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SOME COMMON FIXED POINT THEOREMS FOR SELFMAPPINGS IN UNIFORM SPACE

MEMUDU O. OLATINWO

Abstract. In this paper, we establish some common fixed point theorems for selfmappings in uniform spaces by employing the concepts of an A-distance, an E-distance as well as the notion of comparison function. A more general contractive condition than that used to establish some of the results of Aamri and El Moutawakil [1] is employed to obtain our results. Our results are generalizations of some of the results of [1].

1. Introduction

A *uniform space* (X, Φ) is a nonempty set X equipped with a nonempty family Φ of subsets of $X \times X$ satisfying the following properties:

(i) if U is in Φ, then U contains the diagonal $\{(x, x)|x \in X\}$;
(ii) if U is in Φ and V is a subset of $X \times X$ which contains U, then V is in Φ;
(iii) if U and V are in Φ, then $U \bigcap V$ is in Φ;
(iv) if U is in Φ, then there exists V in Φ, such that, whenever (x, y) and (y, z) are in V, then (x, z) is in U;
(v) if U is in Φ, then $\{(y, x)|(x, y) \in U\}$ is also in Φ.

Φ is called the *uniform structure* of X and its elements are called *entourages or neighbourhoods or surroundings.*

The space (X, Φ) is called *quasiuniform* if property (v) is omitted. The definition of uniform space is contained in Bourbaki [4], Zeidler [13] as well as available on the internet (by Wikipedia, the free encyclopedia).

The concept of a W-distance on metric space was introduced by Kada et al [6] to generalize some important results in nonconvex minimizations and in fixed point theory for both W-contractive and W-expansive maps. The theory of fixed point or common fixed point for contractive or expansive selfmappings in complete metric space has been well-developed. Interested readers can consult Berinde [2, 3], Jachymski [5], Kada et al [6], Kang [7], Rhoades [8], Rus

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[10], Rus et al [11], Wang et al [12] and Zeidler [13] for further study of fixed point or common fixed point theory.

Using the ideas of Kang [7], Montes and Charris [9] established some results on fixed and coincidence points of maps by means of appropriate W-contractive or W-expansive assumptions in uniform space. Furthermore, Aamri and El Moutawakil [1] proved some common fixed point theorems for some new contractive or expansive maps in uniform spaces by introducing the notions of an A-distance and an E-distance.

In Aamri and El Moutawakil [1], the following contractive definition was employed: Let $f, g\colon X \to X$ be selfmappings of X. Then, we have

$$p(f(x), f(y)) \leq \psi(p(g(x), g(y))), \forall x, y \in X, \tag{1}$$

where $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a nondecreasing function satisfying

(i) for each $t \in (0, +\infty), 0 < \psi(t)$,
(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \forall t \in (0, +\infty)$.

ψ satisfies also the condition $\psi(t) < t$, for each $t > 0$.

In this paper, we shall establish some common fixed point theorems by employing a more general contractive condition than (1).

We shall employ the concepts of an A-distance, an E-distance as well as the notion of comparison function in this work. Berinde [2, 3] extended the Banach's fixed point theorem using different contractive definitions involving the concept of the comparison functions. Rus [10] and Rus et al [11] also contain various generalizations and extensions of the Banach's fixed point theorem in which the contractive conditions involve some comparison functions.

Our results are generalizations of Theorems 3.1–3.3 of [1].

2. Preliminaries

We shall require the following definitions and lemma in the sequel. The Remark 2.1 and Definitions 2.2–2.7 are contained in Aamri and El Moutawakil [1]. Let (X, Φ) be a uniform space.

Remark 2.1. When topological concepts are mentioned in the context of a uniform space (X, Φ), they always refer to the topological space $(X, \tau(\Phi))$.

Definition 2.2. If $V \in \Phi$ and $(x, y) \in V, (y, x) \in V$, x and y are said to be *V-close.* A sequence $\{x_n\}_{n=0}^{\infty} \subset X$ is said to be a *Cauchy sequence* for Φ if for any $V \in \Phi$, there exists $N \geq 1$ such that x_n and x_m are V-close for $n, m \geq N$.

Definition 2.3. A function $p\colon X \times X \to \mathbf{R}^+$ is said to be an *A-distance* if for any $V \in \Phi$, there exists $\delta > 0$ such that if $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ for some $z \in X$, then $(x, y) \in V$.

Definition 2.4. A function $p\colon X \times X \to \mathbf{R}^+$ is said to be an *E-distance* if

(p_1) p is an A-distance,
(p_2) $p(x, y) \leq p(x, z) + p(z, y), \forall x, y \in X$.

Definition 2.5. A uniform space (X, Φ) is said to be *Hausdorff* if and only if the intersection of all $V \in \Phi$ reduces to the diagonal $\{(x, x)|x \in X\}$, i.e. if $(x, y) \in V$ for all $V \in \Phi$ implies $x = y$. This guarantees the uniqueness of limits of sequences. $V \in \Phi$ is said to be *symmetrical* if $V = V^{-1} = \{(y, x)|(x, y) \in V\}$.

Definition 2.6. Let (X, Φ) be a uniform space and p be an A-distance on X.

(i) X is said to be *S-complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} p(x_n, x) = 0$.

(ii) X is said to be *p-Cauchy complete* if for every p-Cauchy sequence $\{x_n\}_{n=0}^{\infty}$, there exists $x \in X$ with $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$.

(iii) $f\colon X \to X$ is said to be *p-continuous* if $\lim_{n\to\infty} p(x_n, x) = 0$ implies that $\lim_{n\to\infty} p(f(x_n), f(x)) = 0$.

(iv) $f\colon X \to X$ is *$\tau(\Phi)$-continuous* if $\lim_{n\to\infty} x_n = x$ with respect to $\tau(\Phi)$ implies $\lim_{n\to\infty} f(x_n) = f(x)$ with respect to $\tau(\Phi)$.

(v) X is said to be *p-bounded* if $\delta_p(X) = \sup\{p(x, y)|x, y \in X\} < \infty$.

Definition 2.7. Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Two selfmappings f and g on X are said to be *p-compatible* if, for each sequence $\{x_n\}_{n=0}^{\infty}$ of X such that $\lim_{n\to\infty} p(f(x_n), u) = \lim_{n\to\infty} p(g(x_n), u) = 0$ for some $u \in X$, then we have $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

We shall also state the following definition of a comparison function which is required in the sequel to establish some common fixed point results in uniform space.

Definition 2.8 (Berinde [2,3])**.** A function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is called a *comparison function* if:

(i) ψ is monotone increasing;

(ii) $\lim_{n\to\infty} \psi^n(t) = 0, \forall t \geq 0$.

The definition is also contained in [10, 11].

Remark 2.9. Every comparison function satisfies the condition $\psi(0) = 0$.
Also, both conditions (i) and (ii) imply that $\psi(t) < t, \forall t > 0$.

In this paper, we shall employ the following contractive definition:
Let $f, g\colon X \to X$ be selfmappings of X. There exist $L \geq 0$ and a comparison function $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ such that $\forall x, y \in X$, we have

$$p(f(x), f(y)) \leq Lp(x, g(x)) + \psi(p(g(x), g(y))). \tag{2}$$

Remark 2.10. The contractive condition (2) is more general than (1) in the sense that if $L = 0$ in (2), then we obtain (1) stated in this paper which was employed by Aamri and El Moutawakil [1].

The following Lemma shall be required in the sequel.

Lemma 2.11. *Let (X, Φ) be a Hausdorff uniform space and p be an A-distance on X. Let $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ be arbitrary sequences in X and $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty}$ be sequences in $\mathbf{R}^+$ converging to 0. Then, for $x, y, z \in X$, the following hold:*

(a) *If $p(x_n, y) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in \mathbf{N}$, then $y = z$. In particular, if $p(x, y) = 0$ and $p(x, z) = 0$, then $y = z$.*
(b) *If $p(x_n, y_n) \leq \alpha_n$ and $p(x_n, z) \leq \beta_n, \forall n \in \mathbf{N}$, then $\{y_n\}_{n=0}^{\infty}$ converges to z.*
(c) *If $p(x_n, x_m) \leq \alpha_n \forall m > n$, then $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X, Φ).*

Remark 2.12. Lemma 2.11 is contained in [1], [7] and [9].

Remark 2.13. A sequence in X is p-Cauchy if it satisfies the usual metric condition. See [1] for this remark.

3. The Main Results

The main results of this paper are the following:

Theorem 3.1. *Let (X, Φ) be a Hausdorff uniform space and p an A-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \ldots,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) *$f(X) \subseteq g(X)$,*
(ii) *$p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \ldots,$*
(iii) *$f, g \colon X \to X$ satisfy the contractive condition (2).*

Suppose also that $\psi \colon \mathbf{R}^+ \to \mathbf{R}^+$ is a comparison function. Then, f and g have a common fixed point.

Proof. Let $x_0 \in X$. Choose $x_1 \in X$ such that $f(x_0) = g(x_1)$, choose $x_1 \in X$ such that $f(x_1) = g(x_2)$, and in general, choose $x_n \in X$ such that $f(x_{n-1}) = g(x_n)$.

We recall that $x_n = f(x_{n-1}), n = 1, 2, \ldots$, so that by conditions (ii) and (iii) of the Theorem, we obtain

$$\begin{aligned}
p(f(x_n), f(x_{n+m})) \leq & Lp(x_n, g(x_n)) + \psi(p(g(x_n), g(x_{n+m}))) \\
= & Lp(f(x_{n-1}), f(x_{n-1})) + \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
= & \psi(p(f(x_{n-1}), f(x_{n+m-1}))) \\
\leq & \psi(Lp(x_{n-1}, g(x_{n-1})) + \psi(p(g(x_{n-1}), g(x_{n+m-1})))) \\
= & \psi(Lp(f(x_{n-2}), f(x_{n-2})) + \psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
= & \psi(\psi(p(f(x_{n-2}), f(x_{n+m-2}))) \\
= & \psi^2(p(f(x_{n-2}), f(x_{n+m-2})) \\
\leq & \cdots \leq \psi^n(p(f(x_0), f(x_m)) \leq \psi^n(\delta_p(X)),
\end{aligned}$$

from which we have that

$$p(f(x_n), f(x_{n+m})) \leq \psi^n(\delta_p(X)), \tag{3}$$

where $p(f(x_0), f(x_m)) \leq \delta_p(X)$ and $\delta_p(X) = \sup\{p(x, y) | x, y \in X\} < \infty$.

Therefore, using the definition of comparison function in (3) yields

$$\psi^n(\delta_p(X)) \to 0 \text{ as } n \to \infty,$$

from which it follows that

$$p(f(x_n), f(x_{n+m})) \to 0 \text{ as } n \to \infty.$$

Hence, by applying Lemma 2.11(c), we have that $\{f(x_n)\}_{n=0}^{\infty}$ is a p-Cauchy sequence. Since X is S-complete, $\lim_{n\to\infty} p(f(x_n), u)) = 0$, for some $u \in X$, and therefore $\lim_{n\to\infty} p(g(x_n), u)) = 0$.

Since f and g are p-continuous, then

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0.$$

Also, since f and g are commuting, then $fg = gf$, so that we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0,$$

so that by Lemma 2.11(a), we obtain that $f(u) = g(u)$.

Since $f(u) = g(u), fg = gf$, we have $f(f(u)) = f(g(u)) = g(f(u)) = g(g(u))$. Suppose that $p(f(u), f(f(u))) \neq 0$. Using (2) and the condition $\psi(t) < t, \forall t > 0$ in the Remark 2., then, we have

$$\begin{aligned}
p(f(u), f(f(u))) \leq & Lp(u, g(u)) + \psi(p(g(u), g(f(u)))) \\
= & Lp(f(u), f(u)) + \psi(p(f(u), f(f(u)))) \\
= & \psi(p(f(u), f(f(u))) < p(f(u), f(f(u))),
\end{aligned}$$

which is a contradiction. Therefore, $p(f(u), f(f(u))) = 0$.

Condition (ii) of the Theorem yields $p(f(u), f(u)) = 0$. Since $p(f(u), f(f(u))) = 0$ and $p(f(u), f(u)) = 0$, applying Lemma 2.11(a) then yields $f(f(u)) =$

$f(u)$. Thus, we have $g(f(u)) = f(f(u)) = f(u)$. Hence, $f(u)$ is a common fixed point of f and g.

The proof is similar when f and g are $\tau(\Phi)$-continuous as S-completeness implies p-Cauchy completeness. □

Remark 3.2. Theorem 3.1 is a generalization of Theorem 3.1 of Aamri and El Moutawakil [1]

Theorem 3.1 is an existence result for the common fixed point of f and g, while the next two results guarantee the uniqueness of the common fixed point.

Theorem 3.3. *Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \ldots,$$

with $x_0 \in X$. Let f and g be commuting p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) $f(X) \subseteq g(X)$,
(ii) $p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \ldots$,
(iii) $f, g\colon X \to X$ *satisfy the contractive condition (2).*

Suppose also that $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a comparison function. Then, f and g have a unique common fixed point.

Proof. f and g have a common fixed point since an E-distance function p is an A-distance. Suppose that there exist $u, v \in X$ such that $f(u) = g(u) = u$ and $f(v) = g(v) = v$.

Let $p(u, v) \neq 0$. Then, we have

$$\begin{aligned} p(u, v) =& p(f(u), f(v)) \leq Lp(u, g(u)) + \psi(p(g(u), g(v))) \\ =& Lp(u, u) + \psi(p(u, v)) = \psi(p(u, v)) < p(u, v), \end{aligned}$$

which is a contradiction. Therefore, we have $p(u, v) = 0$. By carrying out a similar process, we also have that $p(v, u) = 0$.

Using condition (p_2) of Definition 2.4, we have $p(u, u) \leq p(u, v) + p(v, u)$, from which it follows that $p(u, u) = 0$. Since $p(u, u) = 0$ and $p(u, v) = 0$, then by Lemma 2.11(a), we have that $u = v$. □

Remark 3.4. Theorem 3.3 is a generalization of Theorem 3.2 as well as corollaries 3.1 & 3.2 of Aamri and El Moutawakil [1].

Theorem 3.5. *Let (X, Φ) be a Hausdorff uniform space and p an E-distance on X. Suppose that X is p-bounded and S-complete. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by*

$$x_n = f(x_{n-1}), n = 1, 2, \ldots,$$

with $x_0 \in X$. Let f and g be p-compatible, p-continuous or $\tau(\Phi)$-continuous selfmappings of X such that

(i) *$f(X) \subseteq g(X)$,*

(ii) *$p(f(x_i), f(x_i)) = 0, \forall x_i \in X, i = 0, 1, 2, \dots,$*

(iii) *$f, g\colon X \to X$ satisfy the contractive condition (2).*

Suppose also that $\psi\colon \mathbf{R}^+ \to \mathbf{R}^+$ is a comparison function. Then, f and g have a unique common fixed point.

Proof. Just as in the proof of Theorem 3.1, we have for some $u \in X$ that $\lim_{n\to\infty} p(f(x_n, u)) = \lim_{n\to\infty} p(g(x_n, u)) = 0$. Since f and g are p-continuous, we have

$$\lim_{n\to\infty} p(f(g(x_n)), f(u)) = \lim_{n\to\infty} p(g(f(x_n)), g(u)) = 0,$$

while the assumption that f and g are p-compatible implies the following $\lim_{n\to\infty} p(f(g(x_n)), g(f(x_n))) = 0$.

Furthermore, by condition (p_2) of Definition 2.4, we have that

$$p(f(g(x_n)), g(u)) \leq p(f(g(x_n)), g(f(x_n))) + p(g(f(x_n)), g(u)) \tag{3}$$

Taking limits in (3) and applying Lemma 2.11(a), then we have

$$\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0.$$

Since $\lim_{n\to\infty} p(f(g(x_n)), f(u)) = 0$ and $\lim_{n\to\infty} p(f(g(x_n)), g(u)) = 0$, then by Lemma 2.11(a) we have $f(u) = g(u)$.

The rest of the proof is as in Theorem 3.3. □

Remark 3.6. Theorem 3.5 is a generalization of Theorem 3.3 of Aamri and El Moutawakil [1].

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Department of Mathematics,
Obafemi Awolowo University,
Ile-Ife, Nigeria.
E-mail address: polatinwo@oauife.edu.ng, molaposi@yahoo.com