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FUNDAMENTAL PROBLEMS FOR A WEAKENED INFINITE PLATE BY A CURVILINEAR HOLE IN A HALF-PLANE

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Abstract. Complex variable method (Cauchy integral method) has been applied to derive exact and closed expressions of Goursat functions for the first and second fundamental problems for an infinite plate weakened by a curvilinear hole. The area outside the hole with the hole itself is conformally mapped on the right half-plane by the use of a rational mapping function. This rational mapping consists of complex constants, in order to make the hole take different famous shapes, which can be found throughout the nature.

Many previous works are considered as special cases of this work. Also many new cases can be derived from the problem.

1. Introduction

For many years, contact and mixed problems in the theory of elasticity has been recognized as a rich and challenging subject for study, see Atkin and Fox (1990). These problems can be established from the initial value problems or from the boundary value problems, or from the mixed problems, see Colton and Kress (1983) and Abdou (2003). Also, many different methods are established for solving the contact and mixed problems in elastic and thermoelastic problems, the books edited by Noda et al. (2003), Hetnarski (2004), Parkus (1976) and Popov (1988) contain many different methods to solve the problems in the theory of elasticity in one, two and three dimensions.

It is known that, see Muskhelishvili (1953), the first and second fundamental problems in the plane theory of elasticity are equivalent to finding two analytic functions $\phi_1(z)$ and $\psi_1(z)$ of one complex argument $z = x + iy$. These functions satisfy the boundary conditions,

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$$k\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = f(t), \tag{1.1}$$

where t denotes the affix of a point on the boundary. In terms of $z = c\omega(s)$, $c > 0$, subject to the condition that $\omega(\infty)$ is bounded and $\omega'(s)$ does not vanish on the right half-plane i.e. $\operatorname{Re} s \geq 0$, the infinite region outside a closed contour is conformally mapped to the right half-plane.

For $k = -1$ and $f(t)$ is a given function of stress in (1.1), we have the boundary condition for the first fundamental problem (or in other words the stress boundary value problem);

$$\phi_1(t) + t\overline{\phi_1'(t)} + \overline{\psi_1(t)} = f_1(t), \quad f_1(t) = -f(t). \tag{1.2}$$

While for $k = \varkappa = \frac{\lambda+3\mu}{\lambda+\mu} > 1$, $\varkappa$ is called Muskhelishvili's constant; λ, μ are the Lame's constants and $f(t) = 2\mu g(t)$ is a given function of displacement, we have the principal formula for the second fundamental problem (or the displacement boundary value problem);

$$\varkappa\phi_1(t) - t\overline{\phi_1'(t)} - \overline{\psi_1(t)} = 2\mu g(t). \tag{1.3}$$

The stress problem is ordered the first because any displacement for a body is resulted after a stress effect on this body.

In the absence of body forces, it is known from Muskhelishvili (1953) that stress components, in the plane theory of elasticity, are

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 4\operatorname{Re}\{\phi'(z)\}, \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2\left[\bar{z}\phi''(z) + \psi'(z)\right]. \end{aligned} \tag{1.4}$$

Also, the complex potential functions $\phi_1(z)$ and $\psi_1(z)$, take the form

$$\begin{aligned} \phi_1(z) &= -\frac{X+iY}{2\pi(1+\varkappa)}\ln z + \Gamma z + \phi(z), \\ \psi_1(z) &= \frac{\varkappa(X-iY)}{2\pi(1+\varkappa)}\ln z + \Gamma^* z + \psi(z), \end{aligned} \tag{1.5}$$

where X, Y are the components of the resultant vector of all external forces acting on the boundary and Γ, Γ^* are complex constants. The potential functions $\phi(z)$ and $\psi(z)$, Goursat functions, are single-valued analytic functions within the region on the right half-plane and $\phi(\infty) = 0$, $\psi(\infty) = 0$.

El-Sirafy (1977) used the complex variable method and a rational mapping function to obtain Goursat functions for the first fundamental problem of a stretched infinite plate weakened by a curvilinear hole with the application of a uniform tensile stress of intensity P, making an angle θ with the x-axis and the edge of the hole which is free from external forces is conformally mapped on the right half-plane by the rational mapping

$$z = c\omega(s) = c\frac{(s+1)^2 + m(s-1)^2}{s^2 - 1 - n(s-1)^2}, \quad (c > 0,\ |n| < 1,\ s = \sigma + i\tau), \tag{1.6}$$

where m, n are real parameters subjected to the condition $\omega'(s)$ does not vanish within the right half-plane.

Also, Abdou and Khar Eldin (1994) used the complex variable method to obtain Goursat functions for a plate stated under the same above circumstances with a more complicated rational mapping

$$(1.7)\quad z = c\omega(s) = c\frac{(s+1)^3 + m(s+1)(s-1)^2 + l(s-1)^3}{(s-1)(s+1)^2 - n(s+1)(s-1)^2}, \qquad (c>0,\ |n|<1,\ s=\sigma+i\tau),$$

where m, n, l are real parameters subjected to the condition $\omega'(s)$ does not vanish within the right half-plane.

In this paper the complex variable method will be applied to solve the first and second fundamental problems for a general infinite plate weakened by a curvilinear hole C. The hole is conformally mapped to the right half-plane by the use of a rational mapping function which is simpler than the one's used above in 1.6 and 1.7 but with complex constants as follows

$$(1.8)\qquad z = \omega(s) = l\left(\frac{s+1}{s-1}\right) + m\left(\frac{s-1}{s+1}\right)^2, \quad \left(|l|>1,\ 0 \le \left|\tfrac{m}{l}\right| < \tfrac{1}{2}\right),$$

where $l = l_1 + i\, l_2$, $m = m_1 + i\, m_2$, and $\left|\frac{m}{l}\right|$ is a parameter restricted such that $\omega'(s)$ does not vanish within the right half-plane.

Basically, in the following shape we can take a fast look on the singularity of the rational mapping (1.8) on the plate. (Figure 1.)

2. Mapping function

The parametric equations of the hole C by assuming that τ is the parameter of the curve are obtained from 1.8 as

(2.1)

$$x = \frac{\left(-1+\tau^2\right)\left(1+\tau^2\right)l_1 + 2\tau\left(1+\tau^2\right)l_2 + \left(\left(1-6\tau^2+\tau^4\right)m_1 - 4\tau\left(-1+\tau^2\right)m_2\right)}{\left(1+\tau^2\right)^2},$$

$$y = \frac{-2\tau\left(1+\tau^2\right)l_1 + \left(-1+\tau^2\right)\left(1+\tau^2\right)l_2 + \left(4\tau\left(-1+\tau^2\right)m_1 + \left(1-6\tau^2+\tau^4\right)m_2\right)}{\left(1+\tau^2\right)^2}.$$

Whereas, our present mapping function deals with famous shapes of tunnels, then it is useful to use it in studying stresses and strains around tunnels. In underground engineering the tunnel is assumed to be driven in a homogeneous, isotropic, linear elastic and pre-stressed geometrical situation. Also, the tunnel is considered to be deep enough such that the stress distribution before excavation is homogeneous. Excavating underground openings in soils and rocks are done for several purposes and in multi-sizes. At least, excavation of the opening will cause the soil or rock to deform elastically.

Worth mentioning, that excavation in soil or rock is a complicated, dangerous, and expensive process. The mechanics of this can be very complex. However, the use of conformal mapping that allows us to study stresses and strains around a unit circle makes it useful for engineers and easier for mathematicians.

The physical interest of the mapping (1.8) comes from its different shapes of holes it treats where we find from Fig's. 2-4 the following notations;

(i) If the hole contain corners only three corners will appear.

(ii) The complex constant m works on circling the shape from the symmetry situation and the circling angle is given by $\theta = \tan^{-1}\frac{m_2}{m_1}$ $(m = m_1 + im_2)$. Positive values of θ means that the circling will be in- the positive direction i.e. in the anticlockwise direction and for negative values the circling will be in the negative direction i.e. in clockwise direction.

(iii) The complex constant l works on expanding the corners of the hole shape. (Figures 2–4.)

3. Method of solution

In this section, we will use the transformation mapping (1.8) in the boundary condition (1.1), then we will apply the complex variable method with the residue theorems to obtain closed expressions for Goursat functions. Therefore, the expression $\overline{\omega\,(i\tau)}/\omega'\,(i\tau)$ will be written in the form

$$\frac{\overline{\omega\,(i\tau)}}{\omega'\,(i\tau)} = \overline{\alpha\,(i\tau)} + \beta\,(i\tau)\,, \tag{3.1}$$

where

$$\begin{aligned} \alpha\,(i\tau) &= \frac{\overline{h}}{i\tau + \overline{a}}, \quad a = \frac{\sqrt[3]{2m} + \sqrt[3]{l}}{\sqrt[3]{2m} - \sqrt[3]{l}}, \\ h &= \frac{(a+1)^2\left[\bar{l}(a-1)^3 + \bar{m}(a+1)^3\right]}{\left(\sqrt[3]{4m} - \sqrt[3]{2l}\right)\left(\left(\sqrt[3]{4m}(a-1)\right)^2 + 2\sqrt[3]{ml}(a^2-1) + \left(\sqrt[3]{2l}(a+1)\right)^2\right)}, \end{aligned} \tag{3.2}$$

and

$$\beta\,(i\tau) = \frac{\overline{\omega\,(i\tau)}}{\omega'\,(i\tau)} + \frac{h}{i\tau - a}. \tag{3.3}$$

It is obvious that $\beta\,(i\tau)$ is a regular function within the right half-plane except at infinity.

Using (3.1) and the complex potential function's form in (1.5) transforms the boundary condition (1.1) into

$$k\Phi(i\tau) - \alpha\,(i\tau)\,\overline{\Phi'(i\tau)} - \overline{\Psi_*(i\tau)} = f_*\,(i\tau)\,, \tag{3.4}$$

where

$$\Psi_*(i\tau) = \Psi(i\tau) + \beta\,(i\tau)\,\Phi'(i\tau), \tag{3.5a}$$

(3.5b)

$$f_*(i\tau) = F(i\tau) - k\Gamma\omega(i\tau) + \bar{\Gamma}^*\overline{\omega(i\tau)} + \overline{\omega'(i\tau)}\left[\alpha(i\tau) + \overline{\beta(i\tau)}\right] N(i\tau),$$

(3.5c)
$$N(i\tau) = \bar{\Gamma} - \frac{X - iY}{2\pi(1+\varkappa)} \cdot \frac{1}{\overline{\overline{\omega(i\tau)}}},$$

and

(3.5d)
$$F(s) = f(\omega(s)).$$

The function $F(s)$ with its derivatives must satisfy the Hölder condition and we assume that $\Phi(\infty) = \Psi(\infty) = 0$.

Multiplying both sides of 3.4 by $(1/2\pi)\, 1/(s - i\tau)$ and integrating with respect to τ from $-\infty$ to ∞, one has

(3.6)
$$k\Phi(s) - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\alpha(i\tau)\,\overline{\Phi'(i\tau)}d\tau}{s - i\tau} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f_*(i\tau)\,d\tau}{s - i\tau},$$

and by using (3.2) and (3.5b), we have

(3.7)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\alpha(i\tau)\,\overline{\Phi'(i\tau)}d\tau}{s - i\tau} = \frac{\bar{h}b}{s + \bar{a}},$$

(3.8)
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f_*(i\tau)\,d\tau}{s - i\tau}$$
$$= A(s) + \frac{4k\Gamma ms}{(s+1)^2} - \frac{2\Gamma^*\bar{l}}{s+1} + \frac{2m}{(s+1)^2}\left[\frac{X - iY}{2\pi(1+\varkappa)} \cdot \frac{s+1}{\bar{l}} - 2s\right],$$

where b is a complex constant to be determined and

(3.9)
$$A(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(i\tau)\,d\tau}{s - i\tau}.$$

Substituting from (3.7) and (3.8) into (3.6), we get

(3.10) $k\Phi(s)$
$$= A(s) + \frac{4k\Gamma ms}{(s+1)^2} - \frac{2\Gamma^*\bar{l}}{s+1} + \frac{\bar{h}b}{s+\bar{a}} + \frac{2m}{(s+1)^2}\left[\frac{X - iY}{2\pi(1+\varkappa)} \cdot \frac{s+1}{\bar{l}} - 2s\right].$$

Inserting $\overline{\Phi'(i\tau)}$ from (3.10) into (3.7), the complex constant b is determined in the form

(3.11)
$$b = \frac{1}{kT}[\overline{A'(-\bar{a})} - \frac{4k\bar{\Gamma}\bar{m}(\bar{a}-1)}{(\bar{a}+1)^3} + \frac{2\bar{\Gamma}^* l}{(\bar{a}+1)^2} + \bar{N}_1 + \bar{N}_2$$

$$- \frac{h}{k\left(\bar{a}+a\right)^{2}}\left\{A'\left(-\bar{a}\right)-\frac{4k\Gamma m\left(a-1\right)}{\left(a+1\right)^{3}}+\frac{2\Gamma^{*}\bar{l}}{\left(a+1\right)^{2}}+N_{1}+N_{2}\right\}],$$

where

$$T=1-\frac{h\bar{h}}{k^{2}\left(\bar{a}+a\right)^{4}}, \tag{3.12a}$$

$$N_{1}=\frac{2m}{\left(a+1\right)^{2}}\left[\frac{X-iY}{2\pi\left(1+\varkappa\right)}\cdot\frac{1}{\bar{\bar{l}}}-2\right], \tag{3.12b}$$

$$N_{2}=\frac{4m}{\left(a+1\right)}\left[2a-\frac{X-iY}{2\pi\left(1+\varkappa\right)}\cdot\frac{\left(a+1\right)}{\bar{l}}\right]. \tag{3.12c}$$

Substituting from (3.11) into (3.10), we get

$$\begin{aligned} &k\Phi\left(s\right) \\ &=A\left(s\right)+\frac{4k\Gamma ms}{(s+1)^{2}}-\frac{2\Gamma^{*}\bar{l}}{s+1}+\frac{\bar{h}b}{\left(s+\bar{a}\right)}+\frac{2m}{(s+1)^{2}}\left[\frac{X-iY}{2\pi\left(1+\varkappa\right)}\cdot\frac{s+1}{\bar{l}}-2s\right]. \end{aligned} \tag{3.13}$$

Also from the boundary condition (3.4), we obtain $\Psi(s)$ in the form

$$\begin{aligned} \Psi(s)=\;&\overline{A\left(s\right)}-\frac{4k\bar{\Gamma}\bar{m}s}{(s-1)^{2}} \\ &+\frac{2\bar{\Gamma}^{*}l}{s-1}-\frac{h\bar{b}}{\left(s-a\right)}-\frac{(s+1)^{2}[\bar{l}(s-1)^{3}+\bar{m}(s+1)^{3}]}{k[4m(s-1)^{3}-2l(s+1)^{3}]}[A'\left(s\right) \\ &-\frac{8k\Gamma ms}{(s+1)^{3}}+\frac{2\Gamma^{*}\bar{l}}{(s+1)^{2}}-\frac{\bar{h}b}{(s+\bar{a})^{2}}-\frac{4m}{(s+1)^{3}}\left[\frac{X-iY}{2\pi\left(1+\varkappa\right)}\cdot\frac{s+1}{\bar{l}}-2s\right] \\ &+\frac{2m}{(s+1)^{2}}\left[\frac{X-iY}{2\pi(1+\varkappa)}\cdot\frac{1}{\bar{l}}-2\right]]+\frac{2\bar{m}}{(s-1)^{2}}\left[2s-\frac{X+iY}{2\pi\left(1+\varkappa\right)}\cdot\frac{s-1}{l}\right] \\ &-\Gamma^{*}\omega\left(s\right)+\left[k\bar{\Gamma}-\overline{N\left(s\right)}\right]\overline{\omega\left(s\right)}-\overline{F}\left(s\right), \end{aligned} \tag{3.14}$$

Finally, Goursat functions are completely determined in (3.13) and (3.14).

4. Special cases

4.1. In our recent mapping function (1.8) if we consider the reality of the constants with letting $l=1$ and $m=l$ also by considering the substitutions $k=-1$, $\Gamma=\frac{p}{4}$, $\Gamma^{*}=-\frac{1}{2}pe^{-2i\theta}$, $X=Y=f=0$, we have

$$\Phi\left(s\right)=-\frac{pls}{\left(s+1\right)^{2}}-\frac{pe^{-2i\theta}}{s+1}+\frac{4ls}{\left(s+1\right)^{2}}+\frac{h\overline{Q}}{\left(s+a\right)T}, \tag{4.1}$$

$$\Psi(s) = \frac{pls}{(s-1)^2} - \frac{pe^{2i\theta}}{s-1} + \frac{hQ}{(s-a)\,T}$$
$$+ \frac{(s+1)^2\left[(s-1)^3 + l\,(s+1)^3\right]}{4l\,(s-1)^3 - 2\,(s+1)^3}\Big(\frac{2pls}{(s+1)^3}$$
$$- \frac{pe^{-2i\theta}}{(s+1)^2} + \frac{8ls}{(s+1)^3} - \frac{4l}{(s+1)^2} + \frac{h\overline{Q}}{(s+a)\,T}\Big) + \frac{4ls}{(s-1)^2} \tag{4.2}$$
$$- \frac{p}{2}\left[\frac{s-1}{s+1} + l\left(\frac{s+1}{s-1}\right)^2\right] + \frac{pe^{-2i\theta}}{2}\left[\frac{s+1}{s-1} + l\left(\frac{s-1}{s+1}\right)^2\right],$$

where

$$Q = \frac{(a-1)\,pl}{(a+1)^3} - \frac{pe^{-2i\theta}}{(a+1)^2} + \frac{h}{4a^2}\Big\{\frac{(a-1)\,pl}{(a+1)^3} - \frac{pe^{2i\theta}}{(a+1)^2}$$
$$- \frac{4l}{(a+1)^2} + \frac{8la}{(a+1)}\Big\} - \frac{4l}{(a+1)^2} + \frac{8la}{(a+1)}. \tag{4.3}$$

If we substitute $m = n = 0$ in the expressions derived by Abdou and Khar Eldin (1994), we get approximately the same above expressions of Goursat functions because of using the same method (complex variable method).

4.2. For $\varsigma = \frac{s+1}{s-1}$, we get the mapping function

$$z = \omega\,(\zeta) = l\zeta + m\zeta^{-2}. \tag{4.4}$$

This mapping corresponds to the one discussed in Aseeri (2007) when $n = 2$. Thereon, letting $s = \frac{\zeta+1}{\zeta-1}$ in the Goursat functions formulae in (3.13) and (3.14), gives

$$k\Phi\,(\zeta) = A\,(\zeta) + \frac{k\Gamma m\left(\zeta^2-1\right)}{\zeta^2} - \frac{\Gamma^*\bar{l}\,(\zeta-1)}{\zeta}$$
$$+ \frac{m\,(\zeta-1)}{\zeta^2}\left[\frac{\zeta\,(X-iY)}{2\pi\,(1+\varkappa)\,\bar{l}} - (\zeta+1)\right] \tag{4.5}$$
$$+ \frac{\bar{h}\,(\zeta-1)\,b}{[\zeta\,(1+\bar{a}) + (1-\bar{a})]},$$

$$
\begin{aligned}
c\Psi(\zeta) = {} & \overline{A(\zeta)} - k\bar{\Gamma}\bar{m}\zeta(\zeta^2-1) \\
& + \bar{\Gamma}^* l(\zeta-1) - \overline{F(s)} - \Gamma^* \left[l\zeta + m\zeta^{-2}\right] \\
& + \left(k\bar{\Gamma} - \overline{N(\zeta)}\right)\left[\bar{l}\zeta^{-1} + \bar{m}\zeta^2\right] \\
& + \bar{m}(\zeta-1)\left[\zeta + 1 - \frac{X+iY}{2\pi(1+\varkappa)l}\right] \\
& - \frac{\left(\bar{l}\zeta^{-1} + \bar{m}\zeta^2\right)U(\zeta)}{k\left(l - 2m\zeta^{-3}\right)} - \frac{h(\zeta-1)\bar{b}}{((1-a)\zeta + (1+a))},
\end{aligned} \tag{4.6}
$$

where

$$
\begin{aligned}
U(\zeta) = {} & A'(\zeta) + \frac{\Gamma^*\bar{l}(\zeta-1)^2}{2\zeta^2} \\
& - \frac{k\Gamma m\left(\zeta^2-1\right)(\zeta-1)}{\zeta^3} - \frac{\bar{h}(\zeta-1)^2 b}{(\zeta(1+\bar{a}) + (1-\bar{a}))^2} \\
& - \frac{m(\zeta-1)^2}{\zeta^3}\left(\left[\frac{(X-iY)\zeta}{2\pi(1+\varkappa)\bar{l}} - (\zeta+1)\right] + \frac{\zeta}{2}\left[\frac{X-iY}{2\pi(1+\varkappa)\bar{l}} - 2\right]\right).
\end{aligned} \tag{4.7}
$$

If we substitute $n = 2$ in the expressions derived by Aseeri (2007), we get, after neglecting the constant term, the same above expressions of Goursat functions.

4.3. Abdou and Aseeri (2007) used the rational mapping

$$
z = \omega(\zeta) = l\zeta + \sum_{j=1}^{M} m_j \zeta^{-j}, \tag{4.8}
$$

where l, m_j $(j = 1, 2, \ldots M)$ are complex constants to obtain Goursat functions for the same recent problem. The derived expressions of Goursat functions in this case are approximately equivalent to those indicated in (4.5) and (4.6) if we put $M = 2$ and $m_1 = 0$.

5. Applications

5.1. Curvilinear hole for an infinite plate subjected to a uniform tensile stress. For $k = -1$, $\Gamma = \frac{p}{4}$, $\Gamma^* = -\frac{1}{2}P\exp(-2i\theta)$ and $X = Y = f = 0$, we have an infinite plate stretched at infinity by the application of a uniform tensile stress of intensity P, making an angle θ with $x-$axis. The plate is weakened by a curvilinear hole C which is free from stress.

The functions (3.13), (3.14) take the form

$$
\Phi(s) = \frac{(p+4)ms}{(s+1)^2} - \frac{p\bar{l}e^{-2i\theta}}{s+1} + \frac{\bar{h}V}{T(s+\bar{a})}, \tag{5.1}
$$

$$
\begin{aligned}
\Psi(s) = {} & \frac{1}{2} p e^{-2i\theta}\left[l\left(\frac{s+1}{s-1}\right)+m\left(\frac{s-1}{s+1}\right)^{2}\right] \\
& -\frac{p}{2}\left[\bar{l}\left(\frac{s-1}{s+1}\right)+\bar{m}\left(\frac{s+1}{s-1}\right)^{2}\right] \\
& +\frac{(p+4)\,\bar{m} s}{(s-1)^{2}}-\frac{p l e^{2i\theta}}{s-1}+\frac{(s+1)^{2}\left[\bar{l}\,(s-1)^{3}+\bar{m}\,(s+1)^{3}\right]}{4m\,(s-1)^{3}-2l\,(s+1)^{3}} \\
& \times\left[\frac{2\,(p+4)\,m s}{(s+1)^{3}}-\frac{p\bar{l} e^{-2i\theta}+4m}{(s+1)^{2}}+\frac{\bar{h} V}{T\,(s+\bar{a})^{2}}\right]+\frac{h\overline{V}}{\bar{T}\,(s-a)},
\end{aligned} \tag{5.2}
$$

where

$$
\begin{aligned}
V = {} & \Big(\frac{p\bar{m}\,(\bar{a}-1)}{(\bar{a}+1)^{3}}-\frac{p l e^{2i\theta}+4m}{(\bar{a}+1)^{2}}+\frac{8\bar{m}\bar{a}}{\bar{a}+1}+\frac{h}{(\bar{a}+a)^{2}}\Big\{\frac{p m\,(a-1)}{(a+1)^{3}} \\
& -\frac{p\bar{l} e^{-2i\theta}+4m}{(a+1)^{2}}+\frac{8ma}{a+1}\Big\}\Big).
\end{aligned} \tag{5.3}
$$

And for $l = i$, $m = 1 + i$, $P = 0.25$, $\theta = \frac{\pi}{4}$, the stress components σ_{xx}, σ_{yy} and σ_{xy} are obtained in large forms calculated by computer and illustrated in Figures 5–9.

5.2. When the external force acts on the center of the curvilinear. For $\Gamma = \Gamma^* = f = 0$ and $k = \varkappa$, we have the second fundamental problem when the force acts on the curvilinear kernel. It will be assumed that the stresses vanish at infinity and it is easily seen that the kernel does not rotate.

In general the kernel remains in the original position. The Goursat's functions are

$$
\Phi\,(s) = \frac{\bar{h} W}{\varkappa^{2} T\,(s+\bar{a})}+\frac{2m}{\varkappa\,(s+1)^{2}}\left[\frac{(X-iY)\,(s+1)}{2\pi\,(1+\varkappa)\,\bar{l}}-2s\right], \tag{5.4}
$$

$$
\begin{aligned}
\Psi(s) = {} & \frac{(s+1)^2 \left[\bar{l}\,(s-1)^3 + \bar{m}\,(s+1)^3\right]}{\varkappa \left[4m\,(s-1)^3 - 2l\,(s+1)^3\right]} \\
& \times \left[\frac{\bar{h}W}{\varkappa T\,(s+\bar{a})^2} - \frac{4m}{(s+1)^2}\left[\frac{X-iY}{4\pi\,(1+\varkappa)\,\bar{l}} - 1\right]\right. \\
& + \left.\frac{4m}{(s+1)^3}\left[\frac{(X-iY)\,(s+1)}{2\pi\,(1+\varkappa)\,\bar{l}} - 2s\right]\right] \\
& + \frac{2\bar{m}}{(s-1)^2}\left[2s - \frac{(X+iY)\,(s-1)}{2\pi\,(1+\varkappa)\,l}\right]\frac{h\overline{W}}{\varkappa\bar{T}\,(a-s)} \\
& + \left[\frac{(X+iY)\,(s-1)\,(s+1)^2}{2\pi\,(1+\varkappa)\left[l\,(s+1)^3 + m\,(s-1)^3\right]}\right]\overline{\omega\,(s)},
\end{aligned}
\tag{5.5}
$$

where

$$
W = \overline{N_1} + \overline{N_2} - \frac{h\,\{N_1 + N_2\}}{\varkappa\,(\bar{a}+a)^2}. \tag{5.6}
$$

For $l = i$, $m = 1+i$, $X = Y = 10$, $\varkappa = 2$, the stress components σ_{xx}, σ_{yy} and σ_{xy} are obtained in large forms calculated by computer and illustrated in Figures 10–14.

6. Conclusions

From the above results and discussions the following may be concluded:

(1) In the theory of two dimensional linear elasticity one of the most useful techniques for the solutions of boundary value problem for a region weakened by a curvilinear hole is to transform the region into a simpler shape to get solutions without difficulties.

(2) The transformation mapping $z = c\omega\,(s)$, $c > 0$, $s = \sigma + i\ \tau$, transforms the domain of the infinite plate with a curvilinear hole into the domain of the right half-plane. While the mapping $z = c\omega\,(\zeta)$, $|\zeta| > 1$, $\zeta = \rho e^{i\theta}$, transforms the domain of the infinite plate with a curvilinear hole into the domain outside a unit circle.

(3) The transformation $s = \frac{\zeta+1}{\zeta-1}$, transforms the domain of the right half-plane into the domain outside a unit circle. The inverse case can be obtained by using the transformation $\zeta = \frac{s+1}{s-1}$.

(4) The physical interest of the mapping (1.8) comes from its different shapes of holes it treats and different directions it takes as shown in Fig's. 2–4.

(5) The complex variable method (Cauchy method) is considered as one of the best methods for solving the integro-differential equations (1.1), and obtaining the two complex potential functions $\phi\,(z)$ and $\psi\,(z)$ directly.

(6) Stress is an internal force whereas positive values of it mean that stress is in the positive direction, i.e. stress acts as a tension force. On the other side, negative values of stress mean that stress is in the negative direction, i.e. stress acts as a press force.

(7) The most important issue deduced from (Fig's. 5, 6 and 10, 11) is that $\max \sigma_{xx} = -\min \sigma_{yy}$ and vice versa ($\min \sigma_{xx} = -\max \sigma_{yy}$).

(8) By following the intendancy of $\frac{\sigma_{xx}}{\sigma_{yy}}$ and $\frac{\sigma_{yy}}{\sigma_{xx}}$ at (Fig's. 8, 9 and 13, 14), we find that $\frac{\sigma_{xx}}{\sigma_{yy}} \to 0$ at the same points where $\frac{\sigma_{yy}}{\sigma_{xx}} \to \infty$ and vice versa ($\frac{\sigma_{xx}}{\sigma_{yy}} \to \infty$ at the same points where $\frac{\sigma_{yy}}{\sigma_{xx}} \to 0$).

(9) When $\frac{\sigma_{xx}}{\sigma_{yy}} \to 0$ that means that the perpendicular stress on $y-$axis is the maximum value and presents the body interior resistance of treatment (like rocks for example). Whereas, the perpendicular stress on $x-$axis is small according to $y-$axis. Thereon, it is better to treat the problem at points determined by angles that gives minimum values of $\frac{\sigma_{xx}}{\sigma_{yy}}$.

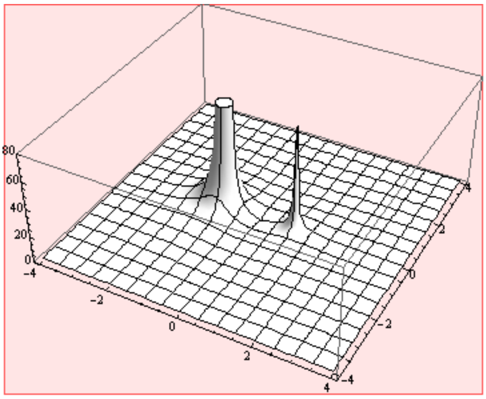

Figure 1

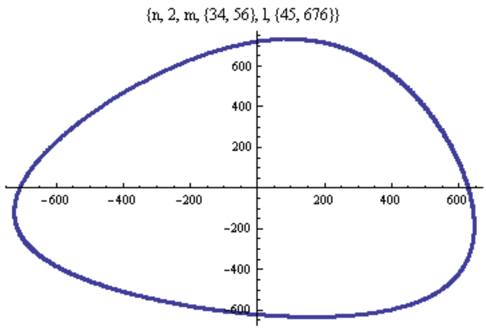

Figure 2

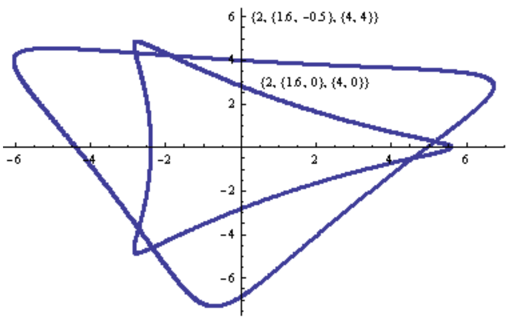

Figure 3

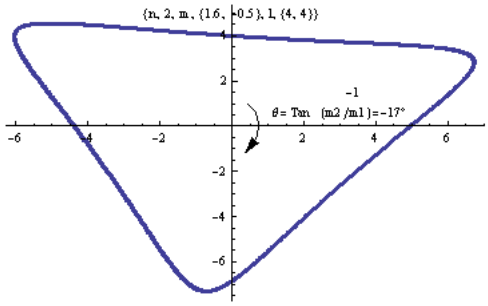

FIGURE 4

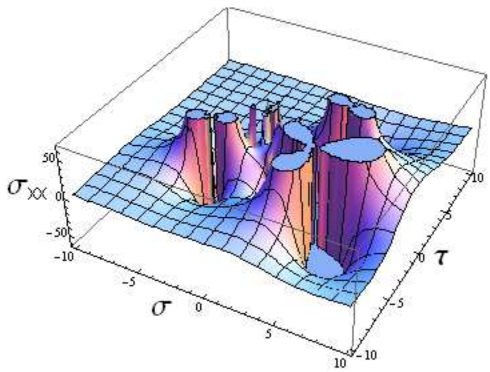

FIGURE 5

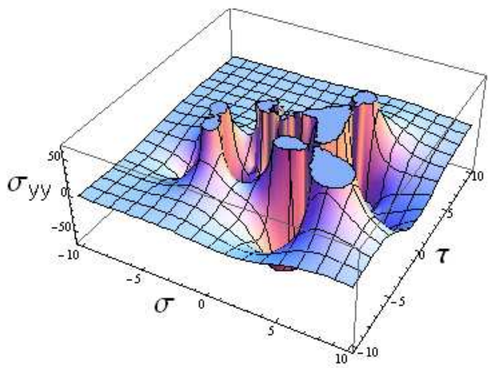

FIGURE 6

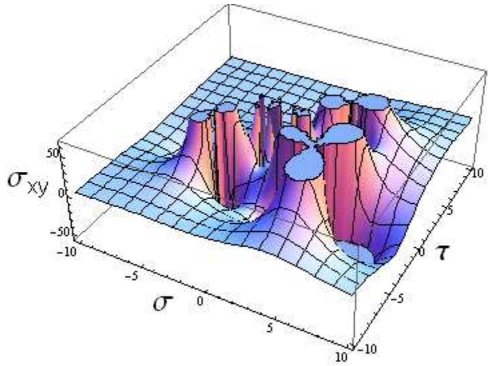

FIGURE 7

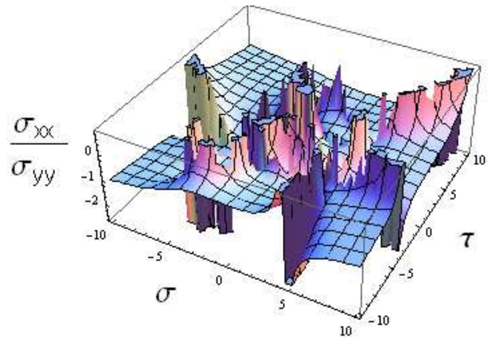

FIGURE 8

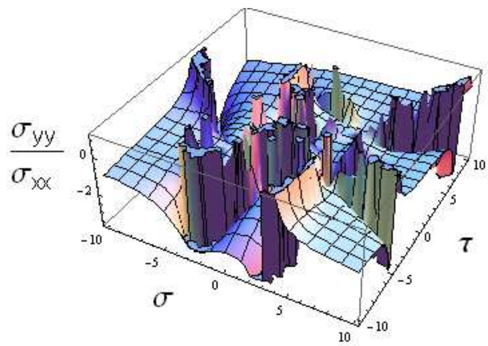

FIGURE 9

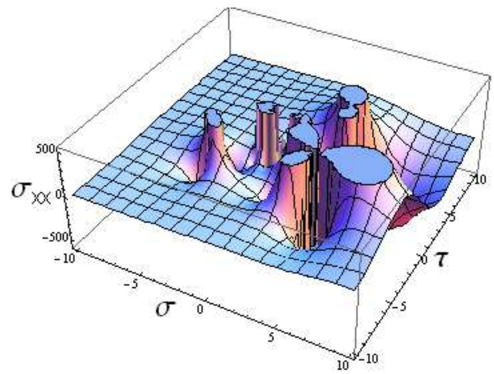

Figure 10

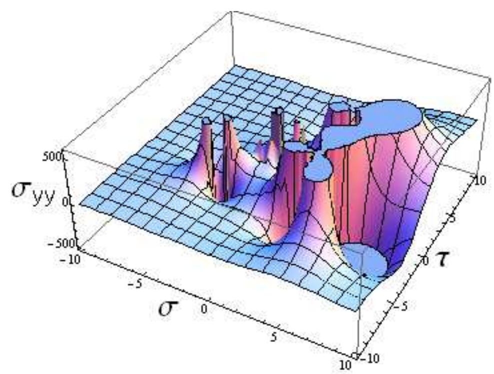

Figure 11

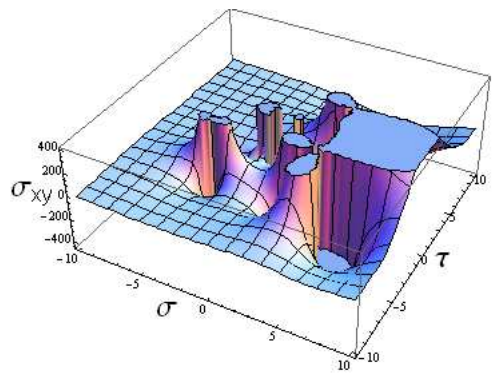

Figure 12

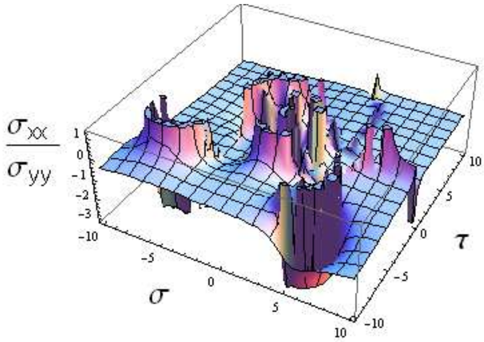

FIGURE 13

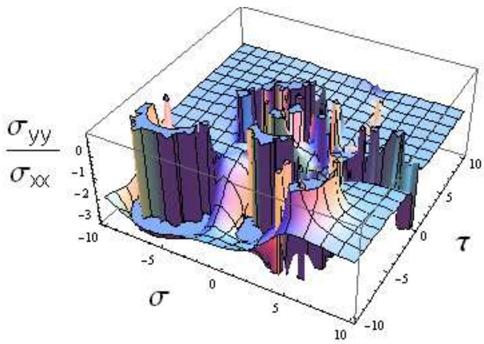

FIGURE 14

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