Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
24 (2008), 25–31
www.emis.de/journals
ISSN 1786-0091

ON THE RECTILINEAR EXTREMALS OF GEODESICS IN *SOL* GEOMETRY

SÁNDOR BÁCSÓ AND BRIGITTA SZILÁGYI

ABSTRACT. In this paper we deal with one of the homogeneous geometries, the *Sol* geometry. The goal of this publication is to show that there does not exist rectilinear extremals of geodesics. During the proof we consider the Riemannian space with *Sol* metric as a special Finsler space and applied Finslerian methods.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold. If for any $x, y \in M$ there does exists an isometry $\Phi\colon M \to M$ such that $y = \Phi(x)$, then the Riemannian manifold is called *homogeneous*.

Homogeneous geometries have main roles in the modern theory of three-manifolds.

Homogeneous spaces are, in a sense, the nicest examples of Riemannian manifolds and have applications in physics (e.g. the *Sol* geometry is useful for studying holography, Yang–Mills theory) [7].

Sol geometry can be obtained by giving a group structure to $T = \mathbb{R} \ltimes \mathbb{R}^2$ as follows:

$$\begin{pmatrix} 1 & a & b & c \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x + ae^{-z} & y + be^{z} & z + c \end{pmatrix}$$

is the right action by translation $(1, x, y, z)$ on $(1, a, b, c)$ expressed in homogeneous coordinates, for (x, y, z) and $(a, b, c) \in T$.

Then an invariant metric on $Sol(O, T)$ is given by

$$(ds)^2 = (dx)^2 e^{2z} + (dy)^2 e^{-2z} + (dz)^2,$$

2000 *Mathematics Subject Classification.* 53B20.
Key words and phrases. Homogeneous Riemannian geometry, geodesic, Finsler metric.

where O denotes the origin.

The well-known equation of geodesics

$$\frac{d^2u^k}{dt^2} + \Gamma^k_{ij}\frac{du^i}{dt}\frac{du^j}{dt} = 0$$

in *Sol* Geometry leads to the system:

$$\begin{aligned}
&\frac{d^2x(t)}{dt^2} + 2\frac{dx(t)}{dt}\frac{dz(t)}{dt} = 0,\\
&\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt}\frac{dz(t)}{dt} = 0,\\
&\frac{d^2z(t)}{dt^2} - e^{2z(t)}\frac{dx(t)}{dt}\frac{dx(t)}{dt} + e^{-2z(t)}\frac{dy(t)}{dt}\frac{dy(t)}{dt} = 0.
\end{aligned}$$

For simplification we abbreviate the notation:

$$\begin{aligned}
&\ddot{x} + 2\dot{x}\dot{z} = 0\\
&\ddot{y} - 2\dot{y}\dot{z} = 0\\
&\ddot{z} - e^{2z}(\dot{x})^2 + e^{-2z}(\dot{y})^2 = 0.
\end{aligned}$$

We solve this differential equation system as a Cauchy problem:

$$\begin{array}{lcl}
x(0) = 0 & & \dot{x}(0) = u\\
y(0) = 0 & \text{and} & \dot{y}(0) = v\\
z(0) = 0 & & \dot{z}(0) = w.
\end{array}$$

Table 1. contains the solutions of the geodesics from the paper [5].

Remarks. 1. While computed the formulas above, we used $u^2+v^2+w^2 = 1$ that can be assumed without loss of generality (meaning arc-length parametrisation).

2. From the conditions $\dot{x}(0)$ and $\dot{y}(0) = 0$ alone, it immediately follows the results of the fourth row.

Naturally arises the question whether these equations can be given 'in a simplier form'. As it is well-known, V. I. Arnold examined the problem whether the solution of

$$\frac{d^2p(\alpha)}{d\alpha^2} = \Phi\left(\alpha, p(\alpha), \frac{dp(\alpha)}{d\alpha}\right),$$

which is a two-parametric array of curves could be transformed to array of rectilinear extremals by a diffeomorphism [2]. We cite here the famous result:

Theorem. *A differential equation of the form* $\frac{d^2p(\alpha)}{d\alpha^2} = \Phi\left(\alpha, p(\alpha), \frac{dp(\alpha)}{d\alpha}\right)$ *can be reduced to the form* $\frac{d^2\bar{p}(\alpha)}{d\bar{\alpha}^2} = 0$ *if and only if the right-hand side is a polynomial in the derivative of order not greater* 3 *both for the equation and for its dual.*

$\dot{x}(0) \neq 0$ $\dot{y}(0) \neq 0$	$x(t) = u \int_0^t e^{-2z(\tau)} d\tau$ $y(t) = v \int_0^t e^{2z(\tau)} d\tau$ $z(t)$ comes from the separable differential equation $\dfrac{dz}{\sqrt{1 - u^2 e^{-2z} - v^2 e^{2z}}} = dt$ whose solution cannot be expressed in terms of a finite number elementary functions
$\dot{x}(0) \neq 0$ $\dot{y}(t) \equiv 0$	$x(t) = \dfrac{1}{u} \left\{ \dfrac{\sinh t + \sqrt{1-u^2} \cosh t}{\cosh t + \sqrt{1-u^2} \sinh t} - \sqrt{1-u^2} \right\}$ $y(t) = 0$ $z(t) = \ln\left(\cosh t + \sqrt{1-u^2} \sinh t\right)$
$\dot{x}(t) \equiv 0$ $\dot{y}(0) \neq 0$	$x(t) = 0$ $y(t) = \dfrac{1}{v} \left\{ \dfrac{\sinh t + \sqrt{1-v^2} \cosh t}{\cosh t + \sqrt{1-v^2} \sinh t} - \sqrt{1-v^2} \right\}$ $z(t) = \ln\left(\cosh t + \sqrt{1-v^2} \sinh t\right)$
$\dot{x}(t) \equiv 0$ $\dot{y}(t) \equiv 0$	$x(t) = 0$ $y(t) = 0$ $z(t) = t$

TABLE 1

The examination for the condition of a differential equation's dual is rather complicated and tedious. So let us consider a Riemannian space with *Sol* metric as a Finsler space.

In the tangent space of each point of a Finsler space, there is a general norm defined, which is not necessarily induced by an inner product.

2. Some important notations and theorems

First of all let us give the concept of Finsler metric precisely:

Definition ([8])**.** Let an n-dimensional differentable manifold M be given with a tangent space T_xM in the point (x^i) $\quad(i = 1, 2, \dots, n)$ of M. Let us denote the coordinates of vectors of T_xM by (y^i). The function $L(x, y) : TM(= \bigcup_x T_xM) \to \mathbb{R}$ is Finsler metric, if the following properties holds:

(1) *Regularity:* $L(x, y)$ is a function C^∞ on the manifold $TM\backslash\{0\}$ of nonzero tangent vectors.
(2) *Positive homogeneity:* $L(x, \lambda y) = \lambda L(x, y)$ for all $\lambda > 0$.
(3) *Strong convexity:* the $n\times n$ matrix $g_{ij}(x, y) = \dfrac{\partial^2 L^2(x, y)}{\partial y^i \partial y^j}$ is positive definite at every $y \neq 0$.

A geodesic is a curve given by the differential equations

$$\frac{d^2x^i}{ds^2} + 2G^i\left(x, dx/ds\right) = 0, \tag{2.1}$$

where s is the normalized parameter, that is the arc-length,

$$2G_j = g_{ij}(x, y)G^i(x, y) = \frac{\partial^2 F(x, y)}{\partial y^j \partial x^i} y^i - \frac{\partial F(x, y)}{\partial x^j} \tag{2.2}$$
$$\text{and} \quad F(x, y) = \left(L(x, y)\right)^2/2.$$

Definition ([8, 1])**.** Let $F^n = (M^n, L(x, y))$ and $\bar{F}^n = \left(M^n, \bar{L}(x, y)\right)$ be Finsler spaces with common differentiable manifold. If any geodesic of F^n coincides with a geodesic of $\bar{F}^n$ as a set of points and vice versa, then the change $L(x, y) \to \bar{L}(x, y)$ of the metric is called projective and F^n is said to be projective to $\bar{F}^n$.

Definition ([1])**.** A Finsler space $F^n = (M^n, L(x, y))$ is said to be with rectilinear extremals (or projectively flat), if M^n is covered by coordinate neighborhoods $\left(U, (x^i)\right)$ in which any geodesic is represented by n linear equations $x^i = x^i_0 + ta^i$ of a parameter t.

The projectively flat spaces are such affine path spaces whose paths are straight. (If we are about to determined all the Finsler spaces which admit a path mapping onto projectively flat space, than we come to Hilbert's fourth problem.)

If a Finsler space $F^n = (M^n, L(x, y))$ is a locally Minkowski space, then we have the covering of M^n by the domains of adapted coordinate systems $\left(U, (x^i)\right)$

in which L is a function of y^i alone, the quantities G^i vanish in U from (2.2) and the equation (2.1) of geodesics reduces to $d^2x^i/ds^2 = 0$, that is why F^n has rectilinear extremals (projectively flat). To sum it up a projectively flat Finsler space is projective to a locally Minkowski space.

In the studies of projective Finsler geometry the Douglas tensor plays a fundamental role. The definition of the Douglas tensor can be found in [6].

Definition ([3])**.** A Finsler space is said a Douglas space, if $D^{ij} = G^i y^j - G^j y^i$ are homogeneous polynomials in y^i of degree three.

The notion of Douglas space may be regarded as a generalization of projectively flat space.

Theorem ([3])**.** *A Finsler space is a Douglas space, if and only if the Douglas tensor vanishes identically.*

Furthermore there are two more invariant tensors. The Weyl torsion and the projective Weyl curvature tensor. The Weyl tensor was introduced by H. Weyl in 1921 [9].

It is well known that a Finsler space is projectively flat, if and only if its Douglas tensor and Weyl tensor vanish identically. Most of the papers in this subject are difficult to understand. On the contrary, Sndor Bácsó and Makoto Matsumoto's method of characterization is easier to comprehend [4].

A projective change $F^n = \big(M^n, L(x,y)\big) \longrightarrow \bar{F}^n = \big(M^n, \bar{L}(x,y)\big)$ of the Finsler metric gives rise to various projective invariants. First we have

$$\begin{aligned}
&Q^0\text{-invariants:} \quad Q^h = G^h - \frac{1}{n+1} G y^h,\\
&Q^1\text{-invariants:} \quad Q^h_i = \frac{\partial Q^h}{\partial y^i} = G^h_i - \frac{1}{n+1}\big(G_i y^h + G\delta^h_i\big),\\
&Q^2\text{-invariants:} \quad Q^h_{ij} = \frac{\partial Q^h_i}{\partial y^j} = G^h_{ij} - \frac{1}{n+1}\big(G_{ij} y^h + G_i \delta^h_j + G_j \delta^h_i\big),
\end{aligned}$$

where $G = G^r_r$, $G_i = G^r_{ri}$ and $G_{ij} = G^r_{rij}$ is the hv-Ricci tensor in Berwald connection.

The above mentioned Douglas tensor is projectively invariant, satisfying $D^h_{ijk} = \dfrac{\partial Q^h_{ij}}{\partial y^k}$.

Starting from the Q^2-invariants we shall introduce the following quantities in a way similar to constructing the h-curvature tensor: Q^3-invariants

$$Q^h_{ijk} = \delta_k Q^h_{ij} + Q^r_{ij} Q^h_{rk} - \delta_j Q^h_{ik} - Q^r_{ik} Q^h_{rj},$$

where $\delta_k Q^h_{ij} = \dfrac{\partial Q^h_{ij}}{\partial x^k} - \dfrac{\partial Q^h_{ij}}{\partial y^r} G^r_k$.

Therefore the Bácsó and Matsumoto with the help of $Q_{ij} = Q^r_{ijr}$ Ricci type tensor produce two tensors, which are very important in projective Finsler geometry:

$$\begin{aligned} \Pi^1\text{-tensor:} \quad & \Pi^h_{ijk} = Q^h_{ijk} + \frac{1}{n-1}\big(\delta^h_j Q_{ik} - Q^h_k Q_{ij}\big), \\ \Pi^2\text{-tensor:} \quad & \Pi_{ijk} = \delta_k Q_{ij} + Q^r_{ij} Q_{rk} - \delta_j Q_{ik} - Q^r_{ik} Q_{rj}. \end{aligned}$$

That is how we can obtain a new characteristic property for the flat projectively space:

Theorem ([4]). *A Finsler space F^n is projectively flat if and only if F^n is a Douglas space and its characteristic satisfies*

$$(1) \quad n > 2: \quad \Pi^h_{ijk} = 0 \qquad \textit{or} \qquad (2) \quad n = 2: \quad \Pi_{ijk} = 0.$$

Proposition ([4]). *The Weyl tensor coincides with the Π^1 tensor.*

Using the Bácsó–Matsumoto paper [4] from the previous theorems and definitions we obtain:

Proposition. *A Finsler space $F^3 = (M^3, L(x, y))$ with* Sol *metric is not projectively flat.*

Proof. Considering the differential equations (2.1) and (2.2), it follows

$$\begin{aligned} & 2G^i = \Gamma^i_{00}, \quad \text{where} \quad \Gamma^i_{00} = \Gamma^i_{jk} y^j y^k, \quad \text{and} \\ & G^1 = \frac{1}{2} y^1 y^3 \\ & G^2 = -\frac{1}{2} y^2 y^3 \\ & G^3 = \frac{1}{2}\left(-e^{2x^3}(y^1)^2 + e^{-2x^3}(y^2)^2\right). \end{aligned}$$

The computation of the Q-invariants lead to the components of Π^1.
The non-vanishing entries are listed below:

$$\begin{aligned} & \Pi^1_{221} = \Pi^1_{212} = \frac{1}{2} e^{-2x^3}, \\ & \Pi^2_{121} = \Pi^2_{112} = \frac{1}{2} e^{2x^3}, \\ & \Pi^3_{113} = \Pi^3_{213} = \Pi^3_{313} = \Pi^3_{123} = \Pi^3_{223} = \Pi^3_{323} = \Pi^3_{133} = \Pi^3_{233} = \Pi^3_{333} = \\ & = -e^{-2x^3}(y^1)^2 - e^{-2x^3}(y^2)^2. \end{aligned}$$

Since our space in question is trivially Douglas type (it is Riemannian), the statement is proved. ☐

References

[1] P. L. Antonelli, R. S. Ingarden, and M. Matsumoto. *The theory of sprays and Finsler spaces with applications in physics and biology*, volume 58 of *Fundamental Theories of Physics*. Kluwer Academic Publishers Group, Dordrecht, 1993.

[2] V. I. Arnold. *Geometrical methods in the theory of ordinary differential equations*, volume 250 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition, 1988. Translated from the Russian by Joseph Szücs [József M. Szűcs].

[3] S. Bácsó and M. Matsumoto. On Finsler spaces of Douglas type—a generalization of the notion of Berwald space. *Publ. Math. Debrecen*, 51(3-4):385–406, 1997.

[4] S. Bácsó and M. Matsumoto. On Finsler spaces of Douglas type. II. Projectively flat spaces. *Publ. Math. Debrecen*, 53(3-4):423–438, 1998.

[5] A. Bölcskei and B. Szilágyi. Geodesics and the frenet formulas in *Sol* geometry. to appear.

[6] J. Douglas. The general geometry of paths. *Ann. of Math. (2)*, 29(1-4):143–168, 1927/28.

[7] J. Gegenberg, S. Vaidya, and J. F. Vázquez-Poritz. Thurston geometries from eleven dimensions. *Classical Quantum Gravity*, 19(23):L199–L204, 2002.

[8] Z. Shen. *Differential geometry of spray and Finsler spaces.* Kluwer Academic Publishers, Dordrecht, 2001.

[9] H. Weyl. Zur infinitesimalgeometrie: Einordnung der projektiven und konformen auffassung. *Nachr. Koenigl. Gesellsch. Wiss. Goettingen, Math.-Phys. Kl.*, pages 99–112, 1921.

Faculty of Informatics,
Debrecen University,
Hungary
E-mail address: `bacsos@inf.unideb.hu`

Department of Geometry,
Budapest University of Technology and Economics,
Hungary
E-mail address: `szilagyi@math.bme.hu`