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COMPUTER GENERATED IMAGES FOR QUADRATIC RATIONAL MAPS WITH A PERIODIC CRITICAL POINT

DUSTIN GAGE AND DANIEL JACKSON

Abstract. We describe an algorithm for distinguishing hyperbolic components in the parameter space of quadratic rational maps with a periodic critical point. We then illustrate computer images of the hyperbolic components of the parameter spaces $V_1 - V_4$, which were produced using our algorithm. We also resolve the singularities of the projective closure of V_5 by blowups, giving an alternative proof that as an algebraic curve, the geometric genus of V_5 is 1. This explains why we are unable to produce an image for V_5.

1. Introduction

V_n is the set of holomorphic conjugacy classes of quadratic rational maps with a critical point of period n. For example, V_1 may be identified with the family of quadratic polynomials

$$z \mapsto z^2 + c,\ c \in \mathbb{C},$$

each map having ∞ as a fixed critical point. The dynamics of the maps in V_1 are encoded by the much studied Mandelbrot set (see e.g. [1],[4],[6]).

V_2 can be taken as the family of functions $f_a(z) = \frac{a}{z^2+2z}$ (along with the function $f(z) = 1/z^2$), each map having the critical 2-cycle $0 \mapsto \infty \mapsto 0$. In [13], Timorin has given a detailed description of the dynamical plane of V_2.

In the case of $n \geq 3$, one can take V_n to be the set of $(b, c) \in \mathbb{C}^2$ such that for the quadratic map

$$f(z) = 1 + b/z + c/z^2,$$

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the critical point 0 has period n (see e.g. [10]). So, for $n \geq 3$, any map f in V_n has the critical cycle

$$0 \mapsto \infty \mapsto 1 \mapsto \ldots \mapsto f^{n-2}(1) = 0.$$

$f^{n-2}(1) = 0$ may be written in the form

$$\frac{P_n}{Q_n} = 0,$$

where $P_n, Q_n \in \mathbb{C}[b, c]$ are polynomials having no common factors. The closure of V_n (in $\mathbb{C}^2$) is a complex algebraic curve supported on $P_n = 0$.

For example, since $f(1) = 1 + b + c$, V_3 identifies with the complex line

$$P_3 = 1 + b + c = 0.$$

For $n = 4$, we have

$$f^2(1) = \frac{1 + 3b + 2b^2 + 3c + 3bc + c^2}{P_3^2}.$$

So V_4 is contained in the zero set of the irreducible conic

$$P_4 = 1 + 3b + 2b^2 + 3c + 3bc + c^2.$$

Now, for $1 \leq n \leq 4$, the projective closure of V_n, which we will denote by $\overline{V_n}$, is birational to $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. However, as is proved in Stimson's thesis [12], and as we will demonstrate in section 5, the geometric genus of $\overline{V_5}$ is 1, i.e. $\overline{V_5}$ is birational to a torus. In [12] it is also shown that $g(V_6) = 6$ and conjectured that $g(V_7) = 22$ (with the irreducibility of V_7 left unfinished). This means that in these higher period cases, we no longer have a parametrization of V_n by $\mathbb{C}$.

A rational map $f : \mathbb{P}^1 = \mathbb{C} \cup \{\infty\} \to \mathbb{P}^1$ is hyperbolic if under iteration, each critical point of f is attracted to some attracting periodic cycle. Hyperbolic maps are an open and (conjecturally) dense set in the space of rational maps (see e.g. [7]). Connected components of the set of hyperbolic maps are called hyperbolic components. Over the past several decades, much progress has been made in describing the hyperbolic components of the V_n's - especially for $n \leq 3$ (see e.g. [8],[9],[10],[13],[14]).

In this paper we will describe an algorithm to draw the hyperbolic components of V_n while distinguishing between types of components. Our algorithm is essentially an implementation of the classification of hyperbolic components for rational maps. We then use this algorithm to generate graphical approximations of the components of V_n for the cases for which we have a parametrization (the genus 0 cases of $V_1 - V_4$). Of course $V_1 - V_3$ have been drawn before (see e.g. [1],[13],[14]), but we have included images of these parameter spaces for the reader's convenience (see Figs. 2, 3, and 4). Our graphical approximations of V_4 are shown in Figs. 5, 6, and 7. Recently, Kiwi and Rees ([5]) have found formulas to count the number of hyperbolic components in V_n. Their results confirm the number of components illustrated in our figures.

To complete the paper, we give an additional proof that the genus of V_5 is 1 (Theorem 1). We do our genus calculation for V_5 by resolving singularities by blowups, whereas the original proof in [12] uses Puiseaux expansions. This additional proof has been included for 2 reasons. The first reason being completeness of this paper - the result explains why we are not able to generate graphics for V_5. The second reason being that the authors hope that a resolution by blowups may shed light on a general genus computation for V_n. During the preparation of this paper, the second author and two undergraduate students performed a similar calculation for V_6. The blowup sequences for V_6 may be found in [3].

This paper is organized as follows: In section 2 we recall the classification of hyperbolic quadratic rational maps with a critical cycle. We then explain how this classification gives an algorithm for approximating the hyperbolic components of V_n, while distinguishing between their types. In section 3 we illustrate some computer generated representations of the hyperbolic components of V_n for $n = 1, \ldots, 4$. Section 4 contains our calculation of the genus of V_5. Finally, section 5 describes the computer program we made to produce the graphics in this paper.

The second author would like to thank Mary Rees for providing some helpful comments and pointing out some useful references during the preparation of this paper.

2. Hyperbolic Quadratic Rational Maps

Since quadratic maps have 2 critical points and any map f in V_n has the critical cycle

$$0 \mapsto \infty \mapsto 1 \mapsto \ldots \mapsto f^{n-2}(1) = 0,$$

we shall refer to the one remaining critical point of f as the free critical point. Hyperbolic rational maps have been classified by their critical orbits (see e.g. [8],[9],[10]). In fact, any hyperbolic map f in V_n must be exactly one of the following four types:

Type 1 The free critical point is attracted to the other critical point, which is a fixed point.

Type 2 The free critical point is in a periodic component of the attracting basin of the critical cycle.

Type 3 The free critical point is in a preperiodic component of the attracting basin of the critical cycle.

Type 4 The free critical point belongs to the attracting basin of a periodic orbit other than the critical cycle.

This classification suggests an algorithm for making an approximation of the hyperbolic components of V_n. Note that it is easy to distinguish type 4 mappings from the other types by testing the orbit of the free critical point for attraction to the critical cycle. Also, if $n > 1$ then V_n has no type 1 maps, and if $n = 1$ then V_n has no maps of types 2 or 3. So the case of $n = 1$ is easy;

and, for $n > 1$ we just need to distinguish between type 2 and type 3 maps. To do this, one must decide if there is a path from the free critical point to the attracting periodic point, entirely contained within the immediate attracting basin. For most type 2 maps in V_n, the line segment between the free critical point and its attractor lies within the immediate basin. However, there are type 2 maps in V_3 and V_4 for which a nonlinear path must be found (see Figs. 1,7, and 8). We solved this problem by flood-filling the immediate attracting basin to test for the free critical point.

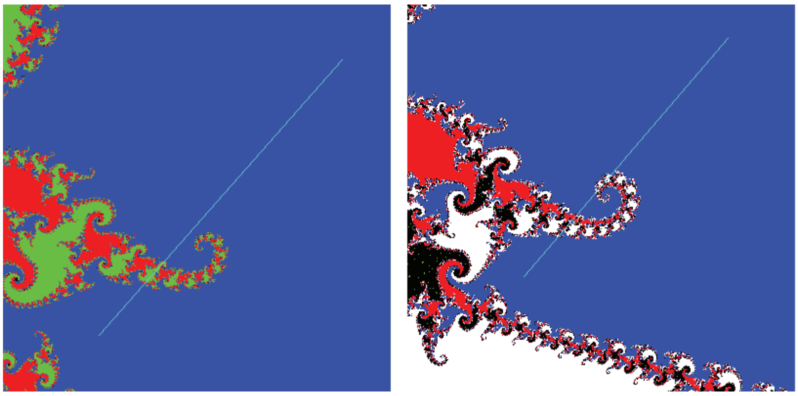

FIGURE 1. The image on the left shows the basins of attraction for the map $f(z) = (z - c)(z - 1)/z^2$ where $c = .16 - 2.2i$. The free critical point is attracted to 1 under iteration of f^3 in this case; but, as is illustrated, the line segment joining the free critical point and 1 is not entirely contained in the immediate basin. The image on the right shows a similar situation for a map in V_4.

3. Computer Generated Images

Using the above algorithm (and a parametrization for V_n), we can generate a graphical approximation of the hyperbolic components of V_n - distinguishing between types. In this section we illustrate such approximations for $n = 1, 2, 3,$ and 4. Approximations of the hyperbolic components of $V_1, V_2,$ and V_3 have previously been illustrated (see e.g. [1],[6], and [14]). However, these approximations (at least for V_2 and V_3) do not give a graphical distinction between the types of components. So we will include these parameter spaces for the reader's convenience. Next we will provide some representations of V_4.

V_1: The set of holomorphic conjugacy classes of quadratic rational maps with a fixed critical point may be identified with the family of polynomials

$$f(z) = z^2 + c \text{ where } c \in \mathbb{C}.$$

There are no type II or III components in this case. The complement of the single type I component is the classical Mandelbrot set (see Fig. 2).

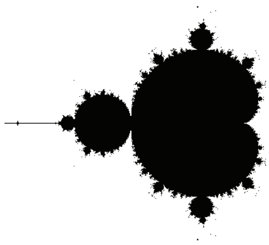

FIGURE 2. V_1: The single type 1 component is in white. Since ∞ is a fixed critical point there are no maps of types 2 or 3. The free critical point, for the maps colored in black, is not attracted to ∞.

V_2: The set of holomorphic conjugacy classes of quadratic rational maps with a period 2 critical point may be identified with the rational map

$$f(z) = \frac{1}{z^2}$$

together with the family

$$f(z) = \frac{a}{z^2 + 2z}, \ a \in \mathbb{C} - \{0\}.$$

V_2 contains one type 2 component (see Fig. 3). A detailed description of the hyperbolic components of V_2 has been given in [13].

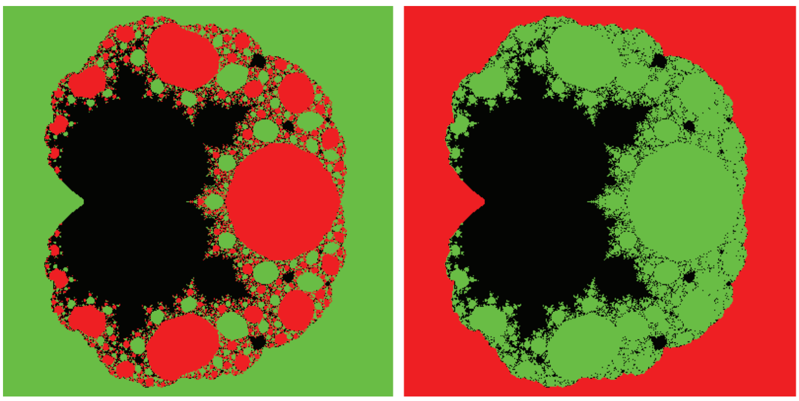

FIGURE 3. V_2: The image on the left shows the maps whose free critical point is attracted to ∞ in red, while the maps with attraction to 0 are colored green. The image on the right shows the type 2 hyperbolic component in red and the type 3 components in green. In both images the free critical point for the maps colored in black is not attracted to the critical cycle.

$V_n, n \geq 3$: In this case, V_n is the collection of quadratic functions

$$f(z) = 1 + \frac{b}{z} + \frac{c}{z^2}$$

that have the critical n-cycle

$$0 \mapsto \infty \mapsto 1 \mapsto \ldots \mapsto f^{n-2}(1) = 0.$$

For example, V_3 is defined by

$$f(1) = 1 + b + c = 0,$$

which gives us the one parameter family of maps

$$f(z) = 1 + \frac{-1-c}{z} + \frac{c}{z^2} = \frac{(z-1)(z-c)}{z^2}.$$

V_3 has 2 type II components: one containing $c = -1$ and the other containing $c = 1$ (see Fig. 4). A nearly complete topological description of the hyperbolic components of V_3 has been given in [10].

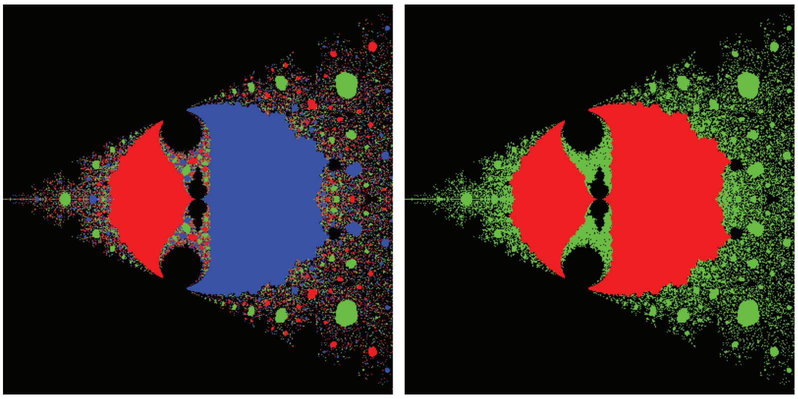

FIGURE 4. V_3: The image on the left shows the maps whose free critical point is attracted to ∞, 0, and 1 in red, green, and blue, respectively. The image on the right shows the type 2 maps in red and the type 3 maps in green. In both images the free critical point for the maps colored in black is not attracted to the critical cycle.

V_4 is determined by $f^2(1) = 0$, which defines the algebraic curve

$$1 + 3b + 2b^2 + 3c + 3bc + c^2 = 0.$$

One may find a rational parametrization for an irreducible conic by projecting from any point on the curve (see e.g. [11]). Figures 5 and 6 show some representations of V_4 using two different parametrizations. Our images show V_4 to have 6 type 2 components - which is confirmed by the formulas given in [5].

Both V_3 and V_4 required the flood-fill algorithm to distinguish their type 2 components from their type 3 components (see Fig. 7). V_2 does not seem to have any such maps.

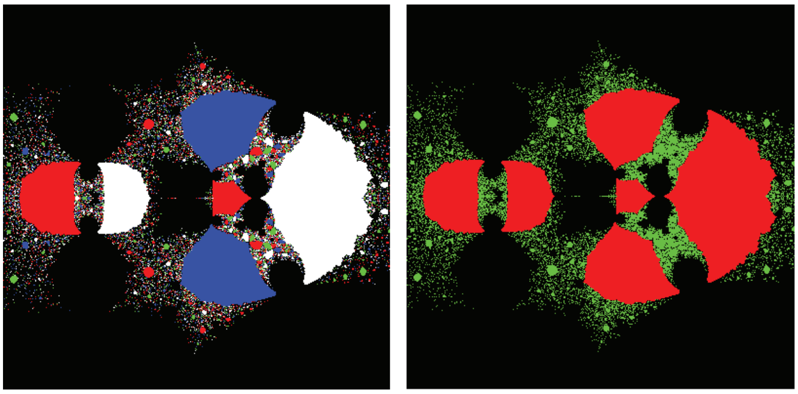

FIGURE 5. V_4 projected from $(-1, 0)$: The image on the left shows the maps whose free critical point is attracted to ∞, 0, 1, and $1 + b + c$ in red, green, blue, and white, respectively. The image on the right shows the type 2 components in red, while the green represents the type 3 components. In both images the free critical point for the maps colored in black is not attracted to the critical cycle.

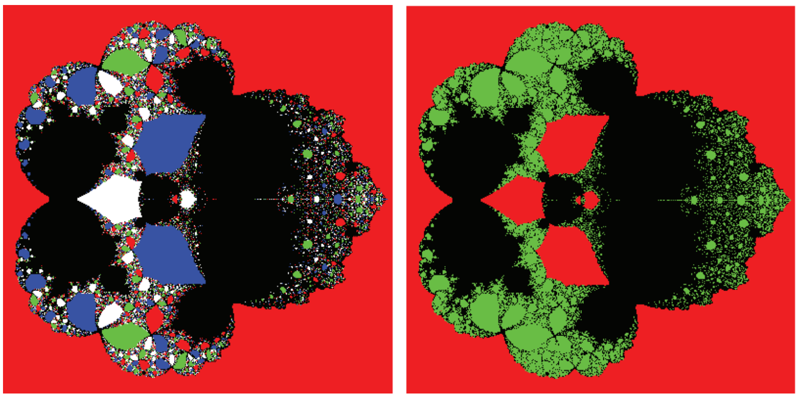

FIGURE 6. V_4 projected from $(0, (3 + \sqrt{5})/2)$: The coloring is the same as in Fig. 5.

4. The Genus of V_5

It is a classical result that any singular point p on an algebraic curve C may be resolved by a sequence of blowups (see e.g. [2] or [11]). The point p and the singular points arising from these blowups are called the infinitely near points to p.

To calculate the geometric genus of an irreducible complex projective plane algebraic curve $C \subset \mathbb{P}^2$ one may use the genus formula:

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum \frac{k_i(k_i - 1)}{2},$$

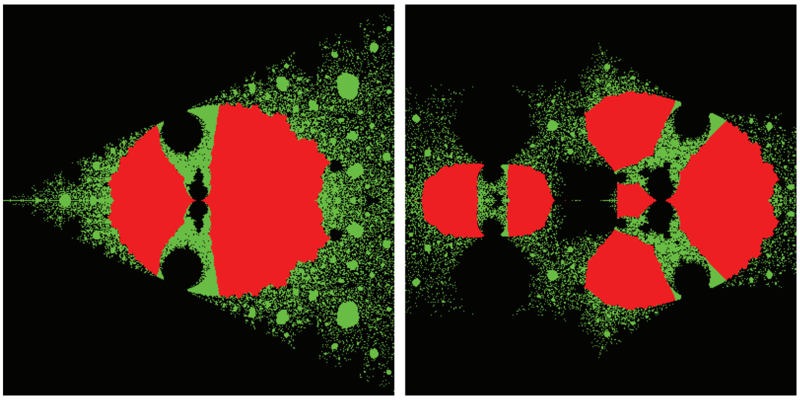

FIGURE 7. The red approximates those type 2 maps for which the line segment between the free critical point and its attracting periodic point is entirely contained within the immediate attracting basin. V_3 is on the left, while V_4 is on the right. Compare with Figs. 4 and 5.

where d is the degree of C and the k_i's are the multiplicities of all the infinitely near points to the singularities of C (see e.g. [11]).

Geometrically, blowing-up a point consists of replacing the point by a line of tangent directions. Algebraically, the blowup of a point in affine space $\mathbb{A}^n = \{(a_1, a_2, \dots, a_n) | a_i \in \mathbb{C}\}$ is described as follows:

After suitable change of coordinates one may arrange that the point to blowup is the origin $(0, 0, \dots, 0) \in \mathbb{A}^n$. In this case, setting

$$B = \{(a_1, a_2, \dots, a_n; x_1 : x_2 : \dots : x_n) | a_i x_j = a_j x_i \text{ for } i, j = 1, \dots, n\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1},$$

the birational map $\pi : B \to \mathbb{A}^n$ given by

$$\pi(a_1, a_2, \dots, a_n; x_1 : x_2 : \dots : x_n) = (a_1, a_2, \dots, a_n),$$

is called the blowup of $\mathbb{A}^n$ centered at the origin.

For more information on plane curves, singularities, and blowups the reader may refer to a resource such as [2] or [11].

For $n \geq 3$, V_n is contained in a complex plane algebraic curve defined by

$$f^{n-2}(1) = 0,$$

where

$$f(z) = 1 + \frac{b}{z} + \frac{c}{z^2}.$$

V_5 is defined by

$$f^3(1) = 0,$$

which simplifies to

$$\frac{P_5}{P_4^2} = 0,$$

where P_4 is the polynomial defining V_4 (see above) and $P_5 \in \mathbb{C}[b,c]$ is a degree 5 polynomial. The homogenization of P_5 is

$$\begin{aligned} H(a,b,c) = {} & a^5 + 7a^4b + 18a^3b^2 + 21a^2b^3 + 11ab^4 + 2b^5 + \\ & 7a^4c + 33a^3bc + 53a^2b^2c + 35ab^3c + 8b^4c + \\ & 15a^3c^2 + 44a^2bc^2 + 42ab^2c^2 + 13b^3c^2 + \\ & 12a^2c^3 + 23abc^3 + 11b^2c^3 + \\ & 5ac^4 + 5bc^4 + c^5, \end{aligned}$$

(i.e. $P_5 = H(1,b,c)$). In what follows we shall refer to the complex projective plane algebraic curve $H = 0$ simply as H. In the projective coordinates $[a : b : c]$, the singularities of H are at

$$p = [0 : 1 : -1] \quad \text{and} \quad q = [-1 : 1 : 0].$$

The infinitely near points to p and q are described by the following 2 lemmas.

Lemma 1. *The infinitely near points to p have multiplicities 3, 1, 1, and 1.*

Proof. After changing coordinates, we may assume that p occurs at the origin $(a,c) = (0,0)$ on the affine patch $\mathbb{A}^2_b = \{b = 1\} \subset \mathbb{P}^2$. In this case, the local equation for H is

$$a^2c + 3ac^2 + c^3 = -(a^5 + 3a^3c + 7a^4c + 8a^2c^2 + 15a^3c^2 + 3ac^3 + 12a^2c^3 + 5ac^4 + c^5).$$

Since $a^2c + 3ac^2 + c^3 = c(a + \frac{3+\sqrt{5}}{2}c)(a + \frac{3-\sqrt{5}}{2}c)$, H has three distinct tangents at p. □

Lemma 2. *The infinitely near points to q have multiplicities 2, 2, 1, and 1.*

Proof. We change coordinates so that q occurs at the origin $(a,c) = (0,0)$ in $\mathbb{A}^2_b$. Then the local equation for H is

$$a^2 = g(a,c), \text{ where}$$

$$\begin{aligned} g(a,c) = {} & 2a^4 + a^5 - 4a^2c + 5a^3c + 7a^4c - \\ & ac^2 - a^2c^2 + 15a^3c^2 - ac^3 + 12a^2c^3 + 5ac^4 + c^5. \end{aligned}$$

Hence q is a cusp of multiplicity 2. We will resolve q by blowing up.

Let $\pi : B \to \mathbb{A}^2_b$ be the blowup of $\mathbb{A}^2_b$ at the origin. We can write B explicitly as

$$B = \{(a,c; x : y) | ay = cx\} \subset \mathbb{A}^2_b \times \mathbb{P}^1,$$

where $[x : y]$ are the projective coordinates on $\mathbb{P}^1$. Then H is birationally equivalent to the projective closure of

$$\begin{aligned} V &= \pi^{-1}(\{a^2 = g(a,c)\} - \{(0,0)\}) \cap (\mathbb{A}^2_b \times \mathbb{P}^1) \\ &= \{(a,c; x : y) | a^2 = g(a,c), ay = cx, (a,c) \neq (0,0)\}. \end{aligned}$$

If we set $\infty = (0,0;1:0)$, then using coordinates for the affine patch $\mathbb{A}_b^2 \times \mathbb{A}_y^1$ we have

$$\begin{aligned}
V = &\ \{(a,c,x) | a^2 = g(a,c), a = cx, c \neq 0\} \cup \infty \\
= &\ \{(a,c,x) | (cx)^2 = g(cx,c), a = cx, c \neq 0\} \cup \infty \\
= &\ \{(a,c,x) | c^2x^2 = c^2(c^3 - cx - c^2x + 5c^3x - 4cx^2 - c^2x^2 + 12c^3x^2 + \\
&\qquad 5c^2x^3 + 15c^3x^3 + 2c^2x^4 + 7c^3x^4 + c^3x^5), a = cx, c \neq 0\} \cup \infty \\
= &\ \{(a,c,x) | x(x+c) = c^3 - c^2x + 5c^3x - 4cx^2 - c^2x^2 + 12c^3x^2 + \\
&\qquad 5c^2x^3 + 15c^3x^3 + 2c^2x^4 + 7c^3x^4 + c^3x^5, a = cx, c \neq 0\} \cup \infty.
\end{aligned}$$

This curve in $\mathbb{A}^3$ has distinct tangent lines $\{a = 0, x = 0\}$ and $\{a = 0, x+c = 0\}$ at $(0,0,0)$; therefore, q will be resolved after one more blowup. $\square$

Proposition. *H is irreducible, and hence $\overline{V_5} = H$.*

Proof. If $H = C_1 \cup C_2$, then $C_1 \cap C_2$ will be singular points of H. By Bezout's theorem, $C_1 \cap C_2$ consists of $\deg C_1 \times \deg C_2$ points. Counting multiplicities, H has 5 singularities and so $\deg C_1 \times \deg C_2 \leq 5$. Thus $\deg C_1 + \deg C_2 = \deg H = 5$ implies that the degrees of C_1 and C_2 must be 1 and 4. Let us suppose C_1 is a line and C_2 is a quartic. Since $C_1 \cap C_2$ consists of 4 points (counting multiplicities) and H has two singularities, C_1 must be tangent to C_2 at one of the singularities of H. From the proof of Lemma 1, the singularity p has 3 distinct tangents and hence C_1 is not tangent to C_2 at p. Thus C_1 must be tangent to C_2 at the double point q. Now, Bezout's theorem implies C_1 must intersect C_2 two more times. Since C_1 is not tangent at p, it must meet C_2 at a point other than p or q. This is contrary to H having only two distinct singular points. $\square$

Combining Lemmas 1 and 2 and applying the genus formula given above, we get:

Theorem. *The geometric genus of V_5 is 1.*

5. Generating the Computer Images

The images in this paper were generated by a Java applet (see Fig. 8) programmed by the authors. Our zoomable fractal generator may be used to graphically explore V_1, V_2, V_3, and V_4 (and other spaces of quadratic maps) in a much more detailed manner than given in this paper. The applet is freely available for use and/or download at either authors' websites (see URLs below). The source code for the applet as well as full screen shots of the images in this paper (including input data) are available on the second author's website.

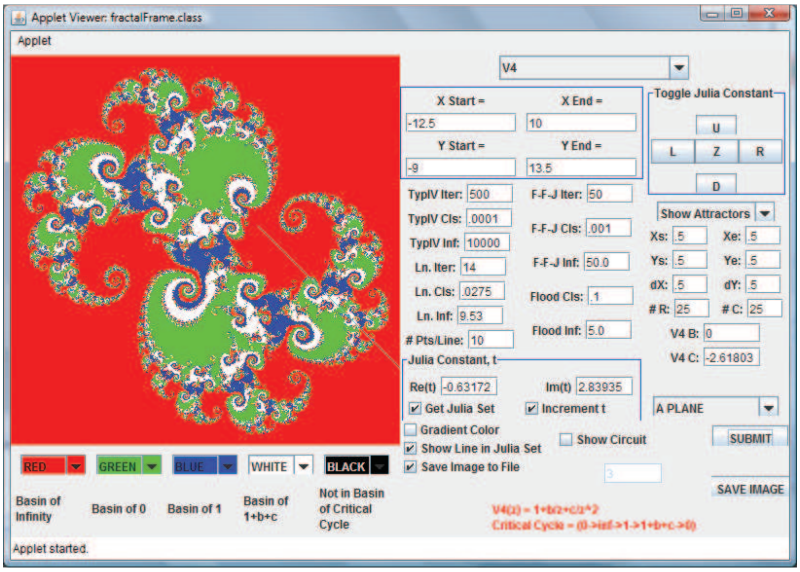

FIGURE 8. Screenshot: The basins of attraction for a type 2 map f in V_4. The free critical point is attracted to ∞ under iteration of f^4. The line segment fails to connect the free critical point with ∞.

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Division of Mathematics and Computer Science,
University of Maine at Farmington,
Farmington, ME 04938
E-mail address: `dustin.gage@maine.edu`
URL: `http://students.umf.maine.edu/dustin.gage/public.www/`

Division of Mathematics and Computer Science,
University of Maine at Farmington,
Farmington, ME 04938
E-mail address: `daniel.jackson1@maine.edu`
URL: `http://faculty.umf.maine.edu/daniel.jackson1/public.www/`