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# SUFFICIENT CONDITIONS FOR OSCILLATORY BEHAVIOUR OF A FIRST ORDER NEUTRAL DIFFERENCE EQUATION WITH OSCILLATING COEFFICIENTS

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Abstract. In this paper, we obtain sufficient conditions so that every solution of neutral functional difference equation

$$\Delta(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n$$

oscillates or tends to zero as $n \to \infty$. Here $\Delta$ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, and $p_n, q_n, f_n$ are the terms of oscillating infinite sequences; $\{\tau_n\}$ and $\{\sigma_n\}$ are non-decreasing sequences, which are less than $n$ and approaches $\infty$ as $n$ approaches $\infty$. This paper generalizes and improves some recent results.

## 1. Introduction

In this work, we find sufficient conditions, so that every solution of neutral functional difference equation

$$\Delta(y_n - p_n y_{\tau(n)}) + q_n G(y_{\sigma(n)}) = f_n \tag{1}$$

oscillates or tends to zero as $n \to \infty$, where $\Delta$ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n$, $\{p_n\}, \{q_n\}$ and $\{f_n\}$ are infinite sequences of real numbers (not necessarily positive) and $G \in C(R, R), \tau(n)$ and $\sigma(n)$ are non-decreasing sequences, which are less than $n$ and approaches $\infty$ as $n$ approaches $\infty$.

Let $n_0$ be a fixed nonnegative integer. Let $\rho = \min\{\tau(n_0), \sigma(n_0)\}$. By a solution of (1) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq \rho$ and satisfies (1) for $n \geq n_0$. Clearly if the initial condition

$$y_n = a_n \quad \text{for} \quad \rho \leq n \leq n_0 \tag{2}$$

is given then the equation (1) has a unique solution satisfying the given initial condition (2). A solution $\{y_n\}$ of (1) is said to be oscillatory if for every

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positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

If we put $\tau(n) = n - m$ and $\sigma(n) = n - k$, where $m, k$ are non-negative integers then (1) reduces to

$$\Delta(y_n - p_n y_{n-m}) + q_n G(y_{n-k}) = f_n. \tag{3}$$

Further, if we put $p_n = 0$ in (3), then we obtain the delay difference equation

$$\Delta(y_n) + q_n G(y_{n-k}) = f_n. \tag{4}$$

Hence (1) is more general than (3) and (4).

Recently the oscillatory and asymptotic behavior of (3) and (4) have been investigated by many authors (see [13]–[23], [25]) when $q_n$ is non-negative. However, the general case, when $q_n$ is allowed to oscillate, it is difficult to study the oscillation of (3) or (4), since the difference $\Delta(y_n - p_n y_{n-m} - \sum_{i=n_0}^{n-1} f_i)$ of any non-oscillatory solution of (3) is always oscillatory. Therefore, the results on oscillation of (4), (3), with oscillating $q_n$, are relatively scarce; see [18, 24, 22]. The motivation of this work is because of the interesting open problem for the above case in [9].

The open problem 7.11.3 of [9, pp197] reads as:

*Problem* 1.1. Extend the following result to difference equation with oscillating coefficients $q_n$.

**Theorem 1.2** ([9, Theorem 7.5.1])**.** *Suppose that $\{q_n\}$ is a non negative sequence of real numbers and $k$ be a positive integer. Then*

$$\liminf_{n\to\infty} \left[\sum_{i=n-k}^{n-1} q_i\right] > \left(\frac{k}{k+1}\right)^{k+1} \tag{5}$$

*is a sufficient condition for every solution of*

$$y_{n+1} - y_n + q_n y_{n-k} = 0 \tag{6}$$

*to be oscillatory.*

Note that if $k = 2$ and $q_n = \left(\frac{1}{e^2} - \frac{1}{e^3}\right)$ then the difference equation (6) admits a positive solution $y_n = e^{-n}$, which approaches zero as $n \to \infty$. In this case (5) does not hold. However,

$$\sum_{n=n_0}^{\infty} q_n = \infty \tag{7}$$

holds and (5) implies (7).

In view of this, we deal in this paper, with the problem which is slightly different from the Problem 1.1. In fact, our main result would be the following theorem, where sufficient conditions are obtained so that every solution of (6) oscillates or tends to zero as $n \to \infty$.

**Theorem 1.3.** *Suppose that $q_n$ changes sign and satisfies the condition*

$$\sum_{n=0}^{\infty} q_n^- < \infty \quad \textit{where} \quad q_n^- = \max(-q_n, 0).$$

*Then the condition*

$$\sum_{n=n_0}^{\infty} |q_n| = \infty \tag{8}$$

*is sufficient for every solution of* (6) *to oscillate or to tend to zero as $n \to \infty$.*

## 2. Main Results

We need the following hypothesis to be used in this article.

(H1) There exists integers $n_0 > 0$,$r \le 0$ and $R \ge 0$ such that $R - r < 1$ and $r \le p_n \le R$ for $n \ge n_0$.
(H2) $G$ is bounded.
(H3) $xG(x) > 0$ for $x \ne 0$.
(H4) $\sum_{n=0}^{\infty} q_n^+ = \infty$, where $q_n^+ = \max(q_n, 0)$.
(H5) $\sum_{n=0}^{\infty} q_n^- < \infty$, where $q_n^- = \max(-q_n, 0)$.
(H6) $\sum_{n=n_0}^{\infty} |f_n| < \infty$.

As a proto type of an infinite sequence satisfying (H4) and (H5), we have

$$q_n = \frac{n|\sin(n\pi/2)| - |\sin((n-1)\pi/2)|}{n^2} \tag{9}$$

Hence, for $n = 1, 2, 3, \dots$ we obtain

$$q_{2n-1}^+ = 1/(2n-1), q_{2n}^+ = 0, q_{2n-1}^- = 0 \text{ and } q_{2n}^- = 1/(2n)^2.$$

A proto type of a function satisfying (H2)-(H3) is $G(u) = ue^{-u^2}$ or $G(u) = \frac{u}{u^2+1}$, which are monotonic non-increasing.

¿From the definitions of $q_n^+$ and $q_n^-$, it follows that $q_n^+ \ge 0$, $q_n^- \ge 0$ and $q_n = q_n^+ - q_n^-$. Then using this decomposition, (1) can be rewritten as

$$\Delta(y_n - p_n y_{\tau(n)}) + q_n^+ G(y_{\sigma(n)}) - q_n^- G(y_{\sigma(n)}) = f_n \tag{10}$$

Now, we present our first result.

**Theorem 2.1.** *Suppose (H1)–(H6) hold. Then every solution of* (1) *oscillates or tends to zero as $n$ tends to $\infty$.*

*Proof.* Let $\{y_n\}$ be any solution of (1). If it oscillates, then we have nothing to prove. If it does not oscillate then assume that $\{y_n\}$ be an eventually positive solution of (1) for $n \ge n_0$. Our intention is to prove that $y_n$ approaches zero as $n \to \infty$. If necessary increment $n_0$ so that

$$y_{\tau(n)} > 0, y_{\sigma(n)} > 0 \quad \text{for } n > n_0. \tag{11}$$

For simplicity of notation, define

$$z_n = y_n - p_n y_{\tau(n)}. \tag{12}$$

Using (H2) and (H5), define

$$T_n = \sum_{i=n}^{\infty} q_i^- G(y_{\sigma(i)}), \tag{13}$$

and

$$w_n = y_n - p_n y_{\tau(n)} + T_n - \sum_{i=n_0}^{n-1} f_i. \tag{14}$$

It may be noted that $T_n > 0$, when $y_n > 0$ and $T_n < 0$, when $y_n < 0$. From (10), using Eqs(12)– Eq(14) for $n > N_1 > N_0$ we obtain

$$\Delta w_n = -q_n^+ G(y_{\sigma(n)}) \leq 0. \tag{15}$$

Hence, $\{w_n\}$ is non-increasing, implying $w_n > 0$ or $w_n < 0$ for large $n$. By (H6), the function $\sum_{i=n_0}^{n} f_i$ being bounded, we have

$$w_{n_0} + \sup_{n \geq n_0} \sum_{i=n_0}^{n} f_i \geq y_n - p_n y_{\tau(n)} \geq y_n - R y_{\tau(n)}. \tag{16}$$

Next, we claim that $\{y_n\}$ is bounded. Otherwise, there exists a sequence $\{y_{n_k}\}$ such that $y_{n_k} \to \infty$ as $k \to \infty$ and $y_{n_k} = \max(y_n : n_0 \leq n \leq n_k)$. Note that $y_{\tau(n_k)} \to \infty$ as $k \to \infty$. Since $\tau(n) \leq n$, from (16) it follows that for each $n_k$

$$w_{n_0} + \sup_{k} \sum_{i=n_0}^{n_k} f_i \geq (1 - R) y_{n_k}.$$

Since $(1 - R) > 0$, the right hand side approaches $\infty$, as $k \to \infty$. This is a contradiction, that proves, $\{y_n\}$ is bounded. Using (H5) and (H6),and noting that, $\{p_n\}$ being bounded, we see that $\{z_n\}$ and $\{w_n\}$ are bounded sequences. Then $\{w_n\}$ must converge, as it is monotonic. By (H5) the sequence $\{T_n\}$ is convergent and tends to zero as $n \to \infty$. This along with(H6), implies $\{z_n\}$ is also convergent. Let

$$l := \lim_{n \to \infty} z_n = \lim_{n \to \infty} w_n. \tag{17}$$

Next, we claim $\liminf_{n \to \infty} y_n = 0$. Otherwise, for $n \geq N_2 > N_1$ we have $y_n > 0$. Then $y_{\sigma(n)} > 0$. From the definition of lim inf, there exists constants $\alpha$ and $N_3 > N_2$ such that $y_{\sigma(n)} \geq \alpha$ for all $n \geq N_3$. Since $\{y_n\}$ is bounded, then we can find a upper bound $\beta$ for $\{y_n\}$. The continuity of $G$ and (H3) imply the existence of a positive lower bound $m$ for $G$ on $[\alpha, \beta]$; i.e., $0 < m < G(y_{\sigma(n)})$ for all $n \geq N_4 \geq N_3$. Then summing (15) from $i = N_4$ to $n-1$, we obtain that

$$w_{N_4} - w_n = \sum_{i=N_4}^{n} q_i^+ G(y_{\sigma(i)}) \geq m \sum_{i=N_4}^{n} q_i^+.$$

Since the left hand side is a member of a bounded sequence, while the right hand side approaches $+\infty$, we have a contradiction. Therefore, $\liminf y_n = 0$. Next, we prove that $\limsup y_n = 0$.

Since $y_n \geq 0$, from assumption (H1), it follows that $y_n - p_n y_{\tau(n)} \geq y_n - R y_{\tau(n)}$. As we know that, for bounded functions,

$$\limsup\{f_n + g_n\} \geq \limsup\{f_n\} + \liminf\{g_n\}.$$

Therefore, by taking $\limsup$ in (12), we obtain that

$$\begin{aligned} l &= \limsup_{n\to\infty}\{y_n - p_n y_{\tau(n)}\} \\ &\geq \limsup_{n\to\infty}\{y_n\} + \liminf_{n\to\infty}\{-R y_{\tau(n)}\} \\ &\geq \limsup_{n\to\infty}\{y_n\} - R \limsup_{n\to\infty}\{y_{\tau(n)}\} \\ &\geq (1-R)\limsup_{n\to\infty}\{y_n\}. \end{aligned} \tag{18}$$

From (H1), it follows that $y_n - p_n y_{\tau(n)} \leq y_n - r y_{\tau(n)}$. Since

$$\liminf\{f_n + g_n\} \leq \liminf\{f_n\} + \limsup\{g_n\},$$

taking $\liminf$ in (12), we obtain

$$\begin{aligned} l &= \liminf_{n\to\infty}\{y_n - r y_{\tau(n)}\} \\ &\leq \liminf_{n\to\infty}\{y_n\} + \limsup_{n\to\infty}\{-r y_{\tau(n)}\} \\ &= 0 - r \limsup_{n\to\infty}\{y_n\}. \end{aligned}$$

From (18) and above inequality, we have

$$(1 - R + r)\limsup_{n\to\infty}\{y_n\} \leq 0.$$

Since $y_n \geq 0$, by (H1), it follows that $\limsup_{n\to\infty} y_n = 0$. Hence, $\lim_{n\to\infty} y_n = 0$. The proof for the case $y_n < 0$ is similar. □

*Remark* 1. It is not difficult to see that if $p_n$ satisfies the condition $0 \leq p_n \leq p < 1$ or $-1 < -p \leq p_n \leq 0$, instead of (H1) then also the above theorem holds. Note that if $q_n \geq 0$, then $q_n^- = 0$ and $q_n^+ = q_n$. Hence, the above theorem improves and generalizes [13, Theorems 2.1 and 2.3], [17, Theorems 2.3 and 2.4], and [20, Corollary 2.5].

Note that, in the above theorem, we assumed that $G$ is bounded. However, equation (6) that is considered in Theorem 1.3, does not satisfy this condition. To address this problem, we introduce the following hypothesis, and state another theorem.

(H7) There exists non-negative constants $a, b$ such that $|G(u)| \leq a|u| + b$, for all $u$.

**Theorem 2.2.** *Assume (H1),(H3)-(H7) hold. Then every solution of* (1) *oscillates or tends to zero as* $n \to \infty$.

*Proof.* As in theorem 2.1, we prove that every non-oscillatory solution converges to zero as $n \to \infty$. Suppose $\{y_n\}$ be an eventually positive solution of (1) for $n \geq n_o$. If necessary increment $n_0$ such that (11) is satisfied, and by (H5),

$$\alpha := (a+b) \sum_{i=n_0}^{\infty} q_i^- < 1 - R. \tag{19}$$

Using that $\tau(n)$ and $\sigma(n)$ are non-decreasing and both tend to $\infty$ as $n \to \infty$, we define $\tau_0 = \tau_{n_0}$ and $\sigma_0 = \sigma_{n_0}$. Select a constant $M$ large enough so that

$$\begin{aligned} &1 \leq M, \\ &|y_n| \leq M \quad \text{ for } \min\{\tau_0, \sigma_0\} \leq n \leq n_0, \\ &\alpha + R \leq \frac{M}{M + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i|}. \end{aligned} \tag{20}$$

By (19), $0 \leq \alpha + R < 1$. Then for $n \leq n_0$

$$0 \leq y_n \leq M + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i| = M_1. \tag{21}$$

Next, we prove $y_n < M_1$ for all $n \geq n_0$, by induction. As per induction hypothesis, assume that (21) holds for all $n < k$. Then summing (10) from $n = n_0$ to $k-1$, we obtain that

$$\begin{aligned} y_k = p_k y_{\tau(k)} + y_{n_0} - p_{n_0} y_{\sigma(n_0)} - \sum_{n=n_0}^{k-1} q_n^+ G(y_{\sigma(n)}) \\ + \sum_{n=n_0}^{k-1} q_n^- G(y_{\sigma(n)}) + \sum_{n=n_0}^{k-1} f_n. \end{aligned} \tag{22}$$

Because $\tau(k) < k$ and $\sigma(k) < k$, we can use (21) to estimate each term in the above expression. Using $p_n \leq R$, we obtain

$$p_k y_{\sigma(k)} \leq R M_1.$$

Applying induction hypothesis,(H7) and the fact that $M_1 > 1$,we obtain for $n \leq k-1$,

$$G(y_{\sigma(n)}) \leq a|y_{\sigma(n)}| + b \leq a M_1 + b \leq (a+b) M_1.$$

Hence,

$$\sum_{n=n_0}^{k-1} q_n^- G(y_{\sigma(n)}) \leq (a+b) M_1 \sum_{n=n_0}^{k-1} q_n^- = M_1 \alpha.$$

Since (20) implies $M_1(R + \alpha) \leq M$, then it follows that

$$y_k \leq M_1(R+\alpha) + y_{n_0} + |p_{n_0} y_{\tau(n_0)}| + \sum_{i=n_0}^{\infty} |f_i| \leq M_1.$$

Thus, by the mathematical principle of induction, $y_n \le M_1$ for $n \ge n_0$. Hence $\{y_n\}$ is bounded. Next, we define $z_n$ and $w_n$ as in Theorem 2.1 and prove $\lim_{n\to\infty} y_n = 0$ by the same method as in the proof of Theorem 2.1. The proof for the case when $y_n$ is eventually negative is similar. □

Since the results in Theorems 2.1 and 2.2 hold for bounded solutions, then we have the following result.

**Theorem 2.3.** *Under assumptions (H1) and (H3)-(H6), every bounded solution of* (1) *oscillates or tends to zero as* $n \to \infty$.

We have the following result in order to answer Theorem 1.3.

**Theorem 2.4.** *Assume that* $q_n$ *changes sign and (H5) holds. If (H4) holds then every solution of* (6) *oscillates or tends to zero as* $n \to \infty$.

*Proof.* The delay equation (6) is a particular case of (1), when $p_n \equiv 0, \sigma(n) = n - k, f_n \equiv 0$. Condition (H2) is not satisfied, but, (H7) is satisfied with $a = 1$ and $b = 0$. Since (H1), (H3)-(H7) are satisfied, we apply Theorem 2.2 and obtain the desired result. □

*Remark* 2. If $q_n$ changes sign then $|q_n| = q_n^+ + q_n^-$. Thus, if (H5) holds then (H4) implies and implied by (8). Hence, Theorem 2.4 proves Theorem 1.3. To emphasize the need of (H5) for Theorem 1.3, we present the following example.

*Example* 1. Consider the delay equation

$$\Delta y_n + q_n y_{n-2} = 0,$$

where

$$q_n = \begin{cases} -1, & \text{if } n \text{ is odd,} \\ \frac{1}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Then, $q_n$ is oscillatory but, does not satisfy (H5). Moreover, $G(u) = u$, does not satisfy (H2). Note that (H1), (H3), (H4), (H6)–(H7) hold, but we cannot apply Theorem 2.1 or Theorem 2.2. In fact,

$$y_n = \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 2, & \text{if } n \text{ is even,} \end{cases}$$

is a solution of the above delay equation which neither oscillates nor tends to zero as $n \to \infty$.

Before we close our article, we present an example to illustrate, one of our main results.

*Example* 2. Consider the NDDE

$$\Delta(y_n) + q_n y_{n-2} = q_n e^{2-n} + e^{-n}(e^{-1} - 1), \tag{23}$$

where $q_n$ is as given by Eq(9). It is easy to verify that Eq(23) satisfies all the conditions of Theorem 2.2. Hence, every solution oscillates or tends to zero as $n \to \infty$. Thus, $y_n = e^{-n}$ is such a solution which $\to 0$ as $n \to \infty$.

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