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G-CONTINUOUS FRAMES AND COORBIT SPACES

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Abstract. A generalized continuous frame is a family of operators on a Hilbert space H which allows reproductions of arbitrary elements of H by continuous superpositions. Generalized continuous frames are natural generalization of continuous and discrete frames in Hilbert spaces which include many recent generalization of frames. In this article,we associate to a generalized continuous frame suitable Banach spaces, called generalized coorbit spaces, provided the frame satisfies a certain integrability condition. Also two classes of generalized coorbit spaces associated to a generalized continuous frame,its standard dual and some results are studied.

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [5]. Frames have very important and interesting properties make them very useful in the characterization of function spaces, signal processing and many other fields. A discrete frame is a countable family of elements in a separable Hilbert spaces allows stable not necessarily unique decomposition of arbitrary elements into expansions of frame elements [4]. Given a separable Hilbert spaces $\mathcal{H}$,a collections of elements $\{f_i\}_{i\in\mathbb{Z}}$ is called a discrete frame if there exist constants $0 < A_1, A_2 < \infty$ such that

$$A_1\|f\|^2 \leq \sum_{i\in\mathbb{Z}} |< f, f_i >|^2 \leq A_2\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

Later, this concept was generalized to continuous frames indexed by a Radon measure space [3, 2, 1] and [7]. For a locally compact Hausdorff space X endowed with a positive Randon measure μ, a family $\{\psi_x\}_{x\in X}$ of vectors in a separable Hilbert spaces $\mathcal{H}$ is called a continuous frame if there exist constants

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$0 < A_1, A_2 < \infty$ such that

$$A_1\|f\|^2 \leq \int_X |< f, \psi_x >|^2 \, d\mu(x) \leq A_2\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

The concept of generalized frames has been introduced by W. Sun [8]. Generalized frames are natural generalizations of frames as members of a Hilbert space to bounded linear operators. A family $\{\Lambda_i\}_{i\in\mathbb{Z}}$ of bonded linear operators from a separable Hilbert space $\mathcal{H}$ into another separable Hilbert space $\mathcal{K}$ is called a generalized frame if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in\mathbb{Z}} \|\Lambda_i(f)\|^2 \leq B\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

M. Fornasier and H. Rauhut have studied a kind of Banach spaces called coorbit spaces that vectors can be decomposed by use continuous frames [6]. Now we are going to extend this action by generalized continuous frames.

2. Generalized Continuous Frames

Let X be a locally compact Hausdorff space endowed with a positive radon measure μ with $\operatorname{supp}\mu = X$.

Definition 2.1. A family $\mathcal{F} = \{\Lambda_x\}_{x\in X}$ of bonded linear operator from a Hilbert space $\mathcal{H}$ into another Hilbert space $\mathcal{K}$ is called generalized continuous frame or simply g-continuous frame for $\mathcal{H}$ with respect to $\mathcal{K}$ if there are positive constants C_1 and C_2 such that

$$C_1\|f\|^2 \leq \int_X \|\Lambda_x(f)\|^2 d\mu(x) \leq C_2\|f\|^2 \text{ for all } f \in \mathcal{H}. \tag{1}$$

If $C_1 = C_2$ then the frame is called tight. We call $\mathcal{F}$ a g-continuous frame for $\mathcal{H}$ if $\mathcal{H} = \mathcal{K}$.

For the sake of simplicity we assume that the mapping $x \mapsto \Lambda_x$ is weakly continuous. Not that, if X is a countable set and μ is counting measure then we obtain the usual definition of (generalized discrete) frame. By the Riesz Representation Theorem, to every functional $\Lambda \in L(\mathcal{H}, \mathbb{C})$, one can find some $g \in \mathcal{H}$ such that $\Lambda(f) =< f, g >$ for all $f \in \mathcal{H}$.Hence a continuous frame is equivalent to a g-continuous frame frame whenever $\mathcal{K} = \mathbb{C}$.

For a g-continuous frame $\mathcal{F}$ define the frame operator $S = S_{\mathcal{F}}$ in weak sense by

$$S \colon \mathcal{H} \to \mathcal{H}, \quad Sf := \int_X \Lambda_x^* \Lambda_x f d\mu(x)$$

where Λ_x^* is adjoint of the operator Λ_x.

Proposition 2.2. *The frame operator S is a bounded, positive, self-adjoint, and invertible.*

Proof. For all $f \in \mathcal{H}$

$$
\begin{aligned}
< Sf, f > &= \int_X < \Lambda_x^* \Lambda_x f, f > d\mu(x) = \int_X < \Lambda_x f, \Lambda_x f > d\mu(x) \\
&= \int_X \| \Lambda_x(f) \|^2 \, d\mu(x) \geq C_1 \| f \|^2 \geq 0 =< Sf, g > \\
&= \int_X < \Lambda_x^* \Lambda_x f, g > d\mu(x) = \int_X < \Lambda_x f, \Lambda_x g > d\mu(x) \\
&= \int_X < f, \Lambda_x^* \Lambda_x g > d\mu(x) =< f, Sg >
\end{aligned}
$$

by (1) we have $C_1 < f, f > \leq < Sf, f > \leq C_2 < f, f >$, $C_1 I \leq S \leq C_2 I$ and $\|I - C_2^{-1} S\| \leq 1 - \frac{C_1}{C_2} < 1$. Hence S is invertible operator. □

Proposition 2.3. *Let $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ is a g-continuous frame for Hilbert space $\mathcal{H}$ with frame operator S and bounds C_1, C_2. Then $\widetilde{\mathcal{F}} = \{\widetilde{\Lambda_x}\}_{x \in X}$ such that $\widetilde{\Lambda_x} = \Lambda_x S^{-1}$, is a frame for $\mathcal{H}$ with bounds C_1^{-1}, C_2^{-1} and frame operator S^{-1}.*

Proof. We show that $S^{-1} f = \int_X (S^{-1} \Lambda_x^* S^{-1} \Lambda_x) f d\mu(x)$.

$$
\begin{aligned}
\int_X (S^{-1} \Lambda_x^* S^{-1} \Lambda_x) f d\mu(x) &= S^{-1} \int_X (\Lambda_x^* S^{-1} \Lambda_x) f d\mu(x) \\
&= S^{-1} \int_X \Lambda_x^* \Lambda_x (S^{-1} f) d\mu(x) \\
&= S^{-1}(S(S^{-1} f)) = S^{-1} f
\end{aligned}
$$

also since $\mathcal{F} = \{\Lambda_x\}_{x \in X}$ is a frame for $\mathcal{H}$ then $C_1 I \leq S \leq C_2 I$. On other hand since I and S are self-adjoint and S^{-1} commutative with I and S,

$$C_1 I S^{-1} \leq S S^{-1} \leq C_2 I S^{-1}$$

and hence

$$C_2^{-1} I \leq S^{-1} \leq C_1^{-1} I.$$

□

If $\mathcal{F}$ is tight frame with bound $A = B = \lambda$ then $S = \lambda I$. Now the set

$$L^2(X, \mathcal{H}) := \{F \colon X \to \mathcal{H} \mid \int_X \|F(x)\|^2 d\mu(x) < \infty\},$$

with inner product $< F, G >:= \int_X < F(x), G(x) > d\mu(x)$, is a Hilbert space.

We define the following two transformations associated to $\mathcal{F}$,

$$
\begin{aligned}
V \colon \mathcal{H} \to L^2(X, \mathcal{H}), \quad & Vf(x) := \Lambda_x(f), \\
W \colon \mathcal{H} \to L^2(X, \mathcal{H}), \quad & Wf(x) := \Lambda_x(S^{-1} f).
\end{aligned}
$$

The operators V and W are well define since by (1) we have

$$\int_X \|Vf(x)\|^2 d\mu(x) = \int_X \|\Lambda_x(f)\|^2 d\mu(x) \leq C_2\|f\|^2 < \infty$$

and

$$\begin{aligned}\int_X \|Wf(x)\|^2 d\mu(x) &= \int_X \|\Lambda_x(S^{-1}f)\|^2 d\mu(x)\\ &= \int_X \|S^{-1}\Lambda_x(f)\|^2 d\mu(x) \leq C_1^{-1}\|f\|^2 < \infty.\end{aligned}$$

In the following we show that adjoint operator of V and W given weakly by

$$V^*\colon L^2(X,\mathcal{H}) \to\to \mathcal{H}, \quad V^*F := \int_X \Lambda_y^* F(y)d\mu(y),$$

$$W^*\colon L^2(X,\mathcal{H}) \to\to \mathcal{H}, \quad W^*F := \int_X S^{-1}\Lambda_y^* F(y)d\mu(y).$$

Since for all $h \in \mathcal{H}$, we have

$$\begin{aligned}< V^*F, h > &= \int_X < \Lambda_y^* F(y), h > d\mu(y) = \int_X < F(y), \Lambda_y(h) > d\mu(y)\\ &= \int_X < F(y), Vh(y) > d\mu(y) =< F, Vh >\end{aligned}$$

and

$$\begin{aligned}< W^*F, h > &= \int_X < S^{-1}\Lambda_y^* F(y), h > d\mu(y)\\ &= \int_X < F(y), \Lambda_y(S^{-1}h) > d\mu(y)\\ &= \int_X < F(y), Wh(y) > d\mu(y) =< F, Wh > .\end{aligned}$$

Proposition 2.4. *Let $\mathcal{F} = \{\Lambda_x\}_{x\in X}$ is a g-continuous frame for Hilbert space $\mathcal{H}$ with frame operator S, then the following holds,*

a) $S = V^*V$, $S^{-1} = W^*W$,
b) $< f, g >=< Vf, Wg >=< Wf, Vg >$,
c) *V and W are unitary if $\mathcal{F}$ is a tight frame,*
d) $\mathrm{Range}\, V = \mathrm{Range}\, W$,
e) *V and W are bijective transformations from $\mathcal{H}$ onto the Hilbert space $\mathcal{M}$ where*

$$\mathcal{M} = \{F \in L^2(X,\mathcal{H}) : \int_X R(x,y)F(y)d\mu = F(x) a.e, R(x,y) = S^{-1}\Lambda_x\Lambda_y^*\}.$$

Proof. Let $f \in \mathcal{H}$, we have

$$(V^*V)(f) = \int_X \Lambda_y^* Vf(y)d\mu(y) = \int_X \Lambda_y^*\Lambda_y f d\mu(y) = Sf,$$

and

$$(V^*W)f = \int_X \Lambda_y^* Wf(y)d\mu(y) = \int_X \Lambda_y^*\Lambda_y(S^{-1}f)d\mu(y) = S(S^{-1}f) = f.$$

In the same argument, $(W^*V)f = f$ and hence for all f and g in $\mathcal{H}$,

$$< f, g >=< Vf, Wg > .$$

Therefore a), b) and c) hold.

Since S is invertible and self-adjoint we have

$$f = SS^{-1}f = \int_X \Lambda_y^*\Lambda_y(S^{-1}f)d\mu(y) = \int_X \Lambda_y^* Wf(y)d\mu(y),$$

and

$$f = S^{-1}Sf = \int_X S^{-1}\Lambda_y^*\Lambda_y f d\mu(y) = \int_X S^{-1}\Lambda_y^* Vf(y)d\mu(y)$$

in the weak sense. Furthermore we have

$$\begin{aligned} Wf(x) &= \Lambda_x(S^{-1}f) = \Lambda_x S^{-1}(\int_X \Lambda_y^* Wf(y)d\mu(y)) \\ &= \int_X \Lambda_x S^{-1}\Lambda_y^* Wf(y)d\mu(y), \end{aligned}$$

$$Vf(x) = \Lambda_x(f) = \Lambda_x(\int_X S^{-1}\Lambda_y^* Vf(y)d\mu(y)) = \int_X \Lambda_x S^{-1}\Lambda_y^* Vf(y)d\mu(y).$$

Therefore

$$Wf(x) = \int_X R(x,y)Wf(y)d\mu(y), \quad Vf(x) = \int_X R(x,y)Vf(y)d\mu(y),$$

and hence Vf and Wf are in $\mathcal{M}$.

Conversely, let F be in $\mathcal{M}$ then

$$\begin{aligned} F(x) &= \int_X R(x,y)F(y)d\mu(y) = \int_X \Lambda_x S^{-1}\Lambda_y^* F(y)d\mu(y) \\ &= \Lambda_x S^{-1}\int_X \Lambda_y^* F(y)d\mu(y) = \Lambda_x S^{-1}(V^*F) = W(V^*F)(x). \end{aligned}$$

Therefore $F \in \operatorname{Range} W$, $\mathcal{M} \subseteq \operatorname{Range} W$ and $\mathcal{M} = \operatorname{Range} W$. The same argument implies that $\mathcal{M} = \operatorname{Range} V$. Finally by (1) V and W are injective and the proof is complete. □

For every kernel function $K\colon X \times X \to L(\mathcal{H})$ and every function $F\colon X \to \mathcal{H}$ corresponds an operator K such that

$$K(F)(x) := \int_X K(x,y)F(y)d\mu(y). \tag{2}$$

Proposition 2.5. *Let* $\mathrm{R}\colon L^2(X,\mathcal{H}) \to L^2(X,\mathcal{H})$ *and*

$$\mathrm{R}(F)(x) := \int_X R(x,y)F(y)d\mu(y),$$

then,

a) $R(x,y) = R(y,x)^*$ *for all* x *and* y *in* X*,*
b) $\mathrm{R}(Vf) = Vf$,$\mathrm{R}(Wf) = Wf$ *for all* f *in* $\mathcal{H}$*,*
c) R *is self-adjoint as an operator on* $L^2(X,\mathcal{H})$*,*
d) R *is orthogonal projection from* $L^2(X,\mathcal{H})$ *onto* $\mathcal{M}$*.*

Proof. a) and b) are trivial. R is self-adjoint as an operator on $L^2(X,\mathcal{H})$, since for all $F, G \in L^2(X,\mathcal{H})$ we have

$$\begin{aligned}
< \mathrm{R}(F), G > &= \int_X < \mathrm{R}(F)(x), G(x) > d\mu(x) \\
&= \int_X < \int_X R(x,y)F(y)d\mu(y), G(x) > d\mu(x) \\
&= \int_X \int_X < R(x,y)F(y), G(x) > d\mu(y)d\mu(x) \\
&= \int_X \int_X < F(y), R(x,y)^* G(x) > d\mu(y)d\mu(x) \\
&= \int_X \int_X < F(y), R(y,x)G(x) > d\mu(y)d\mu(x) \\
&= \overline{\int_X \int_X < R(y,x)G(x), F(y) > d\mu(y)d\mu(x)} \\
&= \overline{\int_X \int_X < R(y,x)G(x), F(y) > d\mu(x)d\mu(y)} \\
&= \overline{\int_X < \int_X R(y,x)G(x)d\mu(x), F(y) > d\mu(y)} \\
&= \overline{\int_X < \mathrm{R}(G)(y), F(y) > d\mu(y)} \\
&= \overline{< \mathrm{R}(G), F >} = < F, \mathrm{R}(G) > .
\end{aligned}$$

For all $F \in L^2(X,\mathcal{H})$ we have $\mathrm{R}(F) \in \mathrm{Range}(\mathrm{R}) = \mathrm{Range}(V)$ then $\mathrm{R}(F) = Vg$, for some $g \in \mathcal{H}$ and hence

$$\mathrm{R}^2(F) = \mathrm{R}(\mathrm{R}(F)) = \mathrm{R}(Vg) = Vg = \mathrm{R}(F)$$

then $\mathrm{R}^2 = \mathrm{R}$ and

$$L^2(X,\mathcal{H}) = N(\mathrm{R}) \bigoplus \mathrm{Range}(\mathrm{R}).$$

$\square$

We assume in the following that $\|\Lambda_x\| \leq C$ for all $x \in X$. This implies $|Vf(x)| \leq C\|f\|$ and $|Wf| \leq C\|S^{-1}\|\|f\|$ for all $x \in X$ and, together with the weak continuity assumption, we conclude $Vf, Wf \in C^b(X, \mathcal{H})$ for all $f \in \mathcal{H}$, where $C^b(X, \mathcal{H})$ denotes the bonded continuous function of X to $\mathcal{H}$.

3. Coorbit Spaces

Associated to a g-continuous frames, there are Banach spaces called coorbit spaces where describe vectors in the Hilbert spaces of kernel functions. M. Fornasier and H. Rauhut [6] associated coorbit spaces to continuous frames. First,to built a weighted algebra, we need to introduce an special weight function.

Definition 3.1. Let m be a real weight function on $X \times X$. m is called admissible if,

a) m is continuous,
b) $1 \leq m(x, y) \leq m(x, z)m(z, y)$ for all $x, y, z \in X$,
c) $m(x, y) = m(y, x)$ for all $x, y \in X$,
d) $m(x, y) \leq C < \infty$ for all $x, y \in X$.

In order to get a weighted algebra we need to make a norm and a multiplication on kernel functions.

Proposition 3.2. *Let*

$$\mathcal{A}_1 := \{K\colon X \times X \to L(\mathcal{H}),\ K \textit{ is measurable}, \|K|\mathcal{A}_1\| < \infty\}$$

where

$$\|K|\mathcal{A}_1\| := \max\{\operatorname*{ess\,sup}_{x\in X} \int_X \|K(x, y)\|d\mu(y), \operatorname*{ess\,sup}_{y\in X} \int_X \|K(x, y)\|d\mu(x)\}$$

is its norm (the norm in integral is uniform norm) and the multiplication in $\mathcal{A}_1$ is given by

$$K_1 \circ K_2(x, y) = \int_X K_1(x, z)K_2(z, y)d\mu(z),$$

such that in weak sense

$$K_1 \circ K_2(x, y) : \mathcal{H} \to \mathcal{H}, \quad K_1 \circ K_2(x, y)f = \int_X K_1(x, z)K_2(z, y)fd\mu(z).$$

Then $\mathcal{A}_1$ with $\|.|\mathcal{A}_1\|$ and the multiplication is a Banach algebra.

Proof. Obviously $\|.|\mathcal{A}_1\|$ is a norm and the conditions of an algebra satisfy. We prove the associativity of multiplication and completeness of the norm. For all $f \in \mathcal{H}$ and $K_1, K_2, K_3 \in \mathcal{A}_1$ we have

$$\begin{aligned}[K_1 \circ (K_2 \circ K_3)](x, y)f &= \int_X K_1(x, z)(K_2 \circ K_3)(z, y)fd\mu(z) \\ &= \int_X K_1(x, z) \int_X K_2(z, t)K_3(t, y)fd\mu(t)d\mu(z)\end{aligned}$$

$$
\begin{aligned}
&= \int_X \int_X K_1(x,z)K_2(z,t)K_3(t,y)f d\mu(t)d\mu(z)\\
&= \int_X \int_X K_1(x,z)K_2(z,t)d\mu(z)K_3(t,y)f d\mu(t)\\
&= \int_X (K_1 \circ K_2)(x,t)K_3(t,y)f d\mu(t)\\
&= [(K_1 \circ K_2)\circ K_3](x,y)f,
\end{aligned}
$$

and then $K_1 \circ (K_2 \circ K_3) = (K_1 \circ K_2) \circ K_3$.

Finally $\|.|\mathcal{A}_1\|$ is Banach since if $\{K_n\}_{n=1}^{\infty}$ is Cauchy sequence in $\mathcal{A}_1$, then $\|K_n - K_m|\mathcal{A}_1\| \to 0$ as $m, n \to \infty$. Hence $\|K_n(x,y) - K_m(x,y)\|_{sup} \to 0$ as $m, n \to \infty$. Since $L(\mathcal{H})$ with uniform norm is Banach then there is $K \in L(\mathcal{H})$ such that $\|K_n(x,y)-K(x,y)\|_{sup} \to 0$ as $m, n \to \infty$ and hence $\|K_n-K|\mathcal{A}_1\| \to 0$ as $m, n \to \infty$. Therefore $\mathcal{A}_1$ is a Banach algebra. □

Now we define a corresponding weighted subalgebra respect to an admissible weight function m.

Proposition 3.3. *Let m be an admissible weight function and let,*

$$\mathcal{A}_m := \{K : X \times X \to L(\mathcal{H}), \quad Km \in \mathcal{A}_1\},$$

with the natural norm $\|K|\mathcal{A}_m\| := \|K_m|\mathcal{A}_1\|$. Then

a) *$\mathcal{A}_m$ is a Banach algebra,*

b) *For every $K \in \mathcal{A}_m$, corresponding operator K on $L^2(X,\mathcal{H})$ defined by*

$$K(F)(x) = \int_X K(x,y)F(y)d\mu(y),$$

is self adjoint.

Proof. Clearly, K is a linear operator on $L^2(X,\mathcal{H})$. For every $F, G \in L^2(X,\mathcal{H})$ we have

$$
\begin{aligned}
< K(F), G > &= \int_X < K(F)(x), G(x) > d\mu(x)\\
&= \int_X \int_X < K(x,y)F(y), G(x) > d\mu(y)d\mu(x)\\
&= \int_X \int_X < F(y), K^*(x,y)G(x) > d\mu(y)d\mu(x)\\
&= \int_X \int_X < F(y), K(y,x)G(x) > d\mu(y)d\mu(x)\\
&= \overline{\int_X \int_X < K(y,x)G(x), F(y) > d\mu(y)d\mu(x)}\\
&= \overline{\int_X < \int_X K(y,x)G(x)d\mu(x), F(y) > d\mu(y)}
\end{aligned}
$$

$$= \overline{\int_X < K(G)(y), F(y) > d\mu(y)}$$
$$= \overline{< K(G), F >} =< F, K(G) > .$$

□

A function space Y that satisfies some properties is other tool for definition coorbit spaces associated to g-continuous frames.

Definition 3.4. Let $(Y, \|.|Y\|)$ be a non-trivial Banach space of functions $F\colon X \to \mathcal{H}$ such that

1) Y is continuously embedded into $L^1_{loc}(X, \mathcal{H})$, where

$$L^1_{loc}(X, \mathcal{H}) := \{F\colon X \to \mathcal{H}, \int_K \|F(x)\| d\mu(x) < \infty \text{ for every compact subset } K \text{ of } X\},$$

2) If F is measurable and $G \in Y$ such that $\|F(x)\| \le \|G(x)\|$ a.e. then $F \in Y$ and $\|F|Y\| \le \|G|Y\|$.
3) There exists an admissible weight function m such that $\mathcal{A}_m(Y) \subset Y$ and

$$\|K(F)|Y\| \le \|K|\mathcal{A}_m\| \|F|Y\|$$

for all $K \in \mathcal{A}_m$, $F \in Y$, then $(Y, \|.|Y\|)$ is called an m-function space.

In the rest of this article, let $(Y, \|.|Y\|)$ with a weight function m be fixed. For fixed point $z \in X$ define a weight function on X by

$$\nu(x) := \nu_z(x) := m(x, z).$$

Now, we define the spaces

$$\mathcal{H}^1_\nu := \mathcal{H}^1_\nu(X, \mathcal{H}) := \{f \in \mathcal{H}, Vf \in L^1_\nu(X, \mathcal{H})\},$$
$$\mathcal{K}^1_\nu := \mathcal{K}^1_\nu(X, \mathcal{H}) := \{f \in \mathcal{H}, Wf \in L^1_\nu(X, \mathcal{H})\}$$

with natural norms

$$\|f|\mathcal{H}^1_\nu\| := \|Vf|L^1_\nu\|, \quad \|f|\mathcal{H}^1_\nu\| := \|Vf|L^1_\nu\|.$$

The frame operator S is an isometric isomorphism between $\mathcal{H}^1_\nu$ and $\mathcal{K}^1_\nu$.

Proposition 3.5. *The spaces $(\mathcal{H}^1_\nu, \|.|\mathcal{H}^1_\nu\|)$ and $(\mathcal{K}^1_\nu, \|.|\mathcal{K}^1_\nu\|)$ are Banach spaces.*

The proof is completely analogous to the proof of proposition 1 in [6] and hence omitted.

Now let R be in $\mathcal{A}_m$ and let $g \in \mathcal{H}$ then,

$$\|\Lambda^*_y g|\mathcal{K}^1_\nu\| = \int_X \|W(\Lambda^*_y g)(x)\| \nu(x) d\mu(x)$$
$$= \int_X \|\Lambda_x S^{-1} \Lambda^*_y g\| \nu(x) d\mu(x) \le \int_X \|R(x, y)\| \|g\| m(x, z) d\mu(x)$$

$$\leq \|g\| m(y,z) \int_X \|R(x,y)\| m(x,y) d\mu(x) \leq \|g\| \|R|\mathcal{A}_m\| \nu(y),$$

and similarly

$$\begin{aligned} \|S^{-1}\Lambda_y^* g|\mathcal{H}_\nu^1\| &= \int_X \|V(S^{-1}\Lambda_y^* g)(x)\| \nu(x) d\mu(x) \\ &= \int_X \|\Lambda_x S^{-1}\Lambda_y^* g\| \nu(x) d\mu(x) \\ &\leq \|g\| m(y,z) \int_X \|R(x,y)\| m(x,y) d\mu(x) \leq \|g\| \|R|\mathcal{A}_m\| \nu(y). \end{aligned}$$

Hence, $\Lambda_y^* g \in \mathcal{K}_\nu^1$ and $S^{-1}\Lambda_y^* g \in \mathcal{H}_\nu^1$ for all $y \in X$.

Now,we define the spaces

$$(\mathcal{H}_\nu^1)^\urcorner := \{f \colon \mathcal{H}_\nu^1 \to \mathcal{H},\ f \text{ is continuous and conjugat-linear}\},$$

$$(\mathcal{K}_\nu^1)^\urcorner := \{f \colon \mathcal{K}_\nu^1 \to \mathcal{H},\ f \text{ is continuous and conjugat-linear}\}.$$

Since $\Lambda_x^* g \in \mathcal{K}_\nu^1$ we may extend the transform V to $(\mathcal{K}_\nu^1)^\urcorner$ by

$$Vf(x) = V_g f(x) = f(\Lambda_x^* g) =< f, \Lambda_x^* g >, \quad f \in \mathcal{K}_\nu^1.$$

By the same argument, the transform W extends to $(\mathcal{H}_\nu^1)^\urcorner$ by

$$Wf(x) = W_g f(x) = f(S^{-1}\Lambda_x^* g) =< f, S^{-1}\Lambda_x^* g >, f \in \mathcal{H}_\nu^1.$$

We may also extend the operator S to an isometric isomorphism between $(\mathcal{K}_\nu^1)^\urcorner$ and $(\mathcal{H}_\nu^1)^\urcorner$ by $< Sf, g >=< f, Sg >$ for $f \in (\mathcal{K}_\nu^1)^\urcorner$ and $g \in \mathcal{H}_\nu^1$.

Definition 3.6. The coorbits of Y with respect the frame $\mathcal{F} = \{\Lambda_x\}_{x\in x}$ are defined as

$$\operatorname{Co} Y := \operatorname{Co}_g(\mathcal{F}, Y) := \{f \in (\mathcal{K}_\nu^1)^\urcorner, \quad Vf = V_g f \in Y\},$$

$$\widetilde{\operatorname{Co}} Y := \operatorname{Co}_g(\tilde{\mathcal{F}}, Y) := \{f \in (\mathcal{H}_\nu^1)^\urcorner, \quad Wf = W_g \in Y\}$$

with natural norm

$$\|f|\operatorname{Co} Y\| := \|Vf|Y\|, \quad \|f|\widetilde{\operatorname{Co}} Y\| := \|Wf|Y\|.$$

The operator S is an isometric isomorphism between $\operatorname{Co} Y$ and $\widetilde{\operatorname{Co}} Y$.

There are some results in what follows and their proofs are similar to correspond results in [6].

Proposition 3.7. *Suppose that $R(Y) \subset L^\infty_{\frac{1}{\nu}}(X, \mathcal{H})$. Then the following statements hold.*

a) *The spaces $(\operatorname{Co} Y, \|.|\operatorname{Co} Y\|)$ and $(\widetilde{\operatorname{Co}} Y, \|.|\widetilde{\operatorname{Co}} Y\|)$ are Banach spaces.*
b) *A function $F \in Y$ is of the form Vf (resp. Wf) for some $f \in \operatorname{Co} Y$ (resp. $\widetilde{Co} Y$) if and only if $F = R(F)$.*

c) *The map* $V\colon \mathrm{Co}\,Y \to Y$ *(resp.* $W : \widetilde{\mathrm{Co}}Y \to Y$*) establishes an isometric isomorphism between* $\mathrm{Co}\,Y$ *(resp.* $\widetilde{Co}Y$*) and the closed subspace* $R(Y)$ *of* Y.

Corollary 1. *If* Y *also is a Hilbert space and* $R(Y) \subset L^{\infty}_{\frac{1}{\nu}}(X, \mathcal{H})$ *then* CoY *and* $\widetilde{Co}Y$ *are Hilbert spaces.*

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