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PURELY PERIODIC NEAREST SQUARE CONTINUED FRACTIONS

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Abstract. We give three sets of conditions to determine whether a real quadratic surd $\xi = (P+\sqrt{D})/Q$ has a purely periodic nearest square continued fraction expansion. One set is a few inequalities involving only ξ and its conjugate $\overline{\xi} = (P-\sqrt{D})/Q$. Another set is a few inequalities involving only $P/\sqrt{D}$ and $Q/\sqrt{D}$. A third set of conditions and additional results are presented.

1. Introduction

The oldest general method for solving Pell equations is the cyclic method, written about by Bhaskara in 1150 AD, although it seems to have been developed by even earlier Indian mathematicians [1, p. 602], [2, p. 21], [9, p. 333], [3, pp. 32–34], [8, pp. 22–24]. The *nearest square continued fraction* introduced by A. A. K. Ayyangar in 1938 [1, 2] encapsulates one variant of the cyclic method. Despite its great age, this continued fraction seems to have been little studied. The nearest square continued fraction is defined in the next section.

Simple tests have long been known for determining whether a real quadratic irrational $\xi = (P+\sqrt{D})/Q$, $D > 0$ not a square, has a purely periodic regular continued fraction expansion, or nearest integer continued fraction expansion. Thus if $\overline{\xi} = (P-\sqrt{D})/Q$, then ξ has a purely periodic regular continued fraction expansion if and only if $\xi > 1$ and $-1 < \overline{\xi} < 0$ [7, pp. 73–74], [3, Thm. 3.8]. Also ξ has a purely periodic nearest integer continued fraction expansion if and only if $\xi > 2$ and $(1-\sqrt{5})/2 < \overline{\xi} \leq (3-\sqrt{5})/2$ [6].

In this paper we derive a test for pure periodicity for the nearest square continued fraction of A. A. K. Ayyangar [2] which is in the spirit of these tests. Our starting point is a recent result in [5].

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This characterization is given in the following theorem. Write

$$\phi = (1+\sqrt{5})/2.$$

Theorem 1. *A real quadratic surd ξ has a purely periodic nearest square continued fraction expansion if and only if*

(i) $\xi > \phi$,
(ii) $N(2\xi+1) \geq -1$,
(iii) *If* $\xi < 2$, *then* $N(\xi - 1) < -1$,
(iv) *If* $2 < \xi \leq \phi + 1$, *then* $N(\xi - 1) \leq -1$,
(v) *If* $\xi > \phi + 1$, *then* $N(2\xi - 1) < -1$.

Here $N(\eta)$ is the norm of $\eta = (P+\sqrt{D})/Q$, i.e., $N(\eta) = \eta\overline{\eta}$ where $\overline{\eta} = (P-\sqrt{D})/Q$ is the conjugate of η.

The region Ω given in the theorem is illustrated in Figure 1. The boundaries are segments of certain hyperbolas. Solid lines along the boundary of the region indicate that these points are in the region, and dotted lines indicate points that are not in the region. Exaggerated points are drawn for $(\xi, \overline{\xi}) = (\phi, 1-\phi) \approx (1.618, -0.618)$ and $(\phi+1, 2-\phi) \approx (2.618, 0.382)$ to emphasize that the former is not in the region, while the latter is in the region.

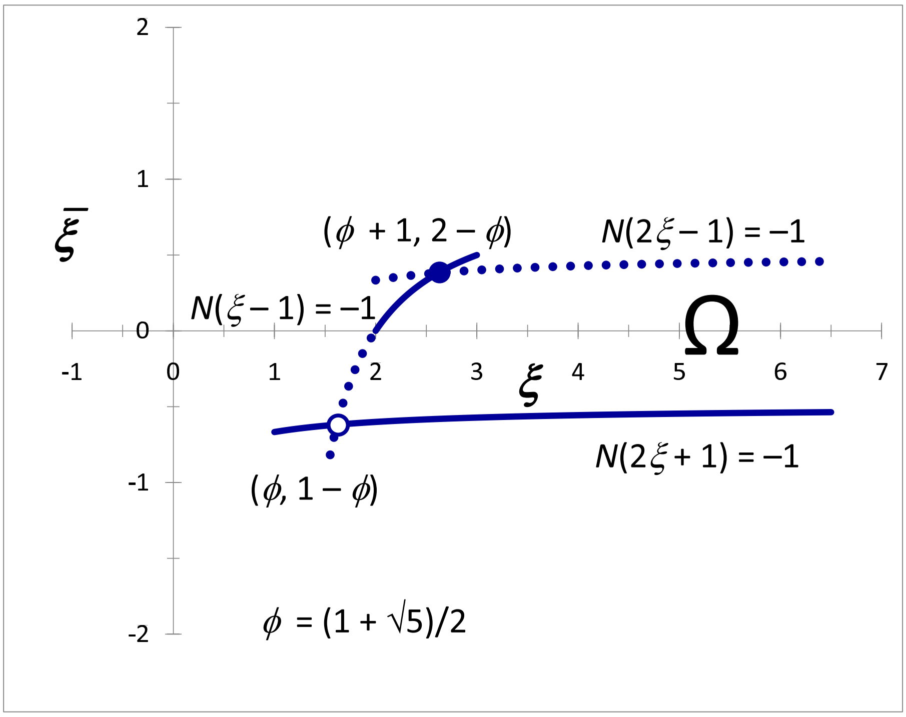

Figure 1. The region Ω

Another characterization is given in the following corollary.

Corollary 1. *Let $\xi = (P+\sqrt{D})/Q$ be a real quadratic surd. Write $x = Q/\sqrt{D}$ and $y = P/\sqrt{D}$. ξ has a purely periodic nearest square continued fraction if and only if either*

$$0 < x < 2/\sqrt{5} \quad \textit{and} \quad -\frac{x}{2} + \sqrt{1-\left(\frac{x}{2}\right)^2} \le y < \frac{x}{2} + \sqrt{1-\left(\frac{x}{2}\right)^2}, \ \textit{or}$$
$$2/\sqrt{5} \le x < 1 \quad \textit{and} \quad x - \sqrt{1-x^2} < y \le x + \sqrt{1-x^2}.$$

The region Ψ given in the corollary is graphed in Figure 2. The boundaries are segments of certain ellipses. Exaggerated points are drawn for $(x, y) = (2/\sqrt{5}, 3/\sqrt{5}) \approx (0.894, 1.342)$ and $(2/\sqrt{5}, 1/\sqrt{5}) \approx (0.894, 0.447)$ to emphasize that the former is in the region, and the latter is not.

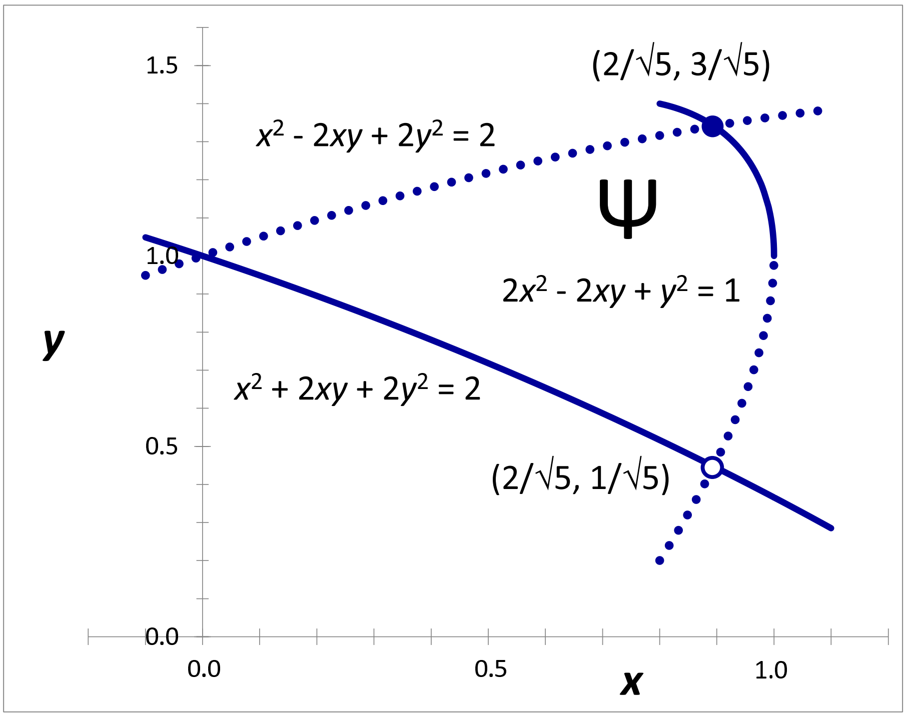

FIGURE 2. The region Ψ

As mentioned, the nearest square continued fraction (NSCF) is defined in the next section. The proof of Theorem 1 is reduced to a few lemmas in the second section following, and these lemmas are proved in subsequent sections. A final section gives additional corollaries.

2. BACKGROUND ON THE NSCF

A (non-terminating) *semi–regular continued fraction* is a continued fraction

$$a_0 + \frac{\epsilon_1}{a_1+} \frac{\epsilon_2}{a_2+} \frac{\epsilon_3}{a_3+} \cdots$$

i	0	1	2	3	4	5	6	7	8	9	10	11	12
P_i	0	9	7	8	3	5	4	5	3	8	7	9	9
Q_i	1	16	3	11	8	9	9	8	11	3	16	1	16
a_i	9	1	5	1	1	1	1	1	1	5	1	18	1
ϵ_i	1	1	1	1	1	1	1	1	1	1	1	1	1

Table 1. The regular continued fraction expansion of $\sqrt{97}$

where the a_i are integers, $a_i > 0$ when $i > 0$, $\epsilon_i = \pm 1$, $a_i + \epsilon_{i+1} \geq 1$ for $i \geq 1$, and $a_i + \epsilon_{i+1} \geq 2$ infinitely often [7, p. 139].

Writing

$$\xi_i = a_i + \frac{\epsilon_{i+1}}{a_{i+1}+}\frac{\epsilon_{i+2}}{a_{i+2}+}\frac{\epsilon_{i+3}}{a_{i+3}+}\cdots$$

we have the familiar recurrence relation

$$\xi_i = a_i + \frac{\epsilon_{i+1}}{\xi_{i+1}}, \quad \text{or} \quad \xi_{i+1} = \frac{\epsilon_{i+1}}{\xi_i - a_i}.$$

The *regular continued fraction* (RCF) of an irrational number ξ_0 is defined by taking $a_i = \lfloor \xi_i \rfloor$ and $\epsilon_{i+1} = 1$ for all $i \geq 0$.

As previously noted, the *nearest square continued fraction* was introduced by A. A. K. Ayyangar in 1938 [1, 2]. Let $\xi_0 = (P + \sqrt{D})/Q$ be a surd in standard form, i.e.,

(i) P, $Q \neq 0$, and $D > 0$ are integers, D not a perfect square,
(ii) $(P^2 - D)/Q$ is an integer,
(iii) $\gcd(P, Q, (D - P^2)/Q) = 1$.

Then with $c = \lfloor \xi_i \rfloor$, the integer part of ξ_i, we can represent ξ_i in two ways, the *positive* and *negative* representations of ξ_i:

$$\xi_i = c + \frac{Q'}{P' + \sqrt{D}} = c + 1 - \frac{Q''}{P'' + \sqrt{D}},$$

where $\frac{P'+\sqrt{D}}{Q'} > 1$ and $\frac{P''+\sqrt{D}}{Q''} > 1$ are also in standard form. Here $P' = cQ - P$, $Q' = (D - P'^2)/Q$, $P'' = (c+1)Q - P$, $Q'' = (P''^2 - D)/Q$.

We choose the partial denominator a_i and numerator ϵ_{i+1} of the nearest square continued fraction expansion as follows:

(i) $a_i = c$ and $\epsilon_{i+1} = 1$ if $|Q'| < |Q''|$, or $|Q'| = |Q''|$ and $Q < 0$.
(ii) $a_i = c + 1$ and $\epsilon_{i+1} = -1$ if $|Q'| > |Q''|$, or $|Q'| = |Q''|$ and $Q > 0$.

We say the NSCF takes the positive representation of ξ_i if the NSCF successor of ξ_i is $\frac{P'+\sqrt{D}}{Q'}$ and we say the NSCF takes the negative representation of ξ_i if the NSCF successor of ξ_i is $\frac{P''+\sqrt{D}}{Q''}$.

Tables 1 and 2 show the first period of the regular and nearest square continued fractions of $\sqrt{97}$.

i	0	1	2	3	4	5	6	7
P_i	0	10	11	13	5	11	10	10
Q_i	1	3	8	9	8	3	1	3
a_i	10	7	3	2	2	7	20	7
ϵ_i	1	-1	-1	-1	1	-1	-1	-1

TABLE 2. The nearest square continued fraction expansion of $\sqrt{97}$

3. Proof of Theorem 1 and Corollary 1

Our proof of Theorem 1 is based on the following theorem from [5]:

Theorem 2. *Let $\xi = (P + \sqrt{D})/Q$ be a quadratic surd in standard form and let $R = (D - P^2)/Q$. Then ξ has a purely periodic nearest square continued fraction expansion if and only if*

(i) $Q^2 + \frac{1}{4}R^2 \leq D, \quad \frac{1}{4}Q^2 + R^2 \leq D,$
(ii) *ξ is the successor of $1/\xi$,*
(iii) *ξ is not of the form $\frac{p+q+\sqrt{p^2+q^2}}{2q}$, $p > 2q > 0$.*

We also need a result by Ayyangar [2, Cor. 1 to Thm. 5]:

Theorem 3. *If a real quadratic surd has a purely periodic nearest square continued fraction, then $\xi > \phi$.*

Given these two theorems, Theorem 1 clearly follows from the following three lemmas:

Lemma 1. *$\xi > \phi$ satisfies Condition (i) of Theorem 2 if and only if*

(i) $N(2\xi + 1) \geq -1$,
(ii) *If $\xi \leq \phi + 1$, then $N(\xi - 1) \leq -1$,*
(iii) *If $\xi > \phi + 1$, then $N(2\xi - 1) \leq -1$.*

Lemma 2. *If $\xi > \phi$ and ξ satisfies Condition (i) of Theorem 2, then ξ is the successor of $1/\xi$ except when $\xi < 2$ and $N(\xi - 1) = -1$.*

Lemma 3. *ξ is of the form $(p+q+\sqrt{p^2+q^2})/2q$ with $p > 2q > 0$ if and only if $\xi > \phi + 1$ and $N(2\xi - 1) = -1$.*

The next three sections prove these lemmas, and so complete the proof of Theorem 1.

Corollary 1 follows from Theorem 1 by considering the birational map $x = 2/(\xi - \overline{\xi})$, $y = (\xi + \overline{\xi})/(\xi - \overline{\xi})$, which has the inverse $\xi = (y+1)/x$, $\overline{\xi} = (y-1)/x$. For example, the segment $\phi \leq \xi \leq \phi + 1$ on the hyperbola $N(\xi - 1) = -1$ in the ξ-$\overline{\xi}$ plane corresponds to the segment $x \geq 2/\sqrt{5}$ on the ellipse $2x^2 - 2xy + y^2 = 1$ in the x-y plane.

4. Proof of Lemma 1

Lemma 1 follows from the following two lemmas:

Lemma 4. *$\xi > \phi$ satisfies Condition (i) of Theorem 2 if and only if*

$$N(2\xi+1) \geq -1,\ N(\xi-1) \leq -1,\ \text{and } N(2\xi-1) \leq -1.$$

Lemma 5.

(i) *For $\phi < \xi < \phi+1$, if $N(\xi-1) \leq -1$, then $N(2\xi-1) < -1$.*
(ii) *For $\xi > \phi+1$, if $N(2\xi-1) \leq -1$, then $N(\xi-1) < -1$.*
(iii) *If $\xi = \phi+1$, then $N(\xi-1) = N(2\xi-1) = -1$.*

Substituting for Q, R, and D in terms of ξ and $\overline{\xi}$ in the formulas in Condition (i) of Theorem 2 yields expressions that can be factored, from which the proof of Lemma 4 is straightforward.

Before getting to that, the following two lemmas are needed:

Lemma 6. *Write*

$$\begin{aligned} H_1(\xi) &= -1 - \frac{1}{\xi+1}, \\ H_2(\xi) &= \frac{1}{2}\left(-1 - \frac{1}{2\xi+1}\right), \\ H_3(\xi) &= \frac{1}{2}\left(1 - \frac{1}{2\xi-1}\right), \\ H_4(\xi) &= 1 - \frac{1}{\xi-1}. \end{aligned}$$

Then for $\xi > \phi$,

(i) *$H_1(\xi) < H_2(\xi)$,*
(ii) *$H_2(\xi) < H_3(\xi)$,*
(iii) *If $\xi < \phi+1$, then $H_4(\xi) < H_3(\xi)$,*
(iv) *If $\xi > \phi+1$, then $H_3(\xi) < H_4(\xi)$,*
(v) *Each $H_i(\xi)$ is an increasing function of ξ.*

Lemma 7. *If $\xi > 1$ then*

(i) *$N(\xi+1) \gtreqless -1$ if and only if $\overline{\xi} \gtreqless H_1(\xi)$,*
(ii) *$N(2\xi+1) \gtreqless -1$ if and only if $\overline{\xi} \gtreqless H_2(\xi)$,*
(iii) *$N(2\xi-1) \gtreqless -1$ if and only if $\overline{\xi} \gtreqless H_3(\xi)$,*
(iv) *$N(\xi-1) \gtreqless -1$ if and only if $\overline{\xi} \gtreqless H_4(\xi)$.*

The proofs of Lemmas 6 and 7 are straightforward.

As a final preliminary, $\xi = (P+\sqrt{D})/Q$ and $\overline{\xi} = (P-\sqrt{D})/Q$ give

$$\frac{Q}{\sqrt{D}} = \frac{2}{\xi-\overline{\xi}}, \quad \frac{P}{\sqrt{D}} = \frac{\xi+\overline{\xi}}{\xi-\overline{\xi}}, \quad \frac{R}{\sqrt{D}} = \frac{-2\xi\overline{\xi}}{\xi-\overline{\xi}}. \tag{1}$$

With these preliminaries we turn to the proof of Lemma 4. Write

$$F = \frac{1}{4}\left(\frac{Q}{\sqrt{D}}\right)^2 + \left(\frac{R}{\sqrt{D}}\right)^2 - 1. \tag{2}$$

Substituting into Equation (2) in terms of ξ and $\overline{\xi}$ we get

$$\begin{aligned} F &= \frac{1}{4}\left(\frac{2}{\xi-\overline{\xi}}\right)^2 + \left(\frac{-2\xi\overline{\xi}}{\xi-\overline{\xi}}\right)^2 - 1 \\ &= \frac{((2\xi+1)(2\overline{\xi}+1)+1)((2\xi-1)(2\overline{\xi}-1)+1)}{4(\xi-\overline{\xi})^2} \\ &= \frac{(N(2\xi+1)+1)(N(2\xi-1)+1)}{4(\xi-\overline{\xi})^2}. \end{aligned} \tag{3}$$

By Equation (2), the surd ξ satisfies the second part of Condition (i) of Theorem 2 if and only if $F \le 0$. By Equation (3), because $4(\xi-\overline{\xi})^2 > 0$, $F \le 0$ is equivalent to

$$(N(2\xi+1)+1)(N(2\xi-1)+1) \le 0.$$

This holds if and only if either

$$N(2\xi+1) \le -1 \ \text{ and } \ N(2\xi-1) \ge -1, \ \text{ or} \tag{4}$$

$$N(2\xi+1) \ge -1 \ \text{ and } \ N(2\xi-1) \le -1. \tag{5}$$

We show that if $\xi > \phi$ then ξ cannot satisfy both conditions in Equation (4). Assume that $N(2\xi+1) \le -1$. By Lemma 7(ii), for $\xi > \phi$, $N(2\xi+1) \le -1$ implies that $\overline{\xi} \le H_2(\xi)$. By Lemma 6(ii), $H_2(\xi) < H_3(\xi)$, so $\overline{\xi} < H_3(\xi)$. Lemma 7(iii) gives $N(2\,\xi-1) < -1$. We conclude that ξ satisfies the second part of Condition (i) of Theorem 2 if and only ξ satisfies Equation (5).

Similarly, using the first part of Condition (i) of Theorem 2 we have either

$$N(\xi+1) \le -1 \ \text{ and } \ N(\xi-1) \ge -1, \ \text{ or}$$

$$N(\xi+1) \ge -1 \ \text{ and } \ N(\xi-1) \le -1. \tag{6}$$

From Equation (5) we have $N(2\xi+1) \ge -1$, which for $\xi > \phi$ implies that $\overline{\xi} \ge H_2(\xi)$. By Lemma 6(i), $H_2(\xi) > H_1(\xi)$, so $N(\xi+1) > -1$. This says that $\xi > \phi$ satisfies the first part of Condition (i) of Theorem 2 if and only if ξ satisfies Equation (6).

As such, $\xi > \phi$ satisfies Condition (i) of Theorem 2 if and only if

$$N(2\xi+1) \ge -1, \quad N(2\xi-1) \le -1, \quad N(\xi+1) > -1, \quad \text{and} \quad N(\xi-1) \le -1.$$

Because, for $\xi > \phi$, $N(2\xi+1) \geq -1$ already implies that $N(\xi+1) > -1$, ξ satisfies Condition (i) of Theorem 2 if and only if

$$N(2\xi+1) \geq -1, \quad N(2\xi-1) \leq -1, \text{ and } \quad N(\xi-1) \leq -1.$$

This establishes Lemma 4.

We turn to the proof of Lemma 5. For Condition (i), if $N(\xi-1) \leq -1$ then $\overline{\xi} \leq H_4(\xi)$. But $H_4(\xi) < H_3(\xi)$, so $\overline{\xi} < H_3(\xi)$, which implies that $N(2\xi-1) < -1$.

For Condition (ii), $\xi > \phi + 1$ gives $H_3(\xi) < H_4(\xi)$.

Finally, direct calculation establishes Condition (iii).

5. Proof of Lemma 2

We begin with a lemma.

Lemma 8. *For $\eta = (P+\sqrt{D})/Q$, if $0 < \eta < 1$, then the successor of η is $1/\eta$ if $|N(\eta)| < |N(1-\eta)|$ or $Q < 0$ and $|N(\eta)| = |N(1-\eta)|$; otherwise the successor of η is $1/(1-\eta) \neq 1/\eta$.*

Proof. It suffices to show that the NSCF takes the positive representation when $|N(\eta)| < |N(1-\eta)|$ or $Q < 0$ and $|N(\eta)| = |N(1-\eta)|$, and that otherwise the NSCF takes the negative representation.

Because $0 < \eta < 1$, $c = \lfloor \eta \rfloor = 0$, so $P' = -P$, $Q' = (D-P^2)/Q$, $P'' = Q-P$, and $Q'' = ((Q-P)^2 - D)/Q$.

Then

$$N(\eta) = \frac{P^2 - D}{Q^2} = \frac{-Q'}{Q},$$

and

$$N(1-\eta) = \frac{(Q-P)^2 - D}{Q^2} = \frac{Q''}{Q}.$$

The lemma follows. □

If $\xi = (P+\sqrt{D})/Q > \phi$ then $1/\xi = (-P+\sqrt{D})/R$ where $R = (D-P^2)/Q$, and $0 < 1/\xi < 1$. Applying Lemma 8 to $1/\xi$, to determine whether ξ is the successor of $1/\xi$ it suffices to determine whether $|N(1/\xi)| < |N(1-1/\xi)|$ or $R < 0$ and $|N(1/\xi)| = |N(1-1/\xi)|$.

But, $|N(1/\xi)| < |N(1-1/\xi)|$ if and only if $1 = |N(1)| < |N(\xi-1)|$. Similarly, $|N(1/\xi)| = |N(1-1/\xi)|$ if and only if $1 = |N(\xi-1)|$.

From Lemma 4(ii) we know that if ξ satisfies Condition (i) of Theorem 2 then $N(\xi-1) \leq -1$ so $|N(\xi-1)| \geq 1$.

So ξ is the successor of $1/\xi$ except when $|N(\xi-1)| = 1$ and $R > 0$.

Now $|N(\xi-1)| = 1$ if and only if $N(\xi-1) = 1$ or $N(\xi-1) = -1$. By Lemma 1 if $\xi > \phi$ and ξ satisfies Condition (i) of Theorem 2, then $N(\xi-1) \leq -1$, so we cannot have $N(\xi-1) = 1$.

If $N(\xi-1)=-1$, then $\overline{\xi}=1-1/(\xi-1)$, so

$$R=\frac{-2\,\xi\,\overline{\xi}}{\xi-\overline{\xi}}=\frac{-2\,\xi\,(\xi-2)}{(\xi-1)^2+1}$$

and $R>0$ when $\phi<\xi<2$, and $R<0$ when $\xi>2$.

6. Proof of Lemma 3

If $\xi=(p+q+\sqrt{p^2+q^2})/2q$ then $2\xi-1=(p+\sqrt{p^2+q^2})/q$ and

$$N(2\xi-1)=-1.$$

If, in addition, if $p>2q>0$, then

$$\frac{p+q+\sqrt{p^2+q^2}}{2q}>\frac{3q+\sqrt{5q^2}}{2q}=\frac{3+\sqrt{5}}{2}=\phi+1.$$

If $\xi>\phi$ and $N(2\xi-1)=-1$ then $\overline{\xi}=H_3(\xi)<1/2<\xi$. In particular, $\xi-\overline{\xi}>0$, so $Q=2\sqrt{D}/(\xi-\overline{\xi})>0$.

If $N(2\xi-1)=-1$ then $2\xi\overline{\xi}-\xi-\overline{\xi}+1=0$. With $\xi=(P+\sqrt{D})/Q$ we get

$$2P^2-2PQ+Q^2=2D, \tag{7}$$

so Q is even.

Write $Q=2q>0$ and $P=p+q$. From Equation (7) we have $2(p+q)^2-2(p+q)(2q)+4q^2=2D$ giving $D=p^2+q^2$.

Next suppose

$$\xi=\frac{p+q+\sqrt{p^2+q^2}}{2q}>\phi+1 \tag{8}$$

with $q>0$. We now show that $p>2q$. First we show that $p\geq 0$. If $p<0$ (so $|p|=-p$), then

$$\frac{p+q+\sqrt{p^2+q^2}}{2q}<\frac{p+q+|p|+q}{2q}=1<\phi+1,$$

a contradiction.

Now, if $0\leq p\leq 2q$ then

$$\frac{p+q+\sqrt{p^2+q^2}}{2q}\leq\frac{2q+q+\sqrt{4q^2+q^2}}{2q}=\phi+1.$$

contradicting (8). This completes the proof that $p>2q>0$.

7. Additional comments

Keith Matthews [4] points out that the following characterization is equivalent to Theorem 1 and avoids the "$\leq$" and "$\geq$."

Theorem 4. *A real quadratic surd ξ has a purely periodic NSCF expansion if and only if either*

(a) *ξ has one of the forms*
 (i) $\xi = (p + q + \sqrt{p^2 + q^2})/q$, $q \geq 2p > 0$,
 (ii) $\xi = (p - q + \sqrt{p^2 + q^2})/2q$, $p > 2q > 0$,

or

(b) (i) $\xi > \phi$,
 (ii) $N(2\xi + 1) > -1$,
 (iii) $N(2\xi - 1) < -1$,
 (iv) $N(\xi - 1) < -1$.

We finish with a few corollaries of Theorem 1. The first gives two conditions ξ must satisfy if it has a purely periodic NSCF expansion.

Corollary 2. *If ξ has a purely periodic NSCF expansion, then*
 (i) $1 - \phi < \overline{\xi} < 1/2$,
 (ii) $\xi - \overline{\xi} > 2$.

Proof. For (i), by Theorem 1, $N(2\xi + 1) \geq -1$, so for $\xi > \phi$, $\overline{\xi} \geq H_2(\xi)$. But $\xi > \phi$ implies that $H_2(\xi) > H_2(\phi) = 1 - \phi$.

By Theorem 2 and Lemma 4, $N(2\xi - 1) \leq -1$, so $\overline{\xi} \leq H_3(\xi) < 1/2$.

For (ii), we have $\xi > \phi > 1/2 > \overline{\xi}$, so $\xi - \overline{\xi} > 0$. From Equation (1), we get $Q > 0$.

From Condition (i) of Theorem 2, $Q^2 + \frac{1}{4}R^2 \leq D$, and $R \neq 0$, so $|Q| < \sqrt{D}$. Because $Q > 0$, in fact

$$0 < Q < \sqrt{D}, \tag{9}$$

or $0 < Q/\sqrt{D} < 1$. Using (1) again establishes that $\xi - \overline{\xi} > 2$. □

There are ξ that satisfy the conclusions of both Theorem 3 and Corollary 2 that do not have purely periodic NSCF expansions, such as $(15 + \sqrt{126})/11$, $(18 + \sqrt{212})/7$, $(10 + \sqrt{198})/7$, $(8 + \sqrt{181})/13$, and $(301 + \sqrt{80661})/284$. But there are partial converses to the combination of Theorem 3 and Corollary 2, such as

Corollary 3. *If $\xi \geq \phi + 1$ and $-1/2 < \overline{\xi} \leq 2 - \phi$, then ξ has a purely periodic NSCF expansion.*

Proof. For $\xi > \phi + 1$ it suffices to show that $N(2\xi + 1) > -1$ and $N(2\xi - 1) < -1$. First, $\overline{\xi} > -1/2$ implies that $2\overline{\xi} + 1 > 0$, and so $N(2\xi + 1) > 0 > -1$.

Second, $\overline{\xi} \le 2-\phi$ implies that $2\overline{\xi}-1 \le 3-2\phi < 0$. When $\xi > \phi+1$ we have $2\xi-1 > 2\phi+1 > 0$ and so $N(2\xi-1) < (2\phi+1)(3-2\phi) = -1$. So if $\xi > \phi+1$ and $-1/2 < \overline{\xi} \le 2-\phi$ then ξ has a purely periodic NSCF expansion.

Finally, $\xi = \phi+1$ has a purely periodic NSCF expansion. □

If $\xi = (P+\sqrt{D})/Q$ has a purely periodic RCF expansion, then $0 < Q \le 2\lfloor\sqrt{D}\rfloor$, and $0 < P < \sqrt{D}$ [7, p. 68]. Similarly, for the NSCF we have

Corollary 4. *If $\xi = (P+\sqrt{D})/Q$ has a purely periodic NSCF expansion, then*
 (i) $0 < Q < \sqrt{D}$,
 (ii) $\sqrt{D/5} < P \le 3\sqrt{D/5}$.

Proof. Figure 2 illustrates this.

Part (i) is Equation (9), in the proof of Corollary 2.

Now we show that $P > \sqrt{D/5}$. From Corollary 2 we have $\overline{\xi} > 1-\phi$, so $\phi\overline{\xi} > \phi(1-\phi) \ge \xi(1-\phi)$. Then $\phi\xi + \phi\overline{\xi} > \xi$, $(2\phi-1)\xi + (2\phi-1)\overline{\xi} > \xi - \overline{\xi}$, and

$$\frac{P}{\sqrt{D}} = \frac{\xi+\overline{\xi}}{\xi-\overline{\xi}} > \frac{1}{2\phi-1} = \frac{1}{\sqrt{5}}.$$

Now we show that $P \le 3\sqrt{D/5}$. It suffices to establish that

$$\overline{\xi} \le (2-\phi)^2\xi \tag{10}$$

because multiplying both sides of Equation (10) by $2(\phi+1)$ (and noticing that $(\phi+1)(2-\phi)=1$) gives $\overline{\xi}(2\phi+2) \le \xi(4-2\phi)$, which gives

$$(\xi+\overline{\xi})(2\phi-1) \le 3(\xi-\overline{\xi})$$

or

$$\frac{P}{\sqrt{D}} = \frac{\xi+\overline{\xi}}{\xi-\overline{\xi}} \le \frac{3}{\sqrt{5}}.$$

Assume first that $\phi < \xi < \phi+1$. Then

$$(2-\phi)^2\xi - H_4(\xi) = \frac{(2-\phi)^2}{\xi-1}(2(\phi+1)-2\xi)(2(\phi+1)-\xi) > 0$$

so $(2-\phi)^2\xi > H_4(\xi) \ge \overline{\xi}$, where the last inequality is from Lemma 7.

Now assume $\xi \ge \phi+1$. Then

$$(2-\phi)^2\xi - H_3(\xi) = \frac{2\,(2-\phi)^2}{2\xi-1}(2\xi-(\phi+1))(\xi-(\phi+1)) \ge 0$$

so $(2-\phi)^2\xi \ge H_3(\xi) \ge \overline{\xi}$. □

The following lemma will be useful in the proof of Corollary 5.

Lemma 9. *For k a nonzero integer and ξ a quadratic surd, at most one of ξ and $\xi+k$ has a purely periodic NSCF expansion.*

Proof. First we claim that ξ and $\xi + k$ have the same NSCF successor. If the positive and negative representations of ξ are

$$\xi = c + \frac{Q'}{P' + \sqrt{D}} = c + 1 - \frac{Q''}{P'' + \sqrt{D}},$$

then the positive and negative representations of $\xi + k$ are

$$\xi + k = c + k + \frac{Q'}{P' + \sqrt{D}} = c + k + 1 - \frac{Q''}{P'' + \sqrt{D}},$$

and the claim follows.

Corollary 1 to Theorem X in [2] states that two different reduced surds cannot have the same successor. A surd is reduced if and only if it has a purely periodic NSCF expansion (see [2, Thm. XI] and the immediately preceding comments). Because ξ and $\xi + k$ have the same NSCF successor, at most one can have a purely periodic NSCF expansion. □

The final corollary explores relations between surds with purely periodic RCF expansions and those with purely periodic NSCF expansions.

Corollary 5. (i) *If ξ has a purely periodic NSCF expansion, then exactly one of ξ or $\xi - 1$ has a purely periodic RCF expansion.*

(ii) *If ξ has a purely periodic RCF expansion, then ξ or $\xi + 1$ has a purely periodic NSCF expansion if and only if either $N(\xi - 1) < -1$ or $N(\xi) \leq -1$.*

(iii) *If $\xi > 2$ has a purely periodic RCF expansion, then either ξ or $\xi + 1$ has a purely periodic NSCF expansion.*

If $\xi < 2$ has purely periodic RCF expansion it might be that one of or neither of ξ or $\xi + 1$ has a purely periodic NSCF expansion. Each of the following three surds $1 < \xi < 2$ has a purely periodic RCF expansion: $(11 + \sqrt{133})/12$, $(5+\sqrt{43})/6$, $(3+\sqrt{93})/7$. For the first, neither ξ nor $\xi+1$ has a purely periodic NSCF expansion; for the second ξ has a purely periodic NSCF expansion, and for the third $\xi + 1$ has a purely periodic NSCF expansion.

Now we prove Corollary 5.

Proof. ξ has a purely periodic RCF expansion if and only if $\xi > 1$ and $-1 < \overline{\xi} < 0$ [7, pp. 73–74], [3, Thm. 3.8].

Item (i) is Lemma 6.2 in [6]. The proof is short, so we give it here. From Theorem 3 we have $\xi > 1$, and from Corollary 2 we have $-1 < 1-\phi < \overline{\xi} < 1/2$. If $\overline{\xi} < 0$, then ξ has a purely periodic RCF expansion.

Now assume $\overline{\xi} > 0$. Because $\overline{\xi} < 1/2$, $-1 < \overline{\xi} - 1 = \overline{\xi - 1} < -1/2 < 0$. Also, $\overline{\xi} > 0$ and Corollary 2(ii) give $\xi > 2$, so $\xi - 1 > 1$.

Lemma 9 gives that not both ξ and $\xi - 1$ have purely periodic NSCF expansions.

Now we prove (ii). If $\xi < 2$ has a purely periodic NSCF expansion, then $N(\xi - 1) < -1$ by Theorem 1(iii). If $\xi > 2$ has a purely periodic NSCF expansion, then $\xi - 1 > 1$ and $\overline{\xi} - 1 < -1$ so $N(\xi - 1) < -1$.

Now suppose that $\xi+1$ has a purely periodic NSCF expansion. Then, using Lemma 4(ii), $N(\xi) = N(\xi+1-1) \leq -1$.

The following sub-cases finish the proof of (ii); here "p.p. NSCF" denotes a purely periodic NSCF expansion.

I. If $N(\xi) \leq -1$ and
   A. $\xi \leq \phi$ then $\xi+1$ has a p.p. NSCF.
   B. $\xi > \phi$ and $N(2\xi+1) < -1$ then $\xi+1$ has a p.p. NSCF.
   C. $\xi > \phi$ and $N(2\xi+1) \geq -1$ then ξ has a p.p. NSCF.

II. If $N(\xi-1) < -1$ and
   A. $N(2\xi+1) < -1$ then $\xi+1$ has a p.p. NSCF.
   B. $N(2\xi+1) \geq -1$ then ξ has a p.p. NSCF.

As an example, we prove I.B. Let $\eta = \xi+1$. We show that η satisfies Theorem 1. From $\xi > \phi$ we have $\eta > \phi+1$, and in particular, η satisfies Theorem 1(i). From $\overline{\xi} > -1$ we have $\overline{\eta} > 0$. Combined with $\eta > \phi+1$ we have $N(2\eta+1) > 0 > -1$, giving Theorem 1(ii). Finally, $N(2\eta-1) = N(2\xi+1) < -1$, so η satisfies Theorem 1(v).

The other sub-cases are proved in an equally straightforward way.

For (iii), $\xi > 2$ and $\overline{\xi} < 0$ imply that $N(\xi-1) < -1$, and the result follows from (ii). $\square$

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