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DUAL CONNECTIONS IN FINSLER GEOMETRY

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Abstract. In the present paper, we generalize the notion of *statistical structure* and its dual connection in Riemannian geometry to Finsler geometry. We shall show that the Berwald connection D of a Finsler manifold is a statistical structure. In particular, as an application of this fact, we shall show that, if the hh-curvature of the Berwald connection D vanishes identically, then the given Finsler metric induces a Hessian metric on the base manifold.

1. Introduction

Let M be a connected smooth manifold of $\dim M = n$ with a (semi-) Riemannian metric h. A *statistical structure* in (M, h) is a symmetric linear connection D satisfying the *Codazzi equation* $(D_X h)(Y, Z) = (D_Y h)(X, Z)$ for all vector fields X, Y and Z on M, that is, Dh is totally symmetric. As is well-known, for an arbitrary symmetric $(0, 3)$-tensor field C, the $(1, 2)$-tensor field S defined by $h(S(X, Y), Z) = C(X, Y, Z)$ induces a statistical structure D on (M, h) by $D = \nabla - S/2$.

A linear connection D^* on (M, h) defined by

$$Xh(Y, Z) = h(D_X Y, Z) + h(Y, D^*_X Z)$$

is called the *dual connection* of D. If D is a statistical structure, then its dual D^* is also a statistical structure on (M, h), and we have the relation $\nabla = (D + D^*)/2$ for the Levi-Civita connection ∇ of (M, h) (cf. [2]). Recently the geometry of statistical structure becomes an interesting topics in differential geometry. In particular, the interest in statistical structure arises from the study of affine geometry and Hessian geometry (e.g., [7] and [10]). In [2], it is proved that the

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flatness of a statistical structure D implies the existence of a Hessian metric on the base manifold M.

Since Finsler geometry includes Riemannian geometry as a special case, it is natural to generalize the notions of statistical structure and the dual connection in Finsler geometry. The aim of the present paper is to define a *statistical structure* D and its *dual connection* D^* in a Finsler manifold.

In Finsler geometry, the Chern connection ∇ is a standard tool for studying Finsler manifolds, since ∇ satisfies the metrical condition and the symmetric property (see Definition 3.1 below). In particular, if the given metric is a Riemannian metric, then ∇ is just the Levi-Civita connection of the given Riemannian metric. In the second and third section, we shall review some fundamental facts in Finsler geometry from [1].

On the other hand, the Berwald connection D satisfies the symmetric property, but not the metrical condition. The Berwald connection D, however, plays an important role for some topics in Finsler geometry. In particular, we shall show that the Berwald connection is a statistical structure in this generalized sense (Theorem 5.1). In the last section, as an application of this fact, we shall show that, if the hh-curvature of the Berwald connection vanishes identically, then the base manifold M admits a flat statistical structure in the original sense, and thus M is a Hessian manifold (Theorem 6.1).

2. Finsler metrics and Cartan tensor

Let $\pi\colon TM \to M$ be the tangent bundle of a connected smooth manifold M. We denote by $v = (x, y)$ the points in TM if $y \in \pi^{-1}(x) = T_xM$. We introduce a coordinate system on TM as follows. Let $U \subset M$ be an open set with local coordinate $(x^1, \ldots, x^n)$. By setting $v = \sum y^i \left(\partial/\partial x^i\right)_x$ for every $v \in \pi^{-1}(U)$, we introduce a local coordinate $(x, y) = (x^1, \ldots, x^n, y^1, \ldots, y^n)$ on $\pi^{-1}(U)$.

Definition 2.1. A function $L\colon TM \longrightarrow \mathbb{R}$ is called a *Finsler metric* on M if

1. $L(x, y) \geq 0$, and $L(x, y) = 0$ if and only if $y = 0$,
2. $L(x, \lambda y) = \lambda L(x, y)$ for ${}^\forall\lambda \in \mathbb{R}^+ = \{\lambda \in \mathbb{R} :\ \lambda > 0\}$,
3. $L(x, y)$ is smooth on $TM^\times = TM\backslash\{0\}$

are satisfied. The pair (M, L) is called a *Finsler manifold.* For each $X \in T_xM$, its norm $\|X\|$ is defined by $\|X\| = L(x, X)$.

The differential π_* of the submersion $\pi\colon TM^\times \to M$ induces an exact sequence

$$0 \longrightarrow V \xrightarrow{\ i\ } T(TM^\times) \xrightarrow{\ \pi_*\ } \widetilde{TM} \longrightarrow 0, \tag{2.1}$$

where V is the *vertical subbundle* which is locally spanned by $\{\partial/\partial y^j\}_{j=1,\ldots,n}$ on $\pi^{-1}(U)$, and $\widetilde{TM} = \{(y, v) \in TM^\times \times TM \,\big|\, v \in T_{\pi(y)}M\}$ is the pullback bundle of TM by π_*.

Since the natural local frame field $\{\partial/\partial x^i\}_{i=1,\dots,n}$ on U is identified with the one of $\widetilde{TM}$ on $\pi^{-1}(U)$, we use the notation $X = \sum(\partial/\partial x^i) \otimes X^i$ for a section X of $\widetilde{TM}$. Furthermore, since $\ker \pi_* = V$, the morphism π_* is given by $\pi_* = \sum(\partial/\partial x^i) \otimes dx^i$.

On the other hand, since $\widetilde{TM}$ is naturally identified with $V \cong \ker \pi_*$, any section X of $\widetilde{TM}$ is considered as a section of V. We denote by X^V the section of V corresponding to $X \in \Gamma(\widetilde{TM})$:

$$X = \sum \frac{\partial}{\partial x^i} \otimes X^i \quad \Longleftrightarrow \quad X^V = \sum \frac{\partial}{\partial y^i} \otimes X^i.$$

A Finsler metric L is said to be *convex* if $F = L^2/2$ is *strictly convex* on each tangent space T_xM, that is, the Hessian (G_{ij}) defined by

$$G_{ij}(x, y) = \frac{\partial^2 F}{\partial y^i \partial y^j} \tag{2.2}$$

is positive-definite. In the sequel, we assume the convexity of L. Then $\widetilde{TM}$ admits a metric G defined by $G(X, Y) = \sum G_{ij} X^i Y^j$ for all $X = \sum(\partial/\partial x^i) \otimes X^i$ and $Y = \sum(\partial/\partial x^j) \otimes Y^j$. We also set

$$C_{ijk} = \frac{1}{2}\frac{\partial G_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 L^2}{\partial y^i\, \partial y^j\, \partial y^k}.$$

Then we define a symmetric tensor field $C\colon \otimes^3 \widetilde{TM} \to \mathbb{R}$ by

$$C(X, Y, Z) = \sum C_{ijk} X^i Y^j Z^k \tag{2.3}$$

for all sections X, Y, Z of $\widetilde{TM}$. It is trivial C vanishes identically if and only if G is a Riemannian metric on M. This tensor field C is called the *Cartan tensor field*.

The multiplier group $\mathbb{R}^+ \cong \{cI \in GL(TM^\times);\ c \in \mathbb{R}^+\} \subset GL(TM^\times)$ acts on the total space by multiplication $m_\lambda\colon TM^\times \ni v = (x, y) \to \lambda v = (x, \lambda y) \in TM^\times$ for $^\forall\lambda \in \mathbb{R}^+$. This action induces a canonical section $\mathcal{E}$ of V defined by $\mathcal{E}(v) = (v, v)$ for all $v \in TM$. We shall consider $\mathcal{E}$ as a section of $\widetilde{TM}$, and we denote it by the same notation $\mathcal{E}$. This section $\mathcal{E}$ is called the *tautological section* of $\widetilde{TM}$. Then it is easily shown that $L = \sqrt{G(\mathcal{E}, \mathcal{E})}$ and

$$C(\mathcal{E}, \cdot, \cdot) \equiv 0. \tag{2.4}$$

3. Chern connection

The vertical subbundle V of the submersion $\pi\colon TM^\times \to M$ is uniquely determined. A subbundle $H \subset T(TM^\times)$ complementary to V is called a *horizontal subbundle*:

$$T(TM^\times) = V \oplus \widetilde{TM} \cong V \oplus H. \tag{3.1}$$

An Ehresmann connection of π is a selection of horizontal subbundles. An Ehresmann connection is given by a $\widetilde{TM}$-valued 1-form θ satisfying

$$\theta(X^V) = X \tag{3.2}$$

for every section X of $\widetilde{TM}$. If an Ehresmann connection θ is given, the subbundle $H := \ker\theta$ is a horizontal subbundle. In the sequel of the present paper, we shall denote by $A^k(\widetilde{TM})$ the space of smooth $\widetilde{TM}$-valued k-forms on $TM^\times$.

Since we are concerned with the tangent bundle, the bundle $\widetilde{TM}$ is also naturally identified with the horizontal subbundle H, and any section $X \in A^0(\widetilde{TM})$ is considered as a section of H. We denote by X^H the section of H corresponding to $X \in A^0(\widetilde{TM})$:

$$X = \sum \frac{\partial}{\partial x^i} \otimes X^i \quad \Longleftrightarrow \quad X^H = \sum \frac{\delta}{\delta x^i} \otimes X^i,$$

where we set $\{\delta/\delta x^1, \dots, \delta/\delta x^n\} = (\partial/\partial x^i)^H$. By the definitions above, we have

$$\pi_*(X^H) = X, \;\; \pi_*(X^V) = 0 \tag{3.3}$$

and

$$\theta(X^H) = 0, \;\; \theta(X^V) = X \tag{3.4}$$

for every $X \in A^0(\widetilde{TM})$.

According to the decomposition (3.1), we get the splitting $d = d^V \oplus d^H$ of the differential operator d on $TM^\times$:

$$d^H f(X) = X^H(f) \quad \text{and} \quad d^V f(Y) = Y^V(f)$$

for every $f \in C^\infty(TM^\times)$. Also any covariant exterior derivation $\nabla\colon A^k(\widetilde{TM}) \to A^{k+1}(\widetilde{TM})$ on $\widetilde{TM}$ has the splitting $\nabla = \nabla^H \oplus \nabla^V$:

$$\nabla^H_X Y = \nabla_{X^H} Y \quad \text{and} \quad \nabla^V_X Y = \nabla_{X^V} Y$$

for all $X, Y \in A^0(\widetilde{TM})$ respectively.

3.1. **Definition of Chern connection.** In this subsection, we shall recall the definition of Chern connection from [4] and [1].

Definition 3.1. The *Chern connection* on (M, L) is a covariant exterior derivation $\nabla\colon A^k(\widetilde{TM}) \to A^{k+1}(\widetilde{TM})$ determined by the following conditions.

(1) ∇ is *symmetric*:

$$\nabla \pi_* = 0. \tag{3.5}$$

(2) ∇ is *almost G-compatible*:

$$\nabla^H G = 0, \tag{3.6}$$

where we take the Ehresmann connection θ defined by

$$\theta = \nabla \mathcal{E}. \tag{3.7}$$

Remark 3.1. From the assumption (3.5), we can easily show that the $\widetilde{TM}$-valued 1-form θ defined by (3.7) is an Ehresmann connection. In [4] or [8], this θ is called the *Cartan's non-linear connection.* It is known that θ defined by (3.7) is uniquely obtained from the given Finsler metric L. The definition (3.7) of θ and the homogeneity of L gives

$$\nabla^H_X \mathcal{E} = 0 \tag{3.8}$$

for every $X \in A^0(\widetilde{TM})$. This equation means that the horizontal subbundle $H = \ker\theta$ is invariant by the action $m_\bullet$ of $\mathbb{R}^+$. □

The assumption (3.5) is equivalent to

$$\nabla^H_X Y - \nabla^H_Y X - \pi_*[X^H, Y^H] = 0 \tag{3.9}$$

and

$$\nabla^V_Y X - \pi_*[Y^V, X^H] = 0, \tag{3.10}$$

and the assumption (3.6) is equivalent to

$$d^H G(X,Y) = G(\nabla^H X, Y) + G(X, \nabla^H Y) \tag{3.11}$$

for all $X, Y \in A^0(\widetilde{TM})$.

3.2. Torsion T_∇ and curvature R_∇ of ∇.

In this subsection, we shall recall the definitions of torsion and curvature of the Chern connection ∇ defined in the previous subsection. We also recall some propositions concerned with torsion and curvature (cf. [1]).

Definition 3.2. The *torsion* $T_\nabla \in A^2(\widetilde{TM})$ of ∇ is defined by

$$T_\nabla = \nabla\theta. \tag{3.12}$$

Because of (3.3), we obtain $T_\nabla(X^V, Y^V) = 0$ for all $X, Y \in A^0(\widetilde{TM})$. If we define $T^{HH}_\nabla(X,Y) := T_\nabla(X^H, Y^H)$ and $T^{HV}_\nabla(X,Y) := T_\nabla(X^H, Y^V)$, then (3.3) and (3.4) give

$$T^{HH}_\nabla(X,Y) = -\theta[X^H, Y^H] \tag{3.13}$$

and

$$T^{HV}_\nabla(X,Y) = \nabla^H_X Y - \theta[X^H, Y^V] \tag{3.14}$$

for all $X, Y \in A^0(\widetilde{TM})$. The following is easily obtained.

Proposition 3.1. *The horizontal part T^{HH}_∇ and the mixed part T^{HV}_∇ satisfy*

$$T^{HH}_\nabla(X,Y) + T^{HH}_\nabla(Y,X) \equiv 0 \tag{3.15}$$

and

$$T^{HV}_\nabla(X,Y) - T^{HV}_\nabla(Y,X) \equiv 0 \tag{3.16}$$

for all $X, Y \in A^0(\widetilde{TM})$. *Furthermore the mixed part* T^{HV}_{∇} *satisfies the following*

$$T^{HV}_{\nabla}(X, \mathcal{E}) = 0. \tag{3.17}$$

Definition 3.3. The *curvature* $R_{\nabla} \in A^2(\mathrm{End}(\widetilde{TM}))$ of ∇ is defined by

$$R_{\nabla} = \nabla^2. \tag{3.18}$$

Similarly to the torsion T_{∇}, we obtain $R(X^V, Y^V) \equiv 0$ for all $X, Y \in A^0(\widetilde{TM})$. We set $R^{HH}_{\nabla}(X, Y)Z := R(X^H, Y^H)Z$ and $R^{HV}_{\nabla}(X, Y)Z := R(X^H, Y^V)Z$. We list up some identities concerning with R_{∇}.

The symmetry assumption (3.5) gives

Proposition 3.2. *The horizontal part* R^{HH}_{∇} *and the mixed part* R^{HV}_{∇} *satisfy the followings:*

$$R^{HH}_{\nabla}(X, Y)Z + R^{HH}_{\nabla}(Y, Z)X + R^{HH}_{\nabla}(Z, X)Y \equiv 0, \tag{3.19}$$

$$R^{HV}_{\nabla}(X, Y)Z - R^{HV}_{\nabla}(Z, Y)X \equiv 0. \tag{3.20}$$

The definition of T_{∇} implies $T_{\nabla} = R_{\nabla}\mathcal{E}$. Thus we get

Proposition 3.3. *The curvature* R_{∇} *and the torsion* T_{∇} *satisfies the relation* $T_{\nabla} = R_{\nabla}\mathcal{E}$*:*

$$R^{HH}_{\nabla}(X, Y)\mathcal{E} = T^{HH}_{\nabla}(X, Y), \tag{3.21}$$

$$R^{HV}_{\nabla}(X, Y)\mathcal{E} = T^{HV}_{\nabla}(X, Y). \tag{3.22}$$

The almost G-compatibility assumption (3.6) gives

Proposition 3.4. *The curvature* R_{∇} *and the torsion* T_{∇} *satisfy the followings:*

$$G(R^{HH}_{\nabla}(X, Y)Z, W) + G(R^{HH}_{\nabla}(X, Y)W, Z) + 2C(T^{HH}_{\nabla}(X, Y), Z, W) = 0 \tag{3.23}$$

$$\begin{aligned} G(R^{HV}_{\nabla}(X, Y)Z, W) \quad &+ \quad G(Z, R^{HV}_{\nabla}(X, Y)W) + \\ &+ \quad 2(\nabla^H_X C)(Y, Z, W) + 2C(T^{HV}_{\nabla}(X, Y), Z, W) = 0 \end{aligned} \tag{3.24}$$

This last identity gives

$$R^{HV}_{\nabla}(X, \mathcal{E}) = 0 \tag{3.25}$$

4. Dual connections

In this section, we shall introduce the notion of dual connection in Finsler geometry. Let (M, L) be a Finsler manifold, and θ the Ehresmann connection defined by (3.7). Let $D = D^H \oplus d^V$ be a symmetric Finsler connection satisfying

$$D\mathcal{E} = \theta. \tag{4.1}$$

Under this assumption, the horizontal part T^{HH}_D of the torsion $T_D = D\theta$ coincides with T^{HH}_{∇}, and therefore we denote it by T^{HH} in the sequel. Furthermore, similarly to the case of Riemannian geometry[10], we call D a *statistical structure* of $(\widetilde{TM}, G)$ if

$$(D^H_X G)(Y,Z) = (D^H_Y G)(X,Z) \tag{4.2}$$

is satisfied for all $X, Y, Z \in A^0(\widetilde{TM})$.

Definition 4.1. A Finsler connection $D^* = D^{*H} \oplus d^V$ is called the *dual connection* of a statistical structure D if D^* satisfies

$$X^H G(Y,Z) = G(D^H_X Y, Z) + G(Y, D^{*H}_X Z) \tag{4.3}$$

for all $X, Y, Z \in A^0(\widetilde{TM})$.

Proposition 4.1. *The dual connection D^* of a statistical structure D is symmetric.*

Proof. It is trivial that $(D^H_X G)(Y,Z)$ is symmetric in Y and Z, and thus we have

$$\begin{aligned}
(D^H_X G)(Y,Z) &= X^H G(Y,Z) - G(D^H_X Y, Z) - G(Y, D^H_X Z) \\
&= X^H G(Z,Y) - G(D^H_X Z, Y) - G(Z, D^H_X Y) \\
&= (G(D^H_X Z, Y) + G(Z, D^{*H}_X Y)) - G(D^H_X Z, Y) - G(Z, D^H_X Y) \\
&= G(Z, D^{*H}_X Y) - G(Z, D^H_X Y)
\end{aligned}$$

and $(D^H_Y G)(X,Z) = G(Z, D^{*H}_Y X) - G(Z, D^H_Y X)$. Consequently we have

$$\begin{aligned}
&(D^H_X G)(Y,Z) - (D^H_Y G)(X,Z) \\
&= G(Z, D^{*H}_X Y) - G(Z, D^H_X Y) - G(Z, D^{*H}_Y X) + G(Z, D^H_Y X) \\
&= G(Z, D^{*H}_X Y - D^{*H}_Y X) - G(Z, D^H_X Y - D^H_Y X) \\
&= G(Z, D^{*H}_X Y - D^{*H}_Y X) - G(Z, \pi_*[X^H, Y^H]) \\
&= G(Z, D^{*H}_X Y - D^{*H}_Y X - \pi_*[X^H, Y^H]) \\
&= G(Z, (D^* \pi_*)(X^H, Y^H))
\end{aligned}$$

for all $X, Y, Z \in A^0(\widetilde{TM})$. Since G is nondegenerate, we have shown that D^* is symmetric if and only if D is a statistical structure. $\square$

From the definition (4.3), the following proposition is obtained immediately.

Proposition 4.2. *Let D^* be the dual connection of a statistical structure D. Then we have*

$$D^H G + D^{*H} G = 0, \tag{4.4}$$

and therefore the dual D^* is also a statistical structure of $(\widetilde{TM}, G)$.

Form Proposition 4.1 and 4.2, for a statistical structure D, the covariant exterior derivation $(D + D^*)/2$ satisfies the symmetric condition (3.5) and the almost G-compatibility (3.6), and thus the uniqueness of Chern connection ∇ gives $\nabla = (D + D^*)/2$. Therefore, we have

Theorem 4.1. *Let D^* be the dual connection of a statistical structure D of $(\widetilde{TM}, G)$. Then the Chern connection ∇ of $(\widetilde{TM}, G)$ is given by*

$$\nabla = \frac{1}{2}(D + D^*) \tag{4.5}$$

Furthermore the dual of D^* coincides with D, that is, $D^{**} = D$.

Let $R_{D^*} = D^{*2}$ be the curvature of the dual connection D^*. We investigate the relation between the curvatures R_D^{HH} and $R_{D^*}^{HH}$.

Theorem 4.2. *Let D^* be the dual connection of a statistical structure D of $(\widetilde{TM}, G)$.*

(1) *The horizontal part R_D^{HH} and $R_{D^*}^{HH}$ of D and D^* are related by*

$$G(R_D^{HH}(X, Y)Z, W) + G(Z, R_{D^*}^{HH}(X, Y)W) + 2C(T^{HH}(X, Y), Z, W) = 0. \tag{4.6}$$

(2) *The dual connection D^* satisfies $R_{D^*}^{HH} = 0$ if and only if D satisfies $R_D^{HH} = 0$.*

Proof. From (4.2), we have

$$\begin{aligned}
&[X^H, Y^H]G(Z, W)\\
&= [X^H, Y^H]^H G(Z, W) + [X^H, Y^H]^V G(Z, W)\\
&= G(D_{[X^H, Y^H]^H} Z, W) + G(Z, D^*_{[X^H, Y^H]^H} W) + [X^H, Y^H]^V G(Z, W)
\end{aligned}$$

for all $X, Y, Z, W \in \Gamma(\widetilde{TM})$. From (3.13) we note that

$$\begin{aligned}
&[X^H, Y^H]^V G(Z, W)\\
&= (D_{[X^H, Y^H]^V} G)(Z, W) + G(D_{[X^H, Y^H]^V} Z, W) + G(Z, D_{[X^H, Y^H]^V} W)\\
&= -2C(T^{HH}(X, Y), Z, W) + G(D_{[X^H, Y^H]^V} Z, W) + G(Z, D^*_{[X^H, Y^H]^V} W),
\end{aligned}$$

since $D^V = D^{*V} = d^V$. Hence we obtain

$$\begin{aligned}
[X^H, Y^H]G(Z, W) &= -2C(T^{HH}(X, Y), Z, W)\\
&\quad + G(D_{[X^H, Y^H]} Z, W) + G(Z, D^*_{[X^H, Y^H]} W).
\end{aligned}$$

On the other hand

$$\begin{aligned}
X^H Y^H G(Z, W) &= G(D_X^H D_Y^H Z, W) + G(D_Y^H Z, D_X^{*H} W)\\
&\quad + G(D_X^H Z, D_Y^{*H} W) + G(Z, D_X^{*H} D_Y^{*H} W)
\end{aligned}$$

implies

$$[X^H, Y^H]G(Z, W) = G(D_X^H D_Y^H Z - D_Y^H D_X^H Z, W) \\ + G(Z, D_X^{*H} D_Y^{*H} W - D_Y^{*H} D_X^{*H} W),$$

and therefore we obtain (4.6).

We suppose $R_D^{HH} = 0$ (resp. $R_{D^*}^{HH} = 0$). Then (4.1) implies

$$T^{HH}(X, Y) = R_D^{HH}(X, Y)\mathcal{E} = R_{D*}^{HH}(X, Y)\mathcal{E} = 0$$

for all $X, Y \in \Gamma(\widetilde{TM})$. Hence we obtain $R_{D^*}^{HH} = 0$ (resp. $R_D^{HH} = 0$) from (4.6). □

5. Berwald connection

There exists another canonical Finsler connection which plays an important role for some topics in Finsler geometry. For the Ehresmann connection θ by (3.7), the Finsler connection $D = d^V \oplus D^H$ on $\widetilde{TM}$ defined by

$$D_X^H Y = \theta[X^H, Y^V] \tag{5.1}$$

is called the *Berwald connection* in a Finsler manifold (M, L). From the equation (3.14) and the definition (5.1), we obtain a relation between the Chern connection ∇ and the Berwald connection D:

$$D_X^H Y = \nabla_X^H Y - T_\nabla^{HV}(X, Y) \tag{5.2}$$

Then, from (4.7) and (4.17), it follows that D defined by (5.1) satisfies (4.1), and thus D is a Finsler connection in the sense of previous section. Furthermore D defined by (5.1) symmetric, namely, $D\pi_* = 0$. In fact, from (3.14) and (3.16) we have

$$\begin{aligned}(D\pi_*)(X^H, Y^H) &= D_X^H Y - D_Y^H X - \pi_*[X^H, Y^H] \\ &= \nabla_X^H Y - T^{HV}(X, Y) - \nabla_Y^H X + T_\nabla^{HV}(Y, X) - \pi_*[X^H, Y^H] \\ &= 0\end{aligned}$$

and $(D\pi_*)(X^V, Y^H) = D_X^V Y - \pi_*[X^V, Y^H] = 0$, since $D^V = d^V$.

The almost G-compatibility and the relation (4.2) induce the following

$$(D_X^H G)(Y, Z) = G(T_\nabla^{HV}(X, Y), Z) + G(Y, T_\nabla^{HV}(X, Z)). \tag{5.3}$$

On the other hand, if we set $X = \mathcal{E}$ in (3.24), then (3.20) and (3.22) give

$$G(T_\nabla^{HV}(Z, Y), W) + G(Z, T_\nabla^{HV}(W, Y)) + 2(\nabla_\mathcal{E}^H C)(Y, Z, W) = 0,$$

and therefore (5.3) is equivalent to

$$(D_X^H G)(Y, Z) = -2(\nabla_\mathcal{E}^H C)(X, Y, Z), \tag{5.4}$$

and from (4.4)

$$(D_X^{*H} G)(Y, Z) = 2(\nabla_\mathcal{E}^H C)(X, Y, Z). \tag{5.5}$$

for all $X, Y, Z \in A^0(\widetilde{TM})$. Thus Proposition 4.1 implies

Theorem 5.1. *The Berwald connection D and its dual connection D^* are statistical structures on the Finsler bundle $(\widetilde{TM}, G)$.*

A Finsler manifold (M, L) is called a *Landsberg space* if the Berwald connection D coincides with the Chern connection ∇. In this case, from (4.5), we have

Proposition 5.1. *If (M, L) is a Landsberg space, then the Berwald connection D and its dual D^* coincide with the Chern connection ∇.*

6. Finsler manifolds satisfying $R_D^{HH} = 0$

The class of Berwald space which is characterized by $R_\nabla^{HV} = 0$ or $R_D^{HV} = 0$ has been studied by [11], and the classification of Berwald spaces is obtained. If a Finsler manifold (M, L) satisfies $R_\nabla^{HH} \equiv 0$, then the metric G on $\widetilde{TM}$ induces a flat Riemannian metric on M, and thus M is locally Euclidean (cf.[1]).

On the other hand, the class of Finsler manifolds satisfying $R_D^{HH} \equiv 0$ has not been studied enough yet. In the sequel, we shall show that, if a Finsler manifold (M, L) satisfies $R_D^{HH} = 0$, then the metric L induces a Hessian metric g on M. Here a Riemannian manifold (M, g) is said to be a *Hessian manifold* if the following conditions are satisfied (cf. [10]).

(1) There exists a flat affine connection $\underline{D}$ on M,
(2) the metric $g = \sum g_{ij} dx^i \otimes dx^j$ is given by the Hessian of some function ψ with respect to the affine coordinate $(x^1, \dots, x^n)$ of $\underline{D}$, that is, g is given by the covariant derivative $\underline{D} d\psi$ of the 1-from $d\psi$:

$$g_{ij} = \frac{\partial^2 \psi}{\partial x^i \partial x^j}.$$

We suppose that $R_D^{HH} = 0$. Then, the Ricci identity $T_D = R_D \mathcal{E}$ gives $T^{HH} = 0$, and thus the horizontal subbundle H is integrable. Hence there exists a section $v \colon M \to TM^\times$ satisfying $v^*\theta = 0$. Such a section v is called a *horizontal section* (cf. [1]). Since v satisfies $v^* \circ d^V = 0$, we obtain

$$d \circ v^* = v^* \circ d^H. \tag{6.1}$$

Lemma 6.1. *Suppose that the horizontal part R_D^{HH} of the Berwald connection D vanishes identically. Then the induced connection v^*D by a horizontal section v is a flat affine connection on M.*

Proof. We denote by ω_j^i the connection form of D with respect to the local frame field $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$. If we set

$$R_D^{HH} \frac{\partial}{\partial x^j} = \sum \frac{\partial}{\partial x^i} \otimes \Omega_j^i,$$

the curvature form Ω^i_j is given by $\Omega^i_j = d^H\omega^i_j + \sum \omega^i_l \wedge \omega^l_j$, and thus the assumption means that $\Omega^i_j = 0$.

On the other hand, the connection form of v^*D is given by $v^*\omega^i_j$. Hence, from (6.1), the curvature of v^*D is given by

$$d(v^*\omega^i_j) + \sum (v^*\omega^i_j) \wedge (v^*\omega^l_j) = v^*(d^H\omega^i_j + \sum \omega^i_l \wedge \omega^l_j) = v^*\Omega^i_j.$$

Consequently v^*D is flat. □

By Theorem 4.2 and Lemma 6.1, if D satisfies $R^{HH}_D = 0$, then the induced connection v^*D^* is also a flat affine connection on M. We set $\underline{D} = v^*D$ and $\underline{D}^* = v^*D^*$. We also denote by $g = \sum G_{ij}(x, v(x))dv^i \otimes dv^j = \sum g_{ij} dx^i \otimes dx^j$ the induced Riemannian metric v^*G on M. Then the condition (4.3) implies the following relation:

$$Xg(Y, Z) = g(\underline{D}_X Y, Z) + g(Y, \underline{D}^*_X Z) \tag{6.2}$$

for all $X, Y, Z \in \Gamma(TM)$. Furthermore, $\underline{D}$ and $\underline{D}^*$ are symmetric affine connections such that $\underline{D}g$ and $\underline{D}^*g$ are totally symmetric. Therefore, if $R^{HH}_D = 0$ is satisfied, then $(M, g, \underline{D})$ and $(M, g, \underline{D}^*)$ are flat statistical manifolds. As is well-known (cf. [2]), if $(M, g, \underline{D})$ is flat, then M is locally Hessian.

Theorem 6.1. *If the curvature R^{HH}_D of the Berwald connection D vanishes identically, then M admits a flat statistical structure, and therefore M is locally a Hessian structure $(\underline{D}, g)$.*

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