}\left[\sum_{i=1}^{n} a_i f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} a_i x_i\right)\right]
\end{aligned}$$

which implies our result. □

Remark 2.1. If let $k = 1$, $x_{n+1} = x_1$ and for any $1 \le i \le n$, $c_{i,1} = c_{i,2} = 1/2$, then $a_i = 1$ and from Theorem 2.1, we obtain

$$\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \ge \frac{n-1}{n}\sum_{i=1}^{n} f\left(\frac{1}{2}\sum_{l=1}^{2} x_{i+l-1}\right)$$

which is the inequality (3).

Theorem 2.2. *For any n, let f be a convex function and $x_1, \ldots, x_n$ lie in its domain and let $\{c_i\}_{1\le i\le n}$ be nonnegative real numbers with $\sum_{i=1}^{n} c_i = 1$. In addition, assume that for any $1 \le k \le n-1$, $x_{n+1} = x_1, \ldots, x_{n+k} = x_k$, $c_{n+1} = c_1, \ldots, c_{n+k} = c_k$, then we have*

$$\begin{aligned}
(8) \quad & \sum_{i=1}^{n} a_i f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} a_i x_i\right) \\
& \qquad \ge \frac{n-1}{n}\sum_{i=1}^{n} f(c_i x_i + c_{i+1}x_{i+1} + \cdots + (1 - c_i - \cdots - c_{i+k-1})x_{i+k}),
\end{aligned}$$

where

$$(9) \qquad a_i := \begin{cases} ic_i + (k-i)c_{n+i} + 1 - \sum_{l=1}^{k} c_{n+i-l}, & 1 \le i \le k \\ kc_i + 1 - \sum_{l=1}^{k} c_{i-l} & k+1 \le i \le n. \end{cases}$$

Proof. As the similar proof as Theorem 2.1, for any $1 \le i \le n$,

$$\begin{aligned} f(c_i x_i + c_{i+1} x_{i+1} + \cdots + (1 - c_i - \cdots - c_{i+k-1}) x_{i+k}) \\ \le \sum_{l=1}^{k} c_{i+l-1} f(x_{i+l-1}) + (1 - c_i - \cdots - c_{i+k-1}) f(x_{i+k}). \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\sum_{i=1}^{n} f(c_i x_i + c_{i+1} x_{i+1} + \cdots + (1 - c_i - \cdots - c_{i+k-1}) x_{i+k}) \\ &\quad \le \sum_{i=1}^{n} \left(\sum_{l=1}^{k} c_{i+l-1} f(x_{i+l-1}) + (1 - c_i - \cdots - c_{i+k-1}) f(x_{i+k}) \right) \\ &\quad = \sum_{i=1}^{k} \left(i c_i + (k - i) c_{n+i} + 1 - \sum_{l=1}^{k} c_{n+i-l} \right) f(x_i) \\ &\qquad + \sum_{i=k+1}^{n} \left(k c_i + 1 - \sum_{l=1}^{k} c_{i-l} \right) f(x_i) \triangleq \sum_{i=1}^{n} a_i f(x_i). \end{aligned}$$

Furthermore, noting the fact $\sum_{i=1}^{n} a_i = n$ and using the Jensen's inequality,

$$\begin{aligned} \sum_{i=1}^{n} a_i f(x_i) &= \frac{n}{n-1} \left[\sum_{i=1}^{n} a_i f(x_i) - \frac{1}{n} \sum_{i=1}^{n} a_i f(x_i) \right] \\ &\le \frac{n}{n-1} \left[\sum_{i=1}^{n} a_i f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} a_i x_i \right) \right] \end{aligned}$$

which implies our result. □

Remark 2.2. If let $k = 1$, $x_{n+1} = x_1$ and $\sum_{i=1}^{n} c_i = 1$, then $a_1 = c_1 + (1 - c_n)$, $a_i = c_i + (1 - c_{i-1})$, $2 \le i \le n$ and from Theorem 2.2, we obtain

$$\begin{aligned} \sum_{i=1}^{n} (c_i + (1 - c_{i-1})) f(x_i) - f\left(\frac{1}{n} \sum_{i=1}^{n} (c_i + (1 - c_{i-1})) x_i \right) \\ \ge \frac{n-1}{n} \sum_{i=1}^{n} f(c_i x_i + (1 - c_i) x_{i+1}), \end{aligned} \tag{10}$$

where $c_0 = c_n$.

The variant of the generalized Popovicui inequality is given in the following theorem.

Theorem 2.3. *If f is a convex function and $x_1, x_2 \ldots, x_n$ lie in its domain and let $\{c_i\}_{1 \le i \le n}$ be nonnegative real numbers with $\sum_{i=1}^{n} c_i = 1$. In addition,* *for any* $1 \le i \le n$, *let* $a = \sum_{i=1}^n c_i a_i$ *and* $b_i = (a - c_i a_i)/(1 - c_i)$, *then*

$$(n-1)\sum_{i=1}^n f(b_i) \le n\left[\sum_{j=1}^n K_j c_j f(a_j) - f\left(\frac{1}{n}\sum_{j=1}^n K_j c_j a_j\right)\right], \tag{11}$$

where for any $1 \le j \le n$,

$$K_j = \sum_{i=1, i\neq j}^n \frac{1}{\sum_{j=1, j\neq i}^n c_j}.$$

Proof. By using the Jensen's inequality, for any $1 \le i \le n$, we have

$$\begin{aligned} f(b_i) = f\left(\frac{1}{1-c_i}(a - c_i a_i)\right) &= f\left(\sum_{j=1, j\neq i}^n \frac{c_j}{\sum_{j=1, j\neq i}^n c_j} a_j\right) \\ &\le \sum_{j=1, j\neq i}^n \frac{c_j}{\sum_{j=1, j\neq i}^n c_j} f(a_j). \end{aligned}$$

Thus by summing for i from 1 to n, we have

$$\begin{aligned} \sum_{i=1}^n f(b_i) \le& \sum_{i=1}^n \sum_{j=1, j\neq i}^n \frac{c_j}{\sum_{j=1, j\neq i}^n c_j} f(a_j) \\ =& \sum_{j=1}^n \left(\sum_{i=1, i\neq j}^n \frac{1}{\sum_{j=1, j\neq i}^n c_j}\right) c_j f(a_j) \\ \triangleq& \sum_{j=1}^n K_j c_j f(a_j). \end{aligned}$$

Furthermore, noting the fact that $\sum_{j=1}^n K_j c_j = n$ and using Jensen's inequality,

$$\begin{aligned} \sum_{j=1}^n K_j c_j f(a_j) =& \frac{n}{n-1}\left[\sum_{j=1}^n K_j c_j f(a_j) - \frac{1}{n}\sum_{j=1}^n K_j c_j f(a_j)\right] \\ \le& \frac{n}{n-1}\left[\sum_{j=1}^n K_j c_j f(a_j) - f\left(\frac{1}{n}\sum_{j=1}^n K_j c_j a_j\right)\right]. \end{aligned}$$

which means our result. □

Remark 2.3. For all $1 \le i \le n$, let $c_i = 1/n$, then $a = (a_1 + \cdots + a_n)/n$, $b_i = (na - a_i)/(n-1)$ and $K_i c_i = 1$, therefore we get

$$(n-1)\sum_{i=1}^n f(b_i) \le n\left[\sum_{j=1}^n f(a_j) - f\left(\frac{1}{n}\sum_{j=1}^n a_j\right)\right],$$

which is the inequality (4).

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