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ON THE FAMILY OF DIOPHANTINE TRIPLES $\{k+2, 4k, 9k+6\}$

ALAN FILIPIN AND ALAIN TOGBÉ

Abstract. In this paper, we prove that if k and d are two positive integers such that the product of any two distinct elements of the set

$$\{k+2, 4k, 9k+6, d\}$$

increased by 4 is a perfect square, then $d = 36k^3 + 96k^2 + 76k + 16$.

1. Introduction

Let m, n be two integers, $m > 1$. A set of m positive integers $\{a_1, \ldots, a_m\}$ is called a Diophantine m-tuple with the property $D(n)$ or a $D(n)$-m-tuple (or a P_n-set of size m), if $a_i a_j + n$ is a perfect square for all $1 \leq i \leq j \leq m$. Diophantus was the first who considered the problem of finding such sets and it was in the case $n = 1$. Particularly he found the set of four positive rational numbers $\left\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\right\}$ with the property $D(1)$. However, the first $D(1)$-quadruple was found by Fermat and it was the set $\{1, 3, 8, 120\}$. Moreover Baker and Davenport [1] proved that the set $\{1, 3, 8, 120\}$ cannot be extended to a $D(1)$-quintuple. Several results of the generalization of the result of Baker and Davenport are obtained. In 1997, Dujella [3] proved that $D(1)$-triples of the form $\{k-1, k+1, 4k\}$, for $k \geq 2$, cannot be extended to a $D(1)$-quintuple. In 1998, Dujella and Pethő [5] proved that the $D(1)$-pair $\{1, 3\}$ cannot be extended to a $D(1)$-quintuple. In 2008, Fujita obtained a more general result by proving that the $D(1)$-pairs of the form $\{k-1, k+1\}$, for $k \geq 2$ cannot be extended to a $D(1)$-quintuple. A folklore conjecture is that there does not exist a Diophantine $D(1)$-quintuple. Recently, Dujella [4] proved that there does not exist a $D(1)$-sextuple and that there are only finitely many $D(1)$-quintuples.

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Case $n = 4$ is closely connected to the case $n = 1$. It is easy to see that if we have $D(4)$-m-tuple with even elements, when we divide those elements by 2, we get a $D(1)$-m-tuple. In 2005, Dujella and Ramasamy [6, Conjecture 1] conjectured that there does not exist a $D(4)$-quintuple. Actually there is a stronger version of this conjecture.

Conjecture 1.1. There does not exist a $D(4)$-quintuple. Moreover, if $\{a, b, c, d\}$ is a $D(4)$-quadruple such that $a < b < c < d$, then

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s, t are positive integers defined by

$$ab + 4 = r^2, ac + 4 = s^2, bc + 4 = t^2.$$

If we denote $d_+ = a + b + c + \frac{1}{2}(abc + rst)$, then $\{a, b, c, d_+\}$ is a $D(4)$-quadruple called a regular $D(4)$-quadruple. We also define number $d_- = a + b + c - \frac{1}{2}(abc + rst)$. If $d_- \neq 0$, then $\{a, b, c, d_-\}$ is also a $D(4)$-quadruple, but $d_- < c$.

The first result of nonextendibility of $D(4)$-m-tuples was obtained by Mohanty and Ramasamy in [17]. They proved that $D(4)$-quadruple $\{1, 5, 12, 96\}$ cannot be extended to a $D(4)$-quintuple. Later Kedlaya [16] proved that if $\{1, 5, 12, d\}$ is a $D(4)$-quadruple, then $d = 96$. One generalization of this result was given by Dujella and Ramasamy [6]. They proved Conjecture 1.1 for a parametric family of $D(4)$-quadruples. Precisely, they proved that if k and d are positive integers and $\{F_{2k}, 5F_{2k}4F_{2k} + 2, d\}$ is a $D(4)$-quadruple, then $d = 4L_{2k}F_{4k} + 2$ where F_k and L_k are the Fibonacci and Lucas numbers. A second generalization was given by Fujita in [12]. He proved that if $k \geq 3$ is an integer and $\{k - 2, k + 2, 4k, d\}$ is a $D(4)$-quadruple, then $d = k^3 - 4k$. All these results support Conjecture 1.1. The first author have studied the size of a $D(4)$-m-tuple. He proved that there does not exist a $D(4)$-sextuple and that there are only finitely many $D(4)$-quintuples exist (see [9, 8, 10, 7]).

The aim of this paper is to consider $D(4)$-triples of the form $\{k+2, 4k, 9k+6\}$, giving another polynomial parametric family of $D(4)$-triples (see the example of Fujita [12]) and to prove the following result.

Theorem 1. *If k is a positive integer and d is a positive integer such that the product of any two distinct elements of the set*

$$\{k + 2, 4k, 9k + 6, d\}$$

increased by 4 is a perfect square, then $d = 36k^3 + 96k^2 + 76k + 16$.

Therefore, one can easily see that our quadruple $\{k+2, 4k, 9k+6, d\}$ with $d = 36k^3+96k^2+76k+16$ is regular. This family is the special case of twoparametric family of $D(4)$ Diophantine triples $\{k, A^2 - 4A, (A + 1)^2k - 4(A + 1)\}$, which can be studied on the same way for small A's. The organization of this paper is as follows. In Section 2, we recall some useful results obtained by the first author and adapt them to our case. We use a result due to Bennett [2] on simultaneous approximations of algebraic numbers which are close to 1 to get an upper bound for k. Finally, in Section 4, we use Baker method and Baker-Davenport reduction method to prove Theorem 1. The method, applied in this paper, was used by the second author and Bo He in [15]. The results obtained in this paper are sightly different from those in [15] as the family considered by He-Togbé is a family of $D(1)$ Diophantine triples. But let us mention that their result proves our main theorem for even k. This method can be applied to many Diophantine sets. With Bo He, we applied it to study a family of two-parametric $D(4)$ Diophantine triples, $\{k, A^2k+4A, (A+1)^2k+4(A+1)\}$ for small values of the parameter A (see [11]). In the same way, the second author and Bo He studied a family of two-parametric $D(1)$ Diophantine triples (see [13] and [14]). These families are simplest if we consider them as linear polynomials in k. Notice that the success of the method depends on the bound of k obtained in Section 3. If this bound is too large, it would take too long to run a program like we have done in Section 4.

2. Some useful lemmas

In this section, we will recall or prove some useful lemmas that will be used to prove Theorem 1. So let r, s, t be positive integers defined by

$$ab+4=r^2, \quad ac+4=s^2, \quad bc+4=t^2. \tag{1}$$

To extend the Diophantine triple $\{a,b,c\}$ to a Diophantine $D(4)$-quadruple $\{a,b,c,d\}$, we have to solve the system

$$ad+4=x^2, \quad bd+4=y^2, \quad cd+4=z^2. \tag{2}$$

If we eliminate d, we obtain the following system of Pellian equations.

$$az^2-cx^2=4(a-c), \tag{3}$$

$$bz^2-cy^2=4(b-c). \tag{4}$$

From [8, Lemma 1], there exists a solution $\left(z_0^{(i)}, x_0^{(i)}\right)$ of (3) such that $z=v_m^{(i)}$, where

$$v_0^{(i)}=z_0^{(i)}, \quad v_1^{(i)}=\frac{1}{2}(sz_0^{(i)}+cx_0^{(i)}), \quad v_{m+2}^{(i)}=sv_{m+1}^{(i)}-v_m^{(i)},$$

and $|z_0^{(i)}|<\sqrt{\frac{c\sqrt{c}}{\sqrt{a}}}$. Similarly, there exists a solution $\left(z_1^{(i)}, y_1^{(i)}\right)$ of (4) such that $z=w_n^{(j)}$, where

$$w_0^{(i)}=z_1^{(j)}, \quad w_1^{(j)}=\frac{1}{2}(tz_1^{(j)}+cy_1^{(j)}), \quad w_{n+2}^{(j)}=tw_{n+1}^{(j)}-w_n^{(j)},$$

and $|z_1^{(j)}|<\sqrt{\frac{c\sqrt{c}}{\sqrt{b}}}$.

The initial terms $z_0^{(i)}$ and $z_1^{(j)}$ are almost completely determined in the following lemma [10, Lemma 9].

Lemma 1. (1) *If the equation* $v_{2m} = w_{2n}$ *has a solution, then* $z_0 = z_1$. *Moreover,* $|z_0| = 2$, *or* $|z_0| = \frac{1}{2}(cr - st)$, *or* $|z_0| < 1.608a^{-\frac{5}{14}}c^{\frac{9}{14}}$.

(2) *If the equation* $v_{2m+1} = w_{2n}$ *has a solution, then*

$$|z_0| = t,\ |z_1| = \frac{1}{2}(cr - st),\ z_0 z_1 < 0.$$

(3) *If the equation* $v_{2m} = w_{2n+1}$ *has a solution, then*

$$|z_1| = s,\ |z_0| = \frac{1}{2}(cr - st),\ z_0 z_1 < 0.$$

(4) *If the equation* $v_{2m+1} = w_{2n+1}$ *has a solution, then* $|z_0| = t$, $|z_1| = s$, $z_0 \cdot z_1 > 0$.

Here, we consider

$$a = k + 2, \quad b = 4k, \quad c = 9k + 6,$$

and

$$r = 2k + 2, \quad s = 3k + 4, \quad t = 6k + 2.$$

We have

$$|z_0| < \sqrt{\frac{c\sqrt{c}}{\sqrt{a}}} < \sqrt{3(9k + 6)} \leq 6.71\sqrt{k}$$

and

$$|z_1| < \sqrt{\frac{c\sqrt{c}}{\sqrt{b}}} < \sqrt{\sqrt{3.75} \cdot (9k + 6)} \leq 5.39\sqrt{k}.$$

Using the same procedure as in [15], we can exclude all items of Lemma 1 except the first item with $x_0 = 2$, $y_0 = 2$, and $|z_0| = (cr - st)/2 = 2$. Therefore, we need to solve the system of Pell equations

$$(k + 2)z^2 - (9k + 6)x^2 = 4(-8k - 4), \tag{5}$$

$$4kz^2 - (9k + 6)y^2 = 4(-5k - 6), \tag{6}$$

with $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. This is equivalent to solve the equation

$$z = v_{2m} = w_{2n}. \tag{7}$$

Let $z_0 = z_1 = \pm 2$. We have

$$v_0 = \pm 2,\ v_1 = 9k + 6 \pm (3k + 4),\ v_{m+2} = (3k + 4)v_{m+1} - v_m, \tag{8}$$

$$w_0 = \pm 2,\ w_1 = 9k + 6 \pm (6k + 2),\ w_{n+2} = (6k + 2)w_{n+1} - w_n. \tag{9}$$

For the relations of indices m and n, we have

Lemma 2. *If* $v_{2m} = w_{2n}$, *then* $n \leq m \leq 2n$.

Proof. By [10, Lemma 5], if $v_m = w_n$, then $n - 1 \leq m \leq 2n + 1$. In our even case, we have $2n - 1 \leq 2m \leq 4n + 1$. The result is obtained. □

Now let us recall the following lemma.

Lemma 3. *We have*

$$v_{2m} \equiv z_0 + \frac{1}{2}c(az_0m^2 + sx_0m) \pmod{c^2},$$
$$w_{2n} \equiv z_1 + \frac{1}{2}c(bz_1n^2 + ty_1n) \pmod{c^2}.$$

Proof. See [6, Lemma 3]. □

The next result will help us to determine a lower bound of m depending on k.

Lemma 4. *Assume that $v_{2m} = w_{2n}$ with $m, n \geq 2$, then*

$$m \geq \frac{1}{2}\sqrt{k} - 1.$$

Proof. Using the results of Lemma 3, we have

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{c}. \tag{10}$$

In our case, the congruence becomes

$$\pm(k+2)m^2 + (3k+4)m \equiv \pm 4kn^2 + (6k+2)n \pmod{9k+6}$$

Multiplying the above congruence by ± 9, it results

$$12m^2 \pm 18m \equiv -24n^2 \mp 18n \pmod{9k+6}.$$

Therefore,

$$4m^2 + 8n^2 \pm 6m \pm 6n \equiv 0 \pmod{3k+2}. \tag{11}$$

If $m, n \geq 2$, then

$$4m^2 + 8n^2 + 6m + 6n \geq 4m^2 + 8n^2 - 6m - 6n > 0.$$

Hence from (11) we obtain

$$4m^2 + 8n^2 + 6m + 6n \geq 3k + 2.$$

By Lemma 2, we know that $m \geq n$. So we have $12m^2 + 12m \geq 3k + 2 > 3k$. This completes the proof of the lemma. □

Now we prove the following result.

Lemma 5. *Let x, y, z be positive integer solutions of the system of Pellian equations* (5) *and* (6) *such that $z \notin \{2, 18k^2 + 30k + 10\}$. Then*

$$\log(2z) > (\sqrt{k} - 2)\log(3k).$$

Proof. The proof is similar to that of Lemma 5 in [15] but here we take $z_0 = \pm 2, x_0 = 2$ and $v_{2m} = w_{2n}$. □

3. An application of Diophantine approximation

In this section, we will use a result of Bennett [2] on simultaneous approximations of algebraic numbers which are close to 1 to get a lower bound for k in order to solve the system of Pell equations (3) and (4). So let us consider the numbers

$$\theta_1 = \sqrt{1 + \frac{4}{3k+2}} \quad \text{and} \quad \theta_2 = \sqrt{1 - \frac{2}{3k+2}}.$$

The following result gives us some information on the approximations of the algebraic numbers θ_1 and θ_2.

Lemma 6. *If $k \geq 5$ and $(x, y, z) \neq (2, 2, 2)$, then*

$$\left|\theta_1 - \frac{6x}{2z}\right| < 16z^{-2} \quad \textit{and} \quad \left|\theta_2 - \frac{3y}{2z}\right| < 3z^{-2}.$$

Therefore we obtain

$$\max\left\{\left|\theta_1 - \frac{6x}{2z}\right|, \left|\theta_2 - \frac{3y}{2z}\right|\right\} < 16z^{-2}.$$

Proof. The lemma can be proved exactly like Lemma 7 in [15]. □

We will use the above lemma to prove the following lemma that gives us the information on d.

Proposition 3.1. *If $k \geq 881906$ and if the set $\{k+2, 4k, 9k+6, d\}$ is a Diophantine quadruple, then $d = 36k^3 + 96k^2 + 76k + 16$.*

Proof. If d satisfies the condition, then $z^2 = cd + 4 = (9k+6)d + 4$. As $d > 1$, we have $z \neq 2$. If $d \neq 36k^3 + 96k^2 + 76k + 16$, thus we have $z \neq 18k^2 + 30k + 10$. Therefore, Lemma 5 implies

$$\log(2z) > (\sqrt{k} - 2)\log(3k). \tag{12}$$

Now we apply [2, Lemma 3.2]. We take $a_0 = -2$, $a_1 = 0$, $a_2 = 4$, $M = 4$, $q = 2z$, $p_0 = 3y$, $p_1 = q$, $p_2 = 6x$ and $N = 3k+2$. When $k \geq 881906$, the condition $N > M^9$ holds. Therefore, by [2, Lemma 3.2] and Lemma 6, we have

$$16z^{-2} > (130(3k+2)\gamma)^{-1}(2z)^{-\mu}, \tag{13}$$

where $\gamma = 57.6$ and

$$\begin{aligned}\mu = 1 + \frac{\log(32.04 \cdot 57.6 \cdot (3k+2))}{\log(1.68 \cdot (3k+2)^2/2304)} &< 1 + \frac{\log(3k+2) + 7.521}{2\log(3k+2) - 7.223} \\ &< 1.5 + \frac{11.132}{2\log(3k+2) - 7.223}.\end{aligned}$$

If $k > 881906$, then we have $\mu < 2$. In fact, we get

$$0.5 - \frac{11.132}{2\log(3k+2) - 7.223} < 2 - \mu.$$

From (13) we have

$$(2z)^{2-\mu} < 16 \cdot 130 \cdot 57.6(3k+2).$$

We deduce that

$$\log(2z) < \log(479232(3k+2))/(2-\mu). \tag{14}$$

A combination of (12) and (14) gives

$$\sqrt{k} - 2 < \frac{\log(479232(3k+2))}{(2-\mu)\log(3k)} < \frac{\log(479232(3k+2))}{\left(0.5 - \frac{11.132}{2\log(3k+2)-7.223}\right)\log(3k)}. \tag{15}$$

If $k \geq 881906$, we get a contradiction. □

4. Proof of Theorem 1

This section is devoted to the remaining cases, i.e. $1 \leq k \leq 881906$. A theorem of lower bounds to linear forms in logarithms to helps get an upper bound for m. In fact, let

$$\alpha_1 = \frac{s+\sqrt{ac}}{2} \quad \text{and} \quad \alpha_2 = \frac{t+\sqrt{bc}}{2}.$$

Solving equations (3) and (4), we have

$$v_{2m} = \frac{1}{2\sqrt{a}}\left((z_0\sqrt{a}+x_0\sqrt{c})\alpha_1^{2m} + (z_0\sqrt{a}+x_0\sqrt{c})\alpha_1^{-2m}\right)$$

and

$$w_{2n} = \frac{1}{2\sqrt{b}}\left((z_1\sqrt{b}+y_1\sqrt{c})\alpha_2^{2n} + (z_1\sqrt{b}+y_1\sqrt{c})\alpha_2^{-2n}\right)$$

respectively. Notice $x_0 = y_1 = 2$ and $z_0 = z_1 = \pm 2$. Solving equations (3) and (4) is equivalent to solve $z = v_{2m} = w_{2n}$ with $m, n \neq 0$. So we have (see [10, Lemma 10])

$$0 < \Lambda := 2m\log\alpha_1 - 2n\log\alpha_2 + \log\alpha_3 < 2ac\alpha_1^{-4m}, \tag{16}$$

where

$$\alpha_3 = \frac{\sqrt{b}(\sqrt{c}\pm\sqrt{a})}{\sqrt{a}(\sqrt{c}\pm\sqrt{b})}.$$

Therefore, we get

$$\log|\Lambda| < -4m\log\alpha_1 + \log(2ac) < (2-4m)\log(3k+5). \tag{17}$$

In [8], using Baker's method the first author proved that

$$\frac{2m}{\log(2m+1)} < 6.543 \cdot 10^{15}\log^2 c.$$

As $1 \leq k \leq 881906$, we obtain

$$\frac{2m}{\log(2m+1)} < 1.648 \cdot 10^{18}.$$

This implies $m < 3.8 \cdot 10^{19}$.

To solve the problem for the remaining cases $1 \leq k \leq 881906$, we will use a Diophantine approximation algorithm, so-called the Baker-Davenport reduction method. Lemma [5, Lemma 5a] or [15, Lemma 9] is a slight modification of the original version of Baker-Davenport reduction method. We will apply it with

$$\kappa = \frac{\log \alpha_1}{\log \alpha_2}, \quad \mu' = \frac{\log \alpha_3}{2 \log \alpha_2}, \quad A = \frac{(k+2)(9k+6)}{\log \alpha_2}, \quad B = \alpha_1^4$$

and $M = 3.8 \cdot 10^{19}$.

In [8], the first author has used Baker-Davenport reduction method for $ab^2c < 10^7$, which covered all cases when $0.022 d_+^{4.5} b^{3.5} < 10^{26}$. In our case, it corresponds to $1 \leq k \leq 15$. Therefore, we wrote a program in Mathematica that we ran for $16 \leq k \leq 881906$. In fact, if $z_0 = z_1 = 2$, then for $16 \leq k \leq 47$, we got $m \leq 3$ after the first step of reduction. We ran again the program by taking $M = 3$ and we obtained $m \leq 1$. For $k \geq 48$, we got $m \leq 2$ after the first step of reduction. On the other hand, if $z_0 = z_1 = -2$, the results are similar with $16 \leq k \leq 43$. In the same way, for $k \geq 44$, we got $m \leq 2$ after the first step of reduction.

If $m = 2$, then by Lemma 2 we have $n \leq 2$ and from Lemma 4 we get $k \leq 36$. It is easy to check there is no integer k for which equation (7) is verified. Then we consider $m = 1$ ($m = n = 0$ gives the trivial solution $d = 0$).

Again by Lemma 2, we have $m = n = 1$. When $v_0 = w_0 = -2$, we obtain $v_2 = w_2 = 18k^2 + 30k + 10$. Then we deduce $d = 36k^3 + 96k^2 + 76k + 16$. This completes the proof of Theorem 1.

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Faculty of Civil Engineering
University of Zagreb
Fra Andrije Kačića-Miošića 26
10000 Zagreb, Croatia
E-mail address: filipin@grad.hr

Mathematics Department,
Purdue University North Central,
1401 S, U.S. 421, Westville IN 46391 USA
E-mail address: atogbe@pnc.edu