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ON NONLINEAR CONNECTIONS IN HIGHER ORDER LAGRANGE SPACES

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Abstract. Considering a Lagrangian of order k, we determine a nonlinear connection N on T^kM such that the horizontal and vertical distributions to be Lagrangian subbundles for the presymplectic structure given by the Cartan-Poincaré two-form ω_L^k.

1. Introduction

We denote by (T^kM, π^k, M), $k \geq 1$, the space of tangent bundle of order k over a smooth, real, n-dimensional manifold M, [5]. Local coordinates (x^i) on M induce local coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i})$ on T^kM, where for a k-jet $j_0^k\rho \in T^kM$, the coordinate functions are defined as follows

$$y^{(\alpha)i}(j_0^k\rho) = \frac{1}{\alpha!}\frac{d^\alpha(x^i \circ \rho)}{dt^\alpha}\bigg|_{t=0},\ \alpha \in \{1, \dots, k\}.$$

The tangent structure of order k, J is defined as follows, [6],

$$J = \frac{\partial}{\partial y^{(1)i}} \otimes dx^i + \frac{\partial}{\partial y^{(2)i}} \otimes dy^{(1)i} + \dots + \frac{\partial}{\partial y^{(k)i}} \otimes dy^{(k-1)i}.$$

The foliated structure of T^kM allows for k regular, integrable, vertical distributions, $V_{k-\alpha+1} = \operatorname{Ker} J^\alpha = \operatorname{Im} J^{k-\alpha+1}$, $\alpha \in \{1, \dots, k\}$.

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The following k vertical vector fields are globally defined on T^kM and they are called Liouville vector fields:

$$\begin{aligned}\Gamma_k &= y^{(1)i}\frac{\partial}{\partial y^{(1)i}} + 2y^{(2)i}\frac{\partial}{\partial y^{(2)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k)i}},\\ \Gamma_\alpha &= J^{k-\alpha}\left(\Gamma_k\right),\ \alpha \in \{1,2,\ldots,k\}.\end{aligned}$$

A semispray is a globally defined vector field S on T^kM that satisfies the equation $JS = \Gamma_k$. Therefore, a semispray S, which is a vector field of order $k+1$, which can be expressed as follows

$$\begin{aligned}S = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}} + \cdots + ky^{(k)i}\frac{\partial}{\partial y^{(k-1)i}} -\\ -(k+1)G^i(x,y^{(1)},\ldots,y^{(k)})\frac{\partial}{\partial y^{(k)i}}\end{aligned} \tag{1.1}$$

and it is perfectly determined by its coefficient functions $G^i(x,y^{(1)},\ldots,y^{(k)})$.

A nonlinear connection, or a horizontal distribution on T^kM is a regular distribution $N\colon u \in T^kM \mapsto N(u) \subset T_uT^kM$ such that the following direct sum holds true:

$$T_u(T^kM) = N(u) \oplus N_1(u) \oplus \cdots \oplus N_{k-1}(u) \oplus V_k(u), \tag{1.2}$$

where $N_1 = J(N), N_{\alpha-1} = J^{\alpha-1}(N),\ \alpha \in \{3,\ldots,k\}$. The adapted basis to this decomposition is given by

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, \ldots, \frac{\delta}{\delta y^{(k-1)i}}, \frac{\partial}{\partial y^{(k)i}}\right\},$$

where, [7]:

$$\begin{aligned}\frac{\delta}{\delta x^i} &= \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j_i\frac{\partial}{\partial y^{(1)j}} - \cdots - \underset{(k)}{N}{}^j_i\frac{\partial}{\partial y^{(k)j}},\\ \frac{\delta}{\delta y^{(1)i}} &= \frac{\partial}{\partial y^{(1)i}} - \underset{(1)}{N}{}^j_i\frac{\partial}{\partial y^{(2)j}} - \cdots - \underset{(k-1)}{N}{}^j_i\frac{\partial}{\partial y^{(k)j}},\\ &\vdots\\ \frac{\delta}{\delta y^{(k-1)i}} &= \frac{\partial}{\partial y^{(k-1)i}} - \underset{(1)}{N}{}^j_i\frac{\partial}{\partial y^{(k)j}}.\end{aligned} \tag{1.3}$$

The functions $\underset{(1)}{N}{}^j_i, \underset{(2)}{N}{}^j_i, \ldots, \underset{(k)}{N}{}^j_i$ are called the local coefficients of the nonlinear connection N.

The dual basis of the previous adapted basis is given by $\left\{dx^i, \delta y^{(1)i}, \dots, \delta y^{(k)i}\right\}$, where:

$$
\begin{aligned}
\delta y^{(1)i} &= dy^{(1)i} + \underset{(1)}{M}{}^i{}_j dx^j, \\
\delta y^{(2)i} &= dy^{(2)i} + \underset{(1)}{M}{}^i{}_j dy^{(1)j} + \underset{(2)}{M}{}^i{}_j dx^j, \\
&\vdots \\
\delta y^{(k)i} &= dy^{(k)i} + \underset{(1)}{M}{}^i{}_j dy^{(k-1)j} + \dots + \underset{(k)}{M}{}^i{}_j dx^j.
\end{aligned}
\tag{1.4}
$$

The functions $\underset{(1)}{M}{}^j{}_i, \underset{(2)}{M}{}^j{}_i, \dots, \underset{(k)}{M}{}^j{}_i$ are called the dual coefficients of the nonlinear connection N.

2. Cartan-Poincaré forms for a higher order Lagrangian

Let us consider a regular Lagrangian of order k, $(k > 1)$, $L(x^i, y^{(1)i}, \dots, y^{(k)i})$. The metric tensor is given by the symmetric d-tensor field:

$$
g_{ij} = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}},
\tag{2.1}
$$

which has maximal rank on T^kM, $\operatorname{rank}||g_{ij}|| = n$.

For a regular Lagrangian of order k, we consider the following Cartan-Poincaré one-forms, [4]:

$$
\theta_L^\alpha = d_{J^\alpha}L = \frac{\partial L}{\partial y^{(\alpha)i}}dx^i + \dots + \frac{\partial L}{\partial y^{(k-\alpha)i}}dy^{(\alpha)i}, \quad \alpha \in \{1, \dots, k\}.
\tag{2.2}
$$

We consider also the following Cartan-Poincaré two-forms:

$$
\begin{aligned}
&\omega_L^\alpha = d\theta_L^\alpha = d\left(\frac{\partial L}{\partial y^{(\alpha)i}}\right) \wedge dx^i + \dots + d\left(\frac{\partial L}{\partial y^{(k-\alpha)i}}\right) \wedge dy^{(\alpha)i}, \\
&\alpha \in \{1, \dots, k\}.
\end{aligned}
\tag{2.3}
$$

We remark here that for $k > 1$, the regularity of the Lagrangian L implies the fact that $\operatorname{rank}(\omega_L^k) = 2n < (k+1)n = \dim(T^kM)$. We refer to ω_L^k as to the canonical presymplectic structure of the Lagrangian L.

3. Nonlinear connection

Now, our question is: Can we find an adapted basis such that the horizontal distribution to be Lagrangian subbundle for the presymplectic structure ω_L^k? Taking into account (1.3) this is equivalent with the determination of the coefficients of a nonlinear connection N.

As we know, [1], in the $k = 1$ case, we consider the canonical nonlinear connection and we have $\omega_L = 2g_{ji}\delta y^j \wedge dx^i$. It follows that in the basis $(dx^i, \delta y^i)$ of the cotangent bundle, the matrix of ω_L is

$$\begin{pmatrix} 0 & 2g_{ji} \\ -2g_{ji} & 0 \end{pmatrix}.$$

For $k > 1$, we consider the Cartan-Poincaré two-form:

$$\begin{aligned} \omega_L^k &= d\left(\frac{\partial L}{\partial y^{(k)i}}\right) \wedge dx^i \\ &= \frac{1}{2}\left[\frac{\delta}{\delta x^j}\left(\frac{\partial L}{\partial y^{(k)i}}\right) - \frac{\delta}{\delta x^i}\left(\frac{\partial L}{\partial y^{(k)j}}\right)\right] dx^j \wedge dx^i \\ &+ \frac{\delta}{\delta y^{(1)j}}\left(\frac{\partial L}{\partial y^{(k)i}}\right)\delta y^{(1)j} \wedge dx^i + \cdots \\ &+ \frac{\delta}{\delta y^{(k-1)j}}\left(\frac{\partial L}{\partial y^{(k)i}}\right)\delta y^{(k-1)j} \wedge dx^i \\ &+ 2g_{ji}\delta y^{(k)j} \wedge dx^i. \end{aligned} \tag{3.1}$$

We are looking for a nonlinear connection on T^kM such that the presymplectic structure of the lagrangian L to be: $\omega_L^k = 2g_{ji}\delta y^{(k)j} \wedge dx^i$, i.e. to have the following matrix:

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 2g_{ji} \\ 0 & & & & \\ \vdots & & O_{kn} & & \\ -2g_{ji} & & & & \end{pmatrix}_{(k+1)n\times(k+1)n}.$$

This is equivalent with the vanishes of all coefficients from (3.1), except the last one. We will obtain the coefficients of the nonlinear connection N.

We have:

$$\frac{\delta}{\delta y^{(k-1)j}}\left(\frac{\partial L}{\partial y^{(k)i}}\right) = 0 \quad \Longrightarrow \quad \frac{\partial^2 L}{\partial y^{(k-1)j}\partial y^{(k)i}} - \underset{(1)}{N}{}^m_j 2g_{mi} = 0$$

Therefore, we find the first coefficient for the nonlinear connection:

$$\underset{(1)}{N}{}^m_j = \frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(k-1)j}\partial y^{(k)i}}. \tag{3.2}$$

Now, we take $\dfrac{\delta}{\delta y^{(k-2)j}}\left(\dfrac{\partial L}{\partial y^{(k)i}}\right) = 0$ and we obtain the second coefficient of the nonlinear connection:

$$\underset{(2)}{N}{}^m_j = \frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(k-2)j}\partial y^{(k)i}} - \underset{(1)}{N}{}^r_j \frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(k-1)r}\partial y^{(k)i}}. \tag{3.3}$$

Finally, from $\dfrac{\delta}{\delta y^{(1)j}}\left(\dfrac{\partial L}{\partial y^{(k)i}}\right)=0$ we obtain the $(k-1)^{th}$ coefficient of the nonlinear connection N:

$$\begin{aligned}\underset{(k-1)}{N}{}^{m}_{j} &= \frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(1)j}\partial y^{(k)i}} - \underset{(1)}{N}{}^{r}_{j}\frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(2)r}\partial y^{(k)i}}\\ &\quad - \cdots - \underset{(k-2)}{N}{}^{r}_{j}\frac{1}{2}g^{mi}\frac{\partial^2 L}{\partial y^{(k-1)r}\partial y^{(k)i}}.\end{aligned} \tag{3.4}$$

Consequently, the coefficients $\underset{(1)}{N}{}^{m}_{j}, \underset{(2)}{N}{}^{m}_{j}, \dots, \underset{(k-1)}{N}{}^{m}_{j}$ are unique determined.

We have:

Theorem 3.1. *With respect to a transformation of the local coordinates* $\left(x^i, y^{(1)i}, \dots, y^{(k)i}\right) \longrightarrow \left(\widetilde{x}^i, \widetilde{y}^{(1)i}, \dots, \widetilde{y}^{(k)i}\right)$ *on* T^kM, *the coefficient of the nonlinear connection* N *are transformed by the rule:*

$$\begin{aligned}\underset{(1)}{\widetilde{N}}{}^{i}_{m}\frac{\partial\widetilde{x}^m}{\partial x^j} &= \frac{\partial\widetilde{x}^i}{\partial x^m}\underset{(1)}{N}{}^{m}_{j} - \frac{\partial\widetilde{y}^{(1)i}}{\partial x^j},\\ \underset{(2)}{\widetilde{N}}{}^{i}_{m}\frac{\partial\widetilde{x}^m}{\partial x^j} &= \frac{\partial\widetilde{x}^i}{\partial x^m}\underset{(2)}{N}{}^{m}_{j} + \frac{\partial\widetilde{y}^{(1)i}}{\partial x^m}\underset{(1)}{N}{}^{m}_{j} - \frac{\partial\widetilde{y}^{(2)i}}{\partial x^j},\\ &\ \vdots\\ \underset{(k-1)}{\widetilde{N}}{}^{i}_{m}\frac{\partial\widetilde{x}^m}{\partial x^j} &= \frac{\partial\widetilde{x}^i}{\partial x^m}\underset{(k-1)}{N}{}^{m}_{j} + \frac{\partial\widetilde{y}^{(1)i}}{\partial x^m}\underset{(k-2)}{N}{}^{m}_{j} + \cdots + \frac{\partial\widetilde{y}^{(k-2)i}}{\partial x^m}\underset{(1)}{N}{}^{m}_{j} - \frac{\partial\widetilde{y}^{(k-1)i}}{\partial x^j}.\end{aligned} \tag{3.5}$$

Proof. A transformation of local coordinates

$$\left(x^i, y^{(1)i}, \dots, y^{(k)i}\right) \longrightarrow \left(\widetilde{x}^i, \widetilde{y}^{(1)i}, \dots, \widetilde{y}^{(k)i}\right)$$

on the manifold T^kM is given by, [7]:

$$\begin{aligned}\widetilde{x}^i &= \widetilde{x}^i(x^1, \dots, x^n),\ \operatorname{rank}\left\|\frac{\partial\widetilde{x}^i}{\partial x^j}\right\| = n,\\ \widetilde{y}^{(1)i} &= \frac{\partial\widetilde{x}^i}{\partial x^j}\,y^{(1)j},\\ 2\widetilde{y}^{(2)i} &= \frac{\partial\widetilde{y}^{(1)i}}{\partial x^j}\,y^{(1)j} + 2\frac{\partial\widetilde{y}^{(1)i}}{\partial y^{(1)j}}\,y^{(2)j},\\ &\ \vdots\\ ky^{(k)i} &= \frac{\partial\widetilde{y}^{(k-1)i}}{\partial x^j}\,y^{(1)j} + 2\frac{\partial\widetilde{y}^{(k-1)i}}{\partial y^{(1)j}}\,y^{(2)j} + \cdots + k\frac{\partial\widetilde{y}^{(k-1)i}}{\partial y^{(k-1)j}}\,y^{(k)j}.\end{aligned}$$

Also, we have the identities, [7]:

$$\frac{\partial \widetilde{y}^{(\alpha)i}}{\partial x^j} = \frac{\partial \widetilde{y}^{(\alpha+1)i}}{\partial y^{(1)j}} = \cdots = \frac{\partial \widetilde{y}^{(k)i}}{\partial y^{(k-\alpha)j}}, \ (\alpha = 0, \ldots, k-1; y^{(0)i} = x^i).$$

With respect to the previous transformation of local coordinates, the natural basis is changed by the following rule:

$$\begin{aligned}
\frac{\partial}{\partial x^i} &= \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial}{\partial \widetilde{x}^j} + \frac{\partial \widetilde{y}^{(1)j}}{\partial x^i}\frac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \frac{\partial \widetilde{y}^{(k)j}}{\partial x^i}\frac{\partial}{\partial \widetilde{y}^{(k)j}},\\
\frac{\partial}{\partial y^{(1)i}} &= \frac{\partial \widetilde{y}^{(1)j}}{\partial y^{(1)i}}\frac{\partial}{\partial \widetilde{y}^{(1)j}} + \cdots + \frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(1)i}}\frac{\partial}{\partial \widetilde{y}^{(k)j}},\\
&\vdots\\
\frac{\partial}{\partial y^{(k)i}} &= \frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\frac{\partial}{\partial \widetilde{y}^{(k)j}}.
\end{aligned}$$

Taking into account last formulas, we have

$$\frac{\partial L}{\partial y^{(k)i}} = \frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\frac{\partial L}{\partial \widetilde{y}^{(k)j}}$$

and consequently, we obtain

$$\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k-1)m}} = \frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\frac{\partial \widetilde{y}^{(k-1)r}}{\partial y^{(k-1)m}}\frac{\partial^2 L}{\partial \widetilde{y}^{(k)j}\partial \widetilde{y}^{(k-1)r}} + \frac{\partial \widetilde{y}^{(k)r}}{\partial y^{(k-1)m}}\frac{\partial \widetilde{y}^{(k)j}}{\partial y^{(k)i}}\frac{\partial^2 L}{\partial \widetilde{y}^{(k)j}\partial \widetilde{y}^{(k)r}}.$$

But,

$$\frac{\partial \widetilde{y}^{(k)r}}{\partial y^{(k-1)m}} = \frac{\partial \widetilde{y}^{(1)r}}{\partial x^m} \text{ and } 2g_{jr} = \frac{\partial^2 L}{\partial \widetilde{y}^{(k)j}\partial \widetilde{y}^{(k)r}}.$$

Now, contracting by $\frac{1}{2}g^{ij}$, we obtain

$$\underset{(1)}{N}{}^j_m = \frac{\partial \widetilde{x}^j}{\partial x^i}\frac{\partial \widetilde{x}^r}{\partial x^m}\underset{(1)}{\widetilde{N}}{}^i_r + \frac{\partial \widetilde{y}^{(1)r}}{\partial x^m}\frac{\partial \widetilde{x}^j}{\partial x^r}$$

and finally,

$$\underset{(1)}{\widetilde{N}}{}^r_i \frac{\partial \widetilde{x}^i}{\partial x^m} = \underset{(1)}{N}{}^j_m \frac{\partial \widetilde{x}^r}{\partial x^j} - \frac{\partial \widetilde{y}^{(1)r}}{\partial x^m}.$$

Similarly, in order to check the second relation from (3.5), we calculate $\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k-2)m}}$, for the third relation we calculate $\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k-3)m}}$ and so on, for the last relation we calculate $\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(1)m}}$. □

Now, considering the following notations:

$$\underset{(\alpha)}{M}{}^r_j = \frac{1}{2}g^{ri}\frac{\partial^2 L}{\partial y^{(k-\alpha)j}\partial y^{(k)i}}, \quad \alpha \in \{1, \ldots, k-1\}, \tag{3.6}$$

we obtain:

Proposition 3.1. *The relationship between the coefficients* $\underset{(1)}{N}{}^m_i, \underset{(2)}{N}{}^m_i, \ldots, \underset{(k-1)}{N}{}^m_i$ *of the nonlinear connection N and the coefficents given in (3.6) is expressed by:*

$$\begin{aligned} \underset{(1)}{N}{}^m_i &= \underset{(1)}{M}{}^m_i \\ \underset{(2)}{N}{}^m_i &= \underset{(2)}{M}{}^m_i - \underset{(1)}{M}{}^m_r \underset{(1)}{N}{}^r_i \\ &\vdots \\ \underset{(k-1)}{N}{}^m_i &= \underset{(k-1)}{M}{}^m_i - \underset{(k-2)}{M}{}^m_r \underset{(1)}{N}{}^r_i - \cdots - \underset{(1)}{M}{}^m_r \underset{(k-2)}{N}{}^r_i . \end{aligned} \tag{3.7}$$

Indeed, replacing (3.6) in (3.2),(3.3) and (3.4) we have the conclusion. Therefore, the system of functions $\left\{ \underset{(1)}{M}{}^m_i, \underset{(2)}{M}{}^m_i, \ldots, \underset{(k-1)}{M}{}^m_i \right\}$ is the system of dual coefficients of the nonlinear connection N.

Example 3.1. Let $\mathcal{R} = (M, g_{ij}(x))$ be a Riemannian space and $\operatorname{Prol}^2 \mathcal{R}^n$ its prolongation of order 2, [8].

We consider the Liouville d-vector fields, [7]:

$$\begin{aligned} z^{(1)m} &= y^{(1)m}, \\ z^{(2)m} &= \frac{1}{2}\left[\Gamma z^{(1)m} + \gamma^m_{ij} z^{(1)i} z^{(1)j}\right], \end{aligned} \tag{3.8}$$

where $\gamma^m_{ij}(x)$ are the Christoffel symbols and the operator Γ is given by:

$$\Gamma = y^{(1)i}\frac{\partial}{\partial x^i} + 2y^{(2)i}\frac{\partial}{\partial y^{(1)i}}.$$

Considering the Lagrangian $L = g_{ij}(x) z^{(k)i} z^{(k)j}$, we remark that the first coefficient $\underset{(1)}{N}{}^m_i$ of our nonlinear connection is the same with the first coefficient from nonlinear connections determined by I. Bucataru, [2], M. de Leon, [4], and R. Miron, [7], i.e. is expressed by:

$$\underset{(1)}{N}{}^i_j = \underset{(1)}{M}{}^i_j = \gamma^i_{jh}(x) y^{(1)h}.$$

Now, for the last coefficient of the nonlinear connection $\underset{(k)}{N}{}^i_j$, we have:

Theorem 3.2. *The skew symmetric part of the coefficient* $\underset{(k)}{N}{}^i{}_j$ *is expressed as follows:*

$$
\begin{aligned}
\underset{(k)}{N}{}_{[ji]} &= \frac{1}{4}\left(\frac{\partial^2 L}{\partial x^i \partial y^{(k)j}} - \frac{\partial^2 L}{\partial x^j \partial y^{(k)i}}\right) \\
&\quad - \frac{1}{2} g^{sm} \sum_{\alpha=1}^{k-1} \left(\underset{(\alpha)}{N}{}_{js}\underset{(k-\alpha)}{M}{}_{im} - \underset{(\alpha)}{N}{}_{is}\underset{(k-\alpha)}{M}{}_{jm}\right).
\end{aligned}
\tag{3.9}
$$

where $\underset{(k)}{M}{}_{ji} = g_{im}\underset{(k)}{M}{}^m_j$.

Proof. For the last coefficient of the nonlinear connection $\underset{(k)}{N}{}^i{}_j$ we have to consider the first coefficient from (3.1):

$$
\frac{1}{2}\left[\frac{\delta}{\delta x^j}\left(\frac{\partial L}{\partial y^{(k)i}}\right) - \frac{\delta}{\delta x^i}\left(\frac{\partial L}{\partial y^{(k)j}}\right)\right] = 0
\tag{3.10}
$$

We obtain:

$$
\begin{aligned}
\underset{(k)}{N}{}_{[ji]} &= \frac{1}{2}\left(\underset{(k)}{N}{}_{ji} - \underset{(k)}{N}{}_{ij}\right) \\
&= \frac{1}{4}\left(\frac{\partial^2 L}{\partial x^i \partial y^{(k)j}} - \frac{\partial^2 L}{\partial x^j \partial y^{(k)i}}\right) \\
&\quad - \frac{1}{2}\left(\underset{(1)}{N}{}^m_j \frac{\partial^2 L}{\partial y^{(1)m}\partial y^{(k)i}} - \underset{(1)}{N}{}^m_i \frac{\partial^2 L}{\partial y^{(1)m}\partial y^{(k)j}}\right) \\
&\quad - \cdots - \frac{1}{2}\left(\underset{(k-1)}{N}{}^m_j \frac{\partial^2 L}{\partial y^{(k-1)m}\partial y^{(k)i}} - \underset{(k-1)}{N}{}^m_i \frac{\partial^2 L}{\partial y^{(k-1)m}\partial y^{(k)j}}\right),
\end{aligned}
\tag{3.11}
$$

where $\underset{(k)}{N}{}_{ji} = g_{im}\underset{(k)}{N}{}^m_j$.

The condition (3.10) determines uniquely the skew symmetric part of the coefficient $\underset{(k)}{N}{}^m_j$, only. □

For the symmetric part we need a supplementary condition. In the $k = 1$ case, I. Bucataru proved that the symmetric part is uniquely determined by a metric condition, [3].

So far, we have a whole family of nonlinear connections that are determined by the presymplectic structure ω_L^k. These connections are derived directly from the Lagrangian L and does not use the canonical semispray.

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