Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 93–102
www.emis.de/journals
ISSN 1786-0091

DOUBLY WARPED PRODUCT *CR*-SUBMANIFOLDS IN A LOCALLY CONFORMAL KAEHLER SPACE FORM

KOJI MATSUMOTO AND VITTORIA BONANZINGA

Abstract. Recently, the present authors considered doubly warped product *CR*-submanifolds in a locally conformal Kaehler manifold and got some inequalities about the length of the second fundamental form ([14]).

In this report, we obtain an inequality of the mean curvature of a doubly warped product *CR*-submanifold in a locally conformal Kaehler space form. Then, we consider the equality case of this inequality.

0. Introduction.

On 1976, I. Vaisman redefined the notion of the locally conformal almost Kaehler structure on Hermitian manifolds ([18, 19]). Then, T. Kashiwada characterised this notion by the tensor representation, and she gave the tensor representation of the curvature tensor of a locally conformal Kaehler space form under a certain condition ([9]).

On the other hand, on 1978, A. Bejancu introduced the notion of *CR*-submanifolds which is a generalization of holomorphic and totally real submanifolds in an almost Hermitian manifold. After his definition, we can see many papers and books in this field ([6, 7, 12, 17, 20] etc.).

Next, B. Y. Chen defined the notion of warped product *CR*-submanifolds in Kaehler manifolds and he proved a lot of interesting results in these submanifolds ([7]). This notion was considered in other Hermitian manifolds and gave similar results ([4]). Then we can also see the similar notion in an (almost) contact metric manifolds ([11, 20] etc.).

Now, we can find only few papers about doubly warped product Riemannian manifolds which are the generalization of a warped product Riemannian manifold ([1, 8, 10]).

2000 *Mathematics Subject Classification.* 53C40.
Key words and phrases. Locally conformal Kaehler manifold, doubly warped product *CR*-submanifold, adapted frame.

Recently, one of the authors defined doubly warped product submanifolds in a Riemannian manifold. Then, special doubly warped product (doubly warped product *CR*-) submanifolds in a locally conformal Kaehler manifold were considered ([13, 14]).

In this report, we consider the mean curvature of a doubly warped product *CR*-submanifold in a locally conformal Kaehler space form and prove an inequality about it. Finally, we consider the equality case of the inequality (Theorem 5.1).

In Section 1, we recall doubly warped product Riemannian manifolds and doubly warped product submanifolds. In addition, we show a essential formula in doubly warped product submanifolds for later use. In Sections 2 and 3, we recall locally conformal Kaehler manifolds and their *CR*-submanifolds. In Section 4, we introduce an adapted frame of a locally conformal Kaehler space form and calculate the components of the Riemannian curvature tensor with respect to this frame for the next section. In Section 5, we obtain an inequality of the mean curvature of a doubly warped product *CR*-submanifold in a locally conformal Kaehler space form and finally, we consider the equality case of the inequality.

1. Doubly warped product Riemannian submanifolds.

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and M be its product manifold. And $f_1 > 0$ (resp. $f_2 > 0$) be a differentiable function on M_1 (resp. M_2). A *doubly warped product Riemannian manifold* $M = M_1 \times_{(f_1, f_2)} M_2$ is a product manifold with a Riemannian metric g on M which is defined by

$$g(U, V) = {f_2}^2 g_1(\pi_{1*}U, \pi_{1*}) + {f_1}^2 g_2(\pi_{2*}U, \pi_{2*}V) \tag{1.1}$$

for any $U, V \in TM$, where TM denotes the tangent bundle of M and π_1 (resp. π_2) is the projection operator of M to M_1 (resp. M_2) and π_{1*} (resp. π_{2*}) is the differential map of π_1 (resp. π_2) ([1, 8, 11] etc.). The functions f_1 and f_2 are called the *warping functions* and the pair (f_1, f_2) is called the *pair of warping functions* of a doubly warped product Riemannian manifold.

Remark 1.1. In a doubly warped product Riemannian manifold, if one of the warping functions is equal to 1, then the manifold is called a warped product Riemannian manifold ([16]).

A doubly warped product manifold is said to be *proper* if both of warping functions are not constant.

Next, let $\tilde{M}$ be a Riemannian manifold with a Riemannian metric $\tilde{g}$. A submanifold M of $\tilde{M}$ is called a *doubly warped product submanifold* of $\tilde{M}$ if it satisfies ([13])

(i) M is a Riemannian submanifold of $\tilde{M}$,
(ii) M is a doubly warped product manifold of two submanifolds M_1 and M_2 of $\tilde{M}$,

(iii) two submanifolds are orthogonal, that is, $\tilde{g}(X, Z) = 0$ for any $X \in TM_1$ and $Z \in TM_2$.

We write the above doubly warped product submanifold as

$$M = M_1 \times_{(f_1, f_2)} M_2, \tag{1.3}$$

where (f_1, f_2) is its pair of warping functions.

In a doubly warped product submanifold, the following equation is essential

$$\nabla_X Z = \nabla_Z X = (Z \log f_2)X + (X \log f_1)Z \tag{1.4}$$

for any $X \in TM_1$ and $Z \in TM_2$, where ∇ is the covariant differentiation with respect to the induced metric g of $\tilde{g}$.

Generally, between a Riemannian manifold $(\tilde{M}, \tilde{g})$ and its submanifold (M, g), the Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_U V = \nabla_U V + \sigma(U, V), \tag{1.5}$$

$$\tilde{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi \tag{1.6}$$

for any $U, V \in TM$ and $\xi \in T^\perp M$, where $T^\perp M$ denotes the normal bundle of M, $\tilde{\nabla}$ (resp. ∇) is the covariant differentiation with respect to $\tilde{g}$ (resp. g), σ is the second fundamental form and A_ξ is the shape operator of ξ. Between the second fundamental form σ and the shape operator A_ξ, there is the relation

$$\tilde{g}(\sigma(U, V), \xi) = \tilde{g}(A_\xi U, V) \tag{1.7}$$

for any $U, V \in TM$ and $\xi \in T^\perp M$ ([5]).

Moreover, we know the Gauss equation

$$\begin{aligned} R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) &+ \tilde{g}(\sigma(X, W), \sigma(Y, Z)) \\ &- \tilde{g}(\sigma(X, Z), \sigma(Y, W)) \end{aligned} \tag{1.8}$$

for any $X, Y, Z, W \in TM$, where $\tilde{R}$ (resp. R) is the curvature $(0, 4)$-tensor with respect to $\tilde{g}$ (resp. g) ([5]).

2. Locally conformal Kaehler manifolds and *CR*-submanifolds.

Let $\tilde{M}(J, \tilde{g}, \alpha)$ be a real m-dimensional locally conformal Kaehler (l.c.K.-) manifold with an almost complex structure J, a Hermitian metric $\tilde{g}$ and a closed 1 form α which is called the *Lee form.* Then, they satisfy

$$J^2 = -I, \qquad \tilde{g}(JU, JV) = \tilde{g}(U, V), \tag{2.1}$$

$$(\tilde{\nabla}_V J)U = -\tilde{g}(\alpha^\sharp, U)JV + \tilde{g}(U, V)\beta^\sharp + g(JU, V)\alpha^\sharp - \tilde{g}(\beta^\sharp, U)V \tag{2.2}$$

for any $U, V \in T\tilde{M}$, where I means the identity transformation, $\alpha^\sharp$ is the dual vector field of α which is called the *Lee vector field*, the 1-form β is defined by $\beta(U) = -\alpha(JU)$ for any $U \in T\tilde{M}$ and $\beta^\sharp$ is the dual vector field of β.

An l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$ is called an *l.c.K.-space form* if it has a constant holomorphic sectional curvature. We know that the Riemannian curvature tensor $\tilde{R}$ of an l.c.K.-space form with the constant holomorphic sectional curvature c is given by ([9])

$$\begin{aligned} 4\tilde{R}(X,Y,Z,W) = c\{&\tilde{g}(X,W)\tilde{g}(Y,Z) - \tilde{g}(X,Z)\tilde{g}(Y,W) \\ &+ \tilde{g}(JX,W)\tilde{g}(JY,Z) - \tilde{g}(JX,Z)\tilde{g}(JY,W) \\ &- 2\tilde{g}(JX,Y)\tilde{g}(JZ,W)\} + 3\{P(X,W)\tilde{g}(Y,Z) \\ &- P(X,Z)\tilde{g}(Y,W) + \tilde{g}(X,W)P(Y,Z) \\ &- \tilde{g}(X,Z)P(Y,W)\} - \tilde{P}(X,W)\tilde{g}(JY,Z) \\ &+ \tilde{P}(X,Z)\tilde{g}(JY,W) - \tilde{g}(JX,W)\tilde{P}(Y,Z) \\ &+ \tilde{g}(JX,Z)\tilde{P}(Y,W) + 2\{\tilde{P}(X,Y)\tilde{g}(JZ,W) \\ &+ \tilde{g}(JX,Y)\tilde{P}(Z,W)\} \end{aligned} \tag{2.3}$$

for any $X, Y, Z, W \in T\tilde{M}$, where P and $\tilde{P}$ are respectively defined by

$$P(X,Y) = -(\tilde{\nabla}_X\alpha)Y - \alpha(X)\alpha(Y) - \frac{1}{2}\|\alpha\|^2\tilde{g}(X,Y), \quad \tilde{P}(X,Y) = P(JX,Y) \tag{2.4}$$

for any $X, Y \in T\tilde{M}$, where $\|\alpha\|$ is the length of the Lee form α.

Remark 2.1. To get (2.3), we have to assume that the symmetric (0,2)-tensor P is hybrid or equivalently $\tilde{P}$ is skew-symmetric. This means the Ricci tensor $\tilde{R}_1$ is hybrid.

We write an l.c.K.-space form with the constant holomorphic sectional curvature c by $\tilde{M}(c)$.

For each $x \in \tilde{M}$, we denote by $\mathcal{D}_x$ the maximal holomorphic subspace ($J\mathcal{D}_x = \mathcal{D}_x$) of the tangent space $T_x\tilde{M}$ of $\tilde{M}$ at x. If the dimension of $\mathcal{D}_x$ is same for each $x \in \tilde{M}$, then $\mathcal{D}_x$ gives a holomorphic distribution $\mathcal{D}$ on $\tilde{M}$.

A submanifold M in an almost Hermitian manifold $\tilde{M}$ is called a *CR-submanifold* if there exists on M a differentiable holomorphic distribution $\mathcal{D}$ whose orthogonal complement $\mathcal{D}^\perp$ is a differentiable totally real distribution, i.e., $J\mathcal{D}_x^\perp \subset T_x^\perp M$ for each $x \in M$, where $T_x^\perp M$ is the normal space of M at x.

A *CR*-submanifold M is called *anti-holomorphic* if $J\mathcal{D}_x^\perp = T_x^\perp M$ for each $x \in M$.

For a *CR*-submanifold M of an almost Hermitian manifold $\tilde{M}$, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in the normal bundle $T^\perp M$. Then we have the following direct sum decomposition

$$T^\perp M = J\mathcal{D}^\perp \oplus \nu, \qquad J\mathcal{D}^\perp \perp \nu. \tag{2.5}$$

Remark 2.2. An anti-holomorphic *CR*-submanifold is a *CR*-submanifold with $\nu = \{0\}$.

In a *CR*-submanifold of an l.c.K.-manifold, we proved the following;

Proposition 2.1 ([12]). *Let M be a CR-submanifold in an l.c.K.-manifold $\tilde{M}$, Then we have*

(1) *the totally real distribution $\mathcal{D}^\perp$ is integrable, that is, $[\mathcal{D}^\perp, \mathcal{D}^\perp] \subset \mathcal{D}^\perp$,*
(2) *the holomorphic distribution $\mathcal{D}$ is integrable if and only if*

$$\tilde{g}(\sigma(X, JY) - \sigma(Y, JX) - 2\tilde{g}(JX, Y)\alpha^\sharp, JZ) = 0 \tag{2.6}$$

for any $X, Y \in \mathcal{D}$, where $[,]$ means the Lie bracket.

3. Doubly warped product CR-submanifolds in an l.c.K.-manifold.

In this section, we consider *CR*-submanifold in an l.c.K.-manifold $\tilde{M}$ which is a doubly warped product submanifold of the form $M = M_\top \times_{(f_\top, f_\perp)} M_\perp$, where $M_\top$ (resp. $M_\perp$) is a holomorphic (resp. totally real) submanifold of $\tilde{M}$ and $f_\top$ (resp. $f_\perp$) is a positive differentiable function on $M_\top$ (resp. $M_\perp$). By virtue of our assumption, since the distribution $\mathcal{D} = TM_\top$ is integrable, the second fundamental form σ has to satisfy (2.6), identically. A doubly warped product manifold $M = M_\top \times_{(f_\top, f_\perp)} M_\perp$ is called a *doubly warped product CR-submanifold* in an l.c.K.-manifold $\tilde{M}$ if the metric tensor g on M is the induced metric of $\tilde{g}$ for certain Riemannian metric $g_\top$ (resp, $g_\perp$) on $M_\top$ (resp. $M_\perp$).

In a doubly warped product *CR*-submanifold M in an l.c.K.-manifold $\tilde{M}$, we have

Proposition 3.1 ([13]). *For a doubly warped product CR-submanifold M in an l.c.K.-manifold $\tilde{M}(J, \tilde{g}, \alpha)$, we have*

$$\begin{aligned} \tilde{g}(\sigma(X, JY), JZ) = \tilde{g}(\alpha^\sharp, Z)\tilde{g}(X, Y) + \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X, JY) \\ - (Z \log f_\perp)\tilde{g}(X, Y), \end{aligned} \tag{3.1}$$

$$\tilde{g}(\sigma(X, Y), JZ) = \tilde{g}(\alpha^\sharp, JZ), \qquad \tilde{g}(\alpha^\sharp, Z) = Z \log f_\perp, \tag{3.2}$$

$$\tilde{g}(\sigma(JX, Z), JW) = \{-\tilde{g}(\alpha^\sharp, X) + (X \log f_\top)\}\tilde{g}(Z, W) \tag{3.3}$$

for any $X, Y \in \mathcal{D}$ and $Z, W \in \mathcal{D}^\perp$.

By virtue of $(3.2)_2$, the equation (3.1) is written as

$$\tilde{g}(\sigma(X, JY), JZ) = \tilde{g}(\alpha^\sharp, JZ)\tilde{g}(X, JY). \tag{3.1$'$}$$

We have from $(3.2)_2$

Proposition 3.2. *There is no proper doubly warped product CR-submanifolds in a Kaehler manifold.*

Proposition 3.3. *There is no proper doubly warped product CR-submanifold in an l.c.K.-manifold which the Lee vector field $\alpha^\sharp$ is normal to $\mathcal{D}^\perp$.*

4. Components of the curvature tensors of an l.c.K.-space form.

In this section, we introduce an adapted frame in an l.c.K.-manifold and then using this frame we calculate the components of the curvature tensor of an l.c.K.-space form.

Let M be a doubly warped product CR-submanifold of an l.c.K.-manifold $\tilde{M}$. Now, we put $\dim \tilde{M} = m$, $\dim M = n$, $\dim M_\top = 2p$, $\dim M_\perp = q$ $(2p + q = n)$. Let $\{e_1, \ldots, e_p, e_1^*, \ldots, e_p^*\}$, $\{e_{2p+1}, \ldots, e_{2p+q}\}$, $\{e_{2p+1}^*, \ldots, e_{2p+q}^*\}$ and $\{e_{n+q+1}, \ldots, e_m\}$ be a local orthonormal basis of $\mathcal{D}$, $\mathcal{D}^\perp$ $(= TM_\perp)$, $J\mathcal{D}^\perp$ and ν, respectively, where $e_i^* = Je_i$ for $i \in \{1, \ldots, p\}$ and $e_{2p+a}^* = Je_{2p+a}$ for $a \in \{1, \ldots, q\}$. We call such local basis an *adapted frame* of $\tilde{M}$.

Now, we assume that our ambient manifold $\tilde{M}$ is an l.c.K.-space form $\tilde{M}(c)$. Then the curvature tensor $\tilde{R}$ is written by (2.3). The straightforward calculation gives us the following

$$
\begin{aligned}
&4\tilde{R}_{lkji} = 4\tilde{R}_{l^*k^*j^*i^*}\\
&\;= c(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}) + 3(\delta_{kj}P_{li} - \delta_{ki}P_{lj} + \delta_{li}P_{kj} - \delta_{lj}P_{ki}),\\
&4\tilde{R}_{lkji^*} = 3(\delta_{kj}P_{li^*} - \delta_{lj}P_{ki^*}) - \delta_{ki}P_{lj^*} + \delta_{li}P_{kj^*} - 2\delta_{ji}P_{lk^*},\\
&4\tilde{R}_{lkj^*i^*} = c(\delta_{li}\delta_{kj} - \delta_{lj}\delta_{ki}) - \delta_{kj}P_{li} + \delta_{ki}P_{lj} - \delta_{li}P_{kj} + \delta_{lj}P_{ki},\\
&4\tilde{R}_{lk^*j^*i} = c(\delta_{li}\delta_{kj} + \delta_{lj}\delta_{ki} - 2\delta_{lk}\delta_{ji}) + 3(\delta_{kj}P_{li} + \delta_{lj}P_{ki})\\
&\qquad - \delta_{ki}P_{lj} + \delta_{lj}P_{ki} - 2(\delta_{lk}P_{ji} + \delta_{ji}P_{lk}),\\
&4\tilde{R}_{lk^*j^*i^*} = 3(\delta_{kj}P_{li^*} - \delta_{ki}P_{lj^*}) - \delta_{li}P_{k^*j} + \delta_{lj}P_{k^*i} + 2\delta_{lk}P_{j^*i},\\
&4\tilde{R}_{lkj(2p+a)} = 3\{\delta_{kj}P_{l(2p+a)} - \delta_{lj}P_{k(2p+a)}\},\\
&4\tilde{R}_{lkj^*(2p+a)} = \delta_{lj}P_{k^*(2p+a)} - \delta_{kj}P_{l^*(2p+a)}\\
&4\tilde{R}_{lk^*j(2p+a)} = -3\delta_{lj}P_{k^*(2p+a)} + \delta_{kj}P_{l^*(2p+a)} + 2\delta_{lk}P_{j^*(2p+a)},\\
&4\tilde{R}_{lk^*j^*(2p+a)} = 3\delta_{kj}P_{l(2p+a)} - \delta_{lj}P_{k(2p+a)} - 2\delta_{lk}P_{j(2p+a)},\\
&4\tilde{R}_{l^*k^*j^*(2p+a)} = 3\{\delta_{kj}P_{l^*(2p+a)} - \delta_{lj}P_{k^*(2p+a)}\},\\
&4\tilde{R}_{lk(2p+b)(2p+a)} = 0, \qquad 4\tilde{R}_{lk^*(2p+b)(2p+a)} = -2\delta_{lk}P_{(2p+b)(2p+a)},\\
&4\tilde{R}_{l^*k^*(2p+b)(2p+a)} = 0,\\
&4\tilde{R}_{l(2p+c)(2p+b)(2p+a)} = 3\{\delta_{cb}P_{l(2p+a)} - \delta_{ca}P_{l(2p+b)}\},\\
&4\tilde{R}_{l^*(2p+c)(2p+b)(2p+a)} = 3\{\delta_{cb}P_{l^*(2p+a)} - \delta_{ca}P_{l^*(2p+b)}\},\\
&4\tilde{R}_{(2p+d)(2p+c)(2p+b)(2p+a)} = c(\delta_{da}\delta_{cb} - \delta_{db}\delta_{ca}) + 3\{\delta_{cb}P_{(2p+d)(2p+a)}\\
&\qquad - \delta_{ca}P_{(2p+d)(2p+b)} + \delta_{da}P_{(2p+c)(2p+b)} - \delta_{db}P_{(2p+c)(2p+a)}\},\\
&4\tilde{R}_{j(2p+b)(2p+a)i} = 4\tilde{R}_{j^*(2p+b)(2p+a)i^*}\\
&\;= c\delta_{ji}\delta_{ba} + 3\{P_{ji}\delta_{ba} + \delta_{ji}P_{(2p+b)(2p+a)}\},
\end{aligned}
\tag{4.1}
$$

where the indices $k, j, \ldots, i$ and $c, b, \ldots, a$ run over the range $1, 2, \ldots, p$ and $1, 2, \ldots, q$, respectively. And we write $\tilde{R}(e_\omega, e_\nu, e_\mu, e_\lambda) = \tilde{R}_{\omega\nu\mu\lambda}$, etc., for any $\omega, \nu, \ldots, \lambda \in \{1, 2, \ldots, n.\}$

5. The mean curvature.

Let M be a doubly warped product *CR*-submanifold of an l.c.K.-space form $\tilde{M}(c)$ and $\{e_1, e_2, \ldots, e_m\}$ be an adapted frame of $\tilde{M}(c)$.

Now, the mean curvature vector H and the mean curvature $\|H\|$ are respectively given by

$$H = \frac{1}{n} \sum_{\mu=1}^{n} \sigma_{\mu\mu}, \quad \|H\|^2 = \frac{1}{n^2} \sum_{\nu,\lambda=1}^{n} \tilde{g}(\sigma_{\nu\nu}, \sigma_{\lambda\lambda}). \tag{5.1}$$

The length $\|\sigma\|$ of the second fundamental form σ is given by

$$\|\sigma\|^2 = \sum_{\mu,\lambda=1}^{n} \tilde{g}(\sigma_{\nu\lambda}, \sigma_{\nu\lambda}) = \sum_{\mu,\lambda=1}^{n} \sum_{\tau=n+1}^{m} \tilde{g}(\sigma_{\mu\lambda}, e_\tau)^2. \tag{5.2}$$

By virtue of (5.1), (5.2) and the Gauss equation (1.8), we have

$$4r = 4 \sum_{\omega,\nu=1}^{n} \tilde{R}_{\omega\nu\nu\omega} + 4n^2\|H\|^2 - 4\|\sigma\|^2, \tag{5.3}$$

where r is the scalar curvature with respect to the induced metric g.

Now, we can write

$$\begin{aligned} 4 \sum_{\omega,\nu=1}^{n} \tilde{R}_{\omega\nu\nu\omega} = 8 \sum_{j,i=1}^{p} (\tilde{R}_{jiij} + \tilde{R}_{ji^*i^*j}) + 8 \sum_{j=1}^{p} \sum_{a=1}^{q} \{\tilde{R}_{j(2p+a)(2p+a)j} \\ + \tilde{R}_{j^*(2p+a)(2p+a)j^*}\} + 4 \sum_{b,a=1}^{q} \tilde{R}_{(2p+b)(2p+a)(2p+a)(2p+b)}. \end{aligned}$$

Using (4.1), we have from the above equation

$$4 \sum_{\omega,\nu=1}^{n} \tilde{R}_{\omega\nu\nu\omega} = c(n^2 - 4p - q) + 6(n-1) \sum_{\mu=1}^{n} P_{\mu\mu} - 12 \sum_{j=1}^{p} P_{jj}. \tag{5.4}$$

Next, we have from (5.2)

$$
\begin{aligned}
\|\sigma\|^2 &= \sum_{\mu,\lambda=1}^{n}\sum_{\tau=n+1}^{m}\tilde{g}(\sigma_{\mu\lambda}, e_\tau)^2 = \sum_{\mu,\lambda=1}^{n}\sum_{\tau=n+1}^{n+q}\tilde{g}(\sigma_{\mu\lambda}, e_\tau)^2 \\
&\quad + \sum_{\mu,\lambda=1}^{n}\sum_{\tau=n+q+1}^{m}\tilde{g}(\sigma_{\mu\lambda}, e_\tau)^2 \geq \sum_{\mu,\lambda=1}^{n}\sum_{\tau=n+1}^{n+q}\tilde{g}(\sigma(e_\mu, e_\nu), e_\tau)^2 \\
&= \sum_{a=1}^{q}\sum_{\mu,\lambda=1}^{n}\tilde{g}(\sigma_{\mu\lambda}, e^*_{2p+a})^2 = \sum_{a=1}^{q}\sum_{j,i=1}^{p}\tilde{g}(\sigma_{ji}, e^*_{2p+a})^2 \\
&\quad + 2\sum_{a=1}^{q}\sum_{j,i=1}^{p}\tilde{g}(\sigma_{ji^*}, e^*_{2p+a})^2 + 2\sum_{a,b=1}^{q}\sum_{j=1}^{p}\{\tilde{g}(\sigma_{j(2p+b)}, e^*_{2p+a})^2 \\
&\quad + \tilde{g}(\sigma_{j^*(2p+b)}, e^*_{2p+a})^2\} + \sum_{c,b,a=1}^{q}\tilde{g}(\sigma_{(2p+c)(2p+b)}, e^*_{2p+a})^2 \\
&\geq \sum_{a=1}^{q}\sum_{j,i=1}^{p}\tilde{g}(\sigma_{\mu\lambda}, e^*_{2p+a})^2 + 2\sum_{a=1}^{q}\sum_{j,i=1}^{p}\tilde{g}(\sigma_{ji^*}, e^*_{2p+a})^2 \\
&\quad + 2\sum_{a,b=1}^{q}\sum_{j=1}^{p}\{\tilde{g}(\sigma_{j(2p+b)}, e^*_{2p+a})^2 + \tilde{g}(\sigma_{j^*(2p+b)}, e^*_{2p+a})^2\}.
\end{aligned}
$$

By virtue of Proposition 3.1, the above inequality gives us

$$
\begin{aligned}
\|\sigma\|^2 \geq p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{(2p+a)})\}^2 &+ 2q\sum_{j=1}^{p}\{(\tilde{g}(\alpha^\sharp, e^*_j) - e^*_j \log f_\top)^2 \\
&+ (\tilde{g}(\alpha^\sharp, e_j) - e_j \log f_\top)^2\}.
\end{aligned}
\tag{5.5}
$$

Substituting (5.4) and (5.5) into (5.3), we have

$$
\begin{aligned}
4n^2\|H\|^2 &\geq 4r - (n^2 - 4p - q)c - 6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu} \\
&\quad + 12\sum_{j=1}^{p}P_{jj} + 4[p\sum_{a=1}^{q}\{\tilde{g}(\alpha^\sharp, e_{(2p+a)})\}^2 \\
&\quad + 2q\sum_{j=1}^{p}\{(\tilde{g}(\alpha^\sharp, e^*_j) - e^*_j \log f_\top)^2 \\
&\quad + (\tilde{g}(\alpha^\sharp, e_j) - e_j \log f_\top)^2\}] \\
&\geq 4r - (n^2 - 4p - q)c - 6(n-1)\sum_{\mu=1}^{n}P_{\mu\mu} + 12\sum_{j=1}^{p}P_{jj}.
\end{aligned}
$$

Thus, we have

Theorem 5.1. *In a doubly warped product CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, the mean curvature $\|H\|$ satisfies*

$$\|H\| \geq \frac{1}{4n^2}\{4r - (n^2 - 4p - q)c - 6(n-1)\sum_{\mu=1}^{n} P_{\mu\mu} + 12\sum_{j=1}^{p} P_{jj}\}. \tag{5.6}$$

In particular, the equality case of (5.6) *is that the Lee vector field $\alpha^\sharp$ satisfies*

$$\tilde{g}(\alpha^\sharp, \mathcal{D}^\perp) = \{0\}, \quad \alpha^i = e_i \log f_\top \quad \text{and} \quad \alpha^{i^*} = e_i^* \log f_\top, \tag{5.7}$$

where α^i (resp. α^{i^*}) is an i-th (resp. $(2p+i)$-th) component of $\alpha^\sharp$ with respect to the adapted frame.

Corollary 5.1. *There is no proper doubly warped product CR-submanifold M which the mean curvature $\|H\|$ satisfies the equality in* (5.6) *and the Lee vector field $\alpha^\sharp$ is tangent to M.*

In particular, we have

Corollary 5.2. *In an anti-holomorphic doubly warped product CR-submanifold M in an l.c.K.-space form $\tilde{M}(c)$, if the mean curvature satisfies the equality in* (5.6)*, then M is totally geodesic in $\tilde{M}(c)$, that is, $\sigma = \{0\}$, identically.*

References

[1] Y. Agaoka, I. B. Kim, B. H. Kim, and D. J. Yeon. On doubly warped product manifolds. *Mem. Fac. Integrated Arts and Sci., Hiroshima Univ., Ser. IV.*, 24:1–10, 1998.

[2] A. Bejancu. *CR* submanifolds of a Kaehler manifold. I. *Proc. Amer. Math. Soc.*, 69(1):135–142, 1978.

[3] A. Bejancu. *Geometry of CR-submanifolds*, volume 23 of *Mathematics and its Applications (East European Series).* D. Reidel Publishing Co., Dordrecht, 1986.

[4] V. Bonanzinga and K. Matsumoto. Warped product *CR*-submanifolds in locally conformal Kaehler manifolds. *Period. Math. Hungar.*, 48(1-2):207–221, 2004.

[5] B.-Y. Chen. *Geometry of submanifolds.* Marcel Dekker Inc., New York, 1973. Pure and Applied Mathematics, No. 22.

[6] B.-Y. Chen. *CR*-submanifolds of a Kaehler manifold. I, II. *J. Differential Geom.*, 16(2, 3):305–322, 493–509, 1981.

[7] B.-Y. Chen. Geometry of warped product *CR*-submanifolds in Kaehler manifolds. *Monatsh. Math.*, 133(3):177–195, 2001.

[8] T. Ikawa and J.-B. Jun. On conformally flat spaces with doubly warped product riemannian metric. *J. of General Education, Nihon Univ.*, 21:123–129, 1995.

[9] T. Kashiwada. Some properties of locally conformal Kähler manifolds. *Hokkaido Math. J.*, 8(2):191–198, 1979.

[10] L. Kozma, I. R. Peter, and C. Varga. Doubly warped product of finsler manifolds. preprint.

[11] K. Matsumoto. On contact *CR*-submanifolds of Sasakian manifolds. *Internat. J. Math. Math. Sci.*, 6(2):313–326, 1983.

[12] K. Matsumoto. On *CR*-submanifolds of locally conformal Kähler manifolds. *J. Korean Math. Soc.*, 21(1):49–61, 1984.

[13] K. Matsumoto. Doubly warped product manifolds and submanifolds. In *Global analysis and applied mathematics*, volume 729 of *AIP Conf. Proc.*, pages 218–224. Amer. Inst. Phys., Melville, NY, 2004.

[14] K. Matsumoto and V. Bonanzinga. Doubly warped product *CR*-submanifolds in a locally conformal kaehler space form ii. preprint.

[15] K. Matsumoto and I. Mihai. Warped product submanifolds in Sasakian space forms. *SUT J. Math.*, 38(2):135–144, 2002.

[16] B. O'Neill. *Semi-Riemannian geometry*, volume 103 of *Pure and Applied Mathematics*. Academic Press Inc., New York, 1983. With applications to relativity.

[17] N. Papaghiuc. Some remarks on *CR*-submanifolds of a locally conformal Kaehler manifold with parallel Lee form. *Publ. Math. Debrecen*, 43(3-4):337–341, 1993.

[18] I. Vaisman. On locally conformal almost Kähler manifolds. *Israel J. Math.*, 24(3-4):338–351, 1976.

[19] I. Vaisman. Locally conformal Kähler manifolds with parallel Lee form. *Rend. Mat. (6)*, 12(2):263–284, 1979.

[20] K. Yano and M. Kon. *CR submanifolds of Kaehlerian and Sasakian manifolds*, volume 30 of *Progress in Mathematics*. Birkhäuser Boston, Mass., 1983.

Koji Matsumoto,
2-3-65 Nishi Odori,
Yonezawa, 992-0059,
Yamagata,
Japan
E-mail address: tokiko_matsumoto@yahoo.com

Vittoria Bonanzinga,
University of Reggio Calabria,
Faculty of Engineering, DIMET,
via Graziella (Feo di Vito),
89100 Reggio Calabria,
Italy
E-mail address: vittoria.bonanzinga@unirc.it