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ON BERWALD AND WAGNER MANIFOLDS

CSABA VINCZE

Abstract. Berwald and Wagner manifolds are two important classes of spaces in Finsler geometry. They are closely related to each other via the conformal change of the metric. After discussing the basic definitions and the elements of the theory we present general methods to construct examples of them.

1. Preliminaries

Let M be a connected differentiable manifold of dimension n. If U is a local coordinate neighbourhood with coordinate functions $u^1, \dots, u^n$ then we use notation $x^1, \dots, x^n, y^1, \dots, y^n$ for the induced coordinate functions on the tangent manifold TM.

Definition 1. A function $F\colon TM \to \mathbb{R}$ satisfying the conditions

(F1) $F(v) \geq 0$ and $F(v) = 0 \Leftrightarrow v = 0$,
(F2) F is smooth on the manifold $TM \setminus \{0\}$,
(F3) F is positively homogeneous of degree 1: $F(tv) = tF(v)$, for all $t > 0$,
(F4) the second order partial derivatives $g_{ij} := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ form the coefficients of an inner product at every point of $TM \setminus \{0\}$,

is a *fundamental function* on the manifold M. Manifolds equipped with a fundamental function are called *Finsler manifolds*. The metric with coefficients g_{ij} is the *Riemann-Finsler metric*, $E := \frac{1}{2}F^2$ is the *energy function*.

Remark 1. Riemannian manifolds are Finsler manifolds with quadratic energy functions.

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Remark 2. The fundamental function F restricted to a tangent space is closely related to the concept of a norm on a real vector space, so $F(v)$ may be called as the Finslerian norm or the Finslerian length of the tangent vector v.

Definition 2. A Finsler manifold M is called a *generalized Berwald manifold* if there exists a linear connection ∇ on M such that the parallel transport with respect to ∇ preserves the Finslerian norm of tangent vectors. *Wagner manifolds* are generalized Berwald manifolds with the special form

$$T = \frac{1}{2}\big(1 \otimes d\alpha - d\alpha \otimes 1\big)$$

of the torsion, where α is a smooth function on the manifold M. If the torsion is identically zero then we have a *Berwald manifold.*

Remark 3. In what follows we fix a manifold M as the base manifold. All objects are defined on M or the manifold $TM \setminus \{0\}$ unless otherwise stated. By a Finsler manifold we mean the base manifold M equipped with a fundamental function.

Definition 3. A conformal relation between two Riemann-Finsler metrics means that

$$\tilde{g}_v(w, z) = e^{2f(v)} g_v(w, z) \tag{1}$$

holds for any tangent vectors w, z and v with a common base point. By a conformal relation between two Finsler manifolds we mean that their Riemann-Finsler metrics are conformally related.

The relation

$$\tilde{F}(v) = e^{f(v)} F(v)$$

between the fundamental functions of two conformally related Finsler manifolds can easily be derived using the homogeneity property of the fundamental function. Calculating the coefficients of the Riemann-Finsler metric by the formula

$$g_{ij} := \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}$$

we may easily derive the following theorem of M. S. Knebelman [7].

Theorem 1. *The scale function between two conformally related Riemann-Finsler metrics depends only on the position.*

Therefore relation (1) reduces to

$$\tilde{g}_v(w, z) = e^{2f(p)} g_v(w, z), \tag{2}$$

where p is the common base point of the tangent vectors v, w and z. If the scale function is constant then the conformal change is *homothetic.* This is the trivial case. Berwald and Wagner manifolds are closely related to each other via the conformal change of the metric as the following theorem due to M. Hashiguchi and Y. Ichijyō [5] shows.

Theorem 2. *A Finsler manifold is Wagnerian if and only if it is conformal to a Berwald manifold.*

More precisely a conformal change

$$\tilde{g}_v(w,z) = e^{2f(p)} g_v(w,z)$$

results in a Berwald manifold with the fundamental function $\tilde{F}$ if and only if there exists a linear connection on the base manifold such that it preserves the Finslerian length of tangent vectors with respect to F and the torsion can be expressed by the formula

$$T = \frac{1}{2}\big(1 \otimes d\alpha - d\alpha \otimes 1\big) \quad \text{with} \quad \alpha = 2f.$$

2. Historical remarks

The notion of generalized Berwald manifolds (esp. Wagner manifolds) was introduced by V. Wagner [24] in 1946. The class of these manifolds is quite rich: Wagner himself showed that any two-dimensional Finsler manifold with cubic metric is a generalized Berwald manifold. Japanese and Hungarian geometers also have main contributions to the development of the theory. Some of them are M. Hashiguchi, Y. Ichijyō, M. Matsumoto, T. Aikou, S. Kikuchi, S. Bácsó, J. Szilasi, Sz. Szakál and Cs. Vincze.

The Japanese school of Finsler geometry has been dominated by Matsumoto and his theory of Finsler connections [9]. Having these ideas the first steps in the systematic treatment of generalized Berwald manifolds (esp. Wagner manifolds) were taken by Hashiguchi [3]. Together with Ichijyō, they succesfully connected the theory of Wagner manifolds with the conformal change of the metric. Basic formulas between the canonical data of conformally related Finsler manifolds were also formulated in [4]. Following Hashiguchi's work lots of geometers started to deal with special problems too. Nice results have been obtained for example on Finsler manifolds with (α,β) - metrics, see e.g. [2]. Another approach to the problem was elaborated by Hungarian geometers ([12], [13], [17] and [18]) using Grifone's connection theory, see also [14]. As an illustration of the activity of the recent research we summarize here some basic problems solved by Vincze in [19], [21] and [23] in the last few years.

I. *The question of the unicity*: how many essentially different ways are there to realize conformal equivalence

$$\tilde{F}_1 \quad \longleftarrow \quad F \quad \longrightarrow \quad \tilde{F}_2$$

of a Finsler manifold to a Berwald manifold. According to the transitivity of the conformal equivalence we can also ask whether there are two Berwald manifolds which are conformally equivalent (but not homothetic) to each other? This is Matsumoto's problem posed in 2001, see [10].

II. *A generalization of Matsumoto's problem.* It is well-known that Berwald spaces can be characterized by the vanishing of the mixed curvature tensor

$$\overset{\circ}{\mathbb{P}}(X,Y)Z := \mathbb{K}(hX, JY)JZ$$

of the canonical Berwald connection D, where

$$\mathbb{K}(X,Y)Z := D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

is the curvature of the Berwald connection in the usual sense, J is the vertical endomorphism and h is the canonical horizontal endomorphism of the Finsler manifold. A generalization of the problem above is to find *conformal changes of the metric such that the (not necessarily zero) mixed curvature tensor of the Berwald connection remains invariant.* If $n \geq 3$ then Finsler manifolds admitting such a conformal change of the metric must have a local product structure $N \times \mathbb{R}$ with the fundamental function F satisfying

$$\frac{1}{2}F^2(v,t) = k^*\left(\gamma_p(v,v) + \frac{k}{2}\sqrt{\gamma_p(v,v)}\ t + t^2\right)e^{2f(v,t)}, \tag{3}$$

where γ is a Riemannian metric on the manifold N and

$$f(v,t) = \frac{k}{\sqrt{16-k^2}}\left(\arctan\frac{1}{\sqrt{16-k^2}}\left(\frac{4t}{\sqrt{\gamma_p(v,v)}} + k\right) - \arctan\frac{k}{\sqrt{16-k^2}}\right)$$

with functions k^* and k depending only on the position. The case $n = 2$ is also discussed by Vattamány and Vincze [16].

Remark 4. Similar but not exactly the same Finslerian energies can be found in [1].

Definition 4. A function of the form (3) is called a (non-reversible) *Asanov-type Finslerian energy function.*

Using the special form of the fundamental function the following result can be proved, for the proof see [19].

Theorem 3. *The conformal equivalence between two Berwald manifolds must be homothetic unless they are Riemannian.*

Therefore we have the following unicity theorem of Wagner manifolds.

Theorem 4. *If there exists a linear connection with torsion*

$$T = \frac{1}{2}(1 \otimes d\alpha - d\alpha \otimes 1)$$

on the base manifold such that the parallel transport preserves the Finslerian norm of tangent vectors, then it is uniquely determined.

III. *The intrinsic characterization of Wagner manifolds.* It remains only to answer whether how we can check intrinsically the existence of a linear connection with semi-symmetric torsion

$$T = \frac{1}{2}\big(1 \otimes d\alpha - d\alpha \otimes 1\big)$$

such that the induced parallel transport preserves the Finslerian norm of tangent vectors. Alternatively, *how can we find intrinsically the scale function such that the resulting manifold is Berwaldian?* Concerning the two-dimensional conformality problem the first result due to Wagner [24], where the special apparatus of two-dimensional Finsler spaces such as the Berwald-frame, main scalar and Landsberg angle was applied, see also [8]. Further results with strange regularity conditions can be found in Kikuchi's paper [6]. The multidimensional problem is solved in [21] by giving a differential equation of the form

$$d\alpha = \text{ canonical data of the Finsler manifold}$$

such that the exterior derivative of the right hand side is a conformally invariant 2-form on the base manifold. This also gives a partial solution of Shen's open problem **36**: *find all conformal invariants of a Finsler metric...*, see http://www.math.iupui.edu/ zshen/Research/preprintindex.html. The key tool to solve the problem is an *associated Riemannian metric* on the base manifold constructed as follows. Choosing a local orientation, define the canonical oriented volume form

$$d\mu_p\ (z_1, \ldots, z_n)(v) := \pm\sqrt{det\ g_v(z_i, z_j)}$$

in the tangent spaces as Riemannian manifolds with the Riemann-Finsler metric. The right hand side is affected by the sign $+$ or $-$ according to the basis $z_1, \ldots, z_n$ belongs to the orientation or not. Integrating the Riemann-Finsler metric on the indicatrix hypersurface

$$I_p := \{v \in T_pM \mid F(v) = 1\ \}$$

with respect to the induced volume form μ_p we have a Riemannian metric

$$\gamma_p(X, Y) := \int_{I_p} g(X^v, Y^v);$$

it is called the *associated Riemannian metric.* The importance of the associated structure can be seen from two fundamental facts:

(A) Conformally equivalent Riemann-Finsler metrics have conformally equivalent associated Riemannian metrics.
(B) If the parallel transport induced by a linear connection ∇ preserves the Finslerian norm of tangent vectors then ∇ is metrical with respect to the associated Riemannian metric; for a proof see [20].

In what follows the objects labelled by the symbol $*$ are related to the associated Riemannian structure such as E_* - the Riemannian energy, h_* - the canonical horizontal distribution, S_* - the canonical spray associated with h_*. We also need the gradient of the function $\varphi := \ln E_* - \ln E$ with respect to the Riemann-Finsler metric. It will be denoted by $J\Theta$ because it must be a vertical vector field on the tangent manifold. Let us define the form ρ by the formula

$$\rho := \frac{d_{h_*}E}{E} - \frac{1}{2}\frac{S_*E}{E}\frac{d_J E_*}{E_*},$$

where d_{h_*} and d_J are the differential operators associated with the mappings h_* and the canonical vertical endomorphism J, respectively. Using the transformation formulas between the canonical objects of conformally equivalent Finsler manifolds it follows that

$$d_J\tilde{\rho} = d_J\rho + \frac{1}{2}d\alpha^v \wedge d_J\varphi,$$

where the conformal relation is given by

$$\tilde{g}_v(w,z) = e^{2f(p)}g_v(w,z) \quad \text{with} \quad \alpha := 2f.$$

Then we can express the exterior derivative of the function α as a difference

$$d\alpha = \frac{\eta}{\sigma} - \frac{\tilde{\eta}}{\tilde{\sigma}},$$

where

$$\sigma_p := \int_{I_p} g(J\Theta, J\Theta) \quad \text{and} \quad \eta_p(X) := \int_{I_p} d_J\rho\ (X^h, \Theta) - \frac{1}{2}\frac{SE_*}{E_*}\ X^v\varphi.$$

Taking the exterior derivative of both sides we obtain a *conformally invariant differential form*

$$\vartheta := \frac{1}{\sigma}\big(d\eta - \frac{1}{\sigma}d\sigma \wedge \eta\big)$$

on the base manifold and the main result can be formulated as follows.

Theorem 5. *A Finsler manifold is locally conformally equivalent to a Berwald manifold if and only if $\vartheta = 0$ and the parallel transport induced by the linear connection*

$$\bar{\nabla}_X Y := \nabla_*(X,Y) + \frac{1}{2\sigma}\big(\eta(Y)X - \gamma_*(X,Y)\eta^\sharp\big)$$

preserves the Finslerian norm of tangent vectors. Then the torsion is just

$$T = \frac{1}{2}(1 \otimes \frac{\eta}{\sigma} - \frac{\eta}{\sigma} \otimes 1)$$

where $\frac{\eta}{\sigma}$ has the local form $d\alpha$.

Proof. Note first that if the resulting manifold is Berwaldian then $\tilde{\rho} = 0$ and, consequently, $\tilde{\eta} = 0$ because the horizontal distribution of the Finsler manifold is just the same as that of the associated Riemannian space. Therefore we should solve the equation

$$d\alpha = \frac{\eta}{\sigma}.$$

The (local) solvability is guaranteed by the condition $\vartheta = 0$ because ϑ is just the exterior derivative of the right hand side. The condition for the uniquely determined metrical connection $\bar{\nabla}$ with respect to the associated Riemannian metric with prescribed torsion T guarantees that the resulting manifold is Berwaldian. □

Remark 5. In terms of conformally invariant differential forms the conditions of Theorem 5 take the form

$$\vartheta = 0 \quad \text{and} \quad \frac{1}{E}\, d_{\bar{h}} E = 0,$$

where $\bar{h}$ is the horizontal projector associated with the linear connection $\bar{\nabla}$.

3. Examples

I. Simple but important examples can be constructed in the class of Randers manifolds with a fundamental function of the form

$$F := F_* + \beta,$$

where F_* is a Riemannian fundamental function and β is a 1-form on the base manifold satisfying the condition

$$\sup \{\beta(v) \mid \gamma_*(v,v) = 1\} < 1.$$

It is well-known that a Randers manifold is a Berwald manifold if and only if β is parallel with respect to the Lévi-Civita connection of the Riemannian metric. The differential equation

$$(\nabla_* \beta)(X,Y) = \|\beta^\sharp\|^2 \gamma_*(X,Y) - \beta(X)\beta(Y) \tag{4}$$

characterizing the Wagnerian Randers manifold is more complicated (all of the operators is taken with respect to the Riemannian metric). It was found by Bácsó, Hashiguchi and Matsumoto [2]. The characterization of Riemannian manifolds admitting non-trivial solutions is due to Vincze [22].

Theorem 6. *The local structure of Riemannian spaces admitting the non-trivial solution $\beta := K^2 dt$ of equation (4) is a product $M = N \times \mathbb{R}$ equipped with the Riemannian metric*

$$\gamma_*(v,v) = e^{2K^2 t} \gamma(T\pi(v), T\pi(v)) + K^2 dt \otimes dt(v,v), \tag{5}$$

where γ is a Riemannian metric on the manifold N, $\pi\colon M \to N$ is the canonical projection and K is a real constant.

Taking $K<1$ we can consider the fundamental function

$$F := F_* + K^2 dt$$

on the product $N \times \mathbb{R}$ as the prototype of Wagnerian Randers manifold up to isometry.

II. Conformal flatness. After substitution $t = -\frac{1}{K^2}\log Ks$ we get that

$$\gamma_*(v,v) = \frac{1}{K^2 s^2}\big(\gamma(T\pi(v),T\pi(v)) + ds \otimes ds(v,v)\big) \quad \text{and} \quad \beta = -\frac{1}{s}ds \tag{6}$$

showing that there is only one possible candidate among Riemannian spaces of constant curvature admitting non-trivial solutions of (4): the upper half-space H^n with the metric

$$\gamma_*(v_p,v_p) := \frac{1}{K^2 p_n^2}\big(du^1 \otimes du^1 + \dots du^n \otimes du^n\big)(v,v).$$

The manifold H^n with the fundamental function

$$F(v_p) = \frac{1}{Kp^n}\sqrt{(v_1)^2 + \dots + (v_n)^2} - \frac{1}{p^n}v_p(u^n)$$

is Wagnerian. It is a conformally flat Finsler manifold because the conformal change

$$F \to \tilde{F}(v_p) := e^{\log p^n}F(v_p) = \frac{1}{K}\sqrt{(v_1)^2 + \dots + (v_n)^2} - v_p(u^n)$$

results in a Berwald manifold of zero curvature. Following Szabó's terminology in [11] the upper half-space H^n with the fundamental function F and $\tilde{F}$ is a *Bolyai-Lobatchewsky-Finsler manifold* and a *Hilbert type Bolyai-Lobatchewsky-Finsler manifold* with rectilinear geodesics, respectively. In terms of local coordinates we have the equation

$$(y^1)^2 + \dots + (y^{n-1})^2 + (1-K^2)(y^n - \frac{K^2}{1-K^2})^2 = K^2(1 + \frac{K^2}{1-K^2})$$

for the indicatrix hypersurfaces with respect to $\tilde{F}$. It can be easily seen that all of its intersections with the coordinate planes (y^i, y^n) is an ellipse with one of its foci at the origin.

III. Polyellipses [15] and polyellipsoids.

Definition 5. Let F_* be a Riemannian fundamental function on the manifold M. Tangent vectors $v_1,\dots,v_m$ form an *invariant system* with respect to ∇_* at a point p if

$$\varphi(v_i) \in \{v_1,\dots,v_m\} \quad (i=1,\dots,m)$$

for any element φ of the holonomy group at the point p.

The simplest examples are

(i) the singleton consisting of the zero vector at the point p,
(ii) the zero vector together with N_p and $-N_p$, where N is a covariantly constant vector field,

(iii) In case of a finite holonomy group $\{f_1, \dots, f_m\}$ we have an invariant system $f_1(v), \dots, f_m(v)$ for any tangent vector v.

Theorem 7. *Let F_* be a Riemannian fundamental function and suppose that ∇_* admits finite invariant systems of tangent vectors at some (and therefore all) point p of the base manifold. Then ∇_* is Berwald metrizable by a non-Riemannian fundamental function.*

Proof. Let $v_1, \dots, v_m$ be an invariant system at a single point p and consider a polyellipsoid with foci $v_1, \dots, v_m$. It is a level hypersurface of the function

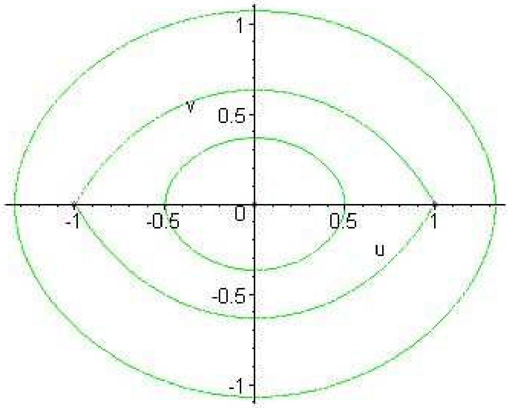

FIGURE 1. Polyellipses with three collinear foci in the plane.

$$\mathcal{P}(v) := d_*(v, v_1) + \cdots + d_*(v, v_m),$$

where d_* is the distance function induced by the inner product γ_* at the point p. Figure 1 illustrates an invariant system of type (ii). Let F_p be the function satisfying (F1)-(F4) at p with such a polyellipsoid as the indicatrix hypersurface. Since the foci form an invariant system with respect to ∇_*, the extension of F_p by parallel transport results in a well-defined fundamental function on the whole manifold. □

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Institute of mathematics,
University of Debrecen,
4010 Debrecen. P.O.Box 12,
Hungary
E-mail address: csvincze@math.klte.hu